A Fourth-Order Runge-Kutta Method for Numerical Solution of the Kuramoto-Sivashinsky Equation

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ABSTRACT

We present a numerical method for the Kuramoto-Sivashinsky equation with corresponding initial and boundary conditions. The exponential time differencing Runge-Kutta 4 method is used to solve the diagonal example of the Kuramoto-Sivashinsky equation.

Keywords: Kuramoto–Sivashinsky equation, Integration factor, Exponential time differencing methods, Runge-Kutta method.

Introduction

Nonlinear coupled partial differential equations are very important in a variety of scientific fields, especially in fluid mechanics, solid-state physics, plasma physics, plasma waves, capillary-gravity waves, and chemical physics. Unfortunately, the explicit solution of these equations in an analytical form is possible only in special simple cases, and, as a result, the possibility of analyzing mathematical models built based on differential equations is provided using approximate numerical solution methods. The numerical method is an algorithm for calculating the approximate values of the desired solution at the points of a finite set of arguments.

In many branches of mathematical physics, when modeling nonlinear evolutionary processes use the Kuramoto-Sivashinsky equation or its natural modifications cations and generalizations. Usually, this equation is considered together with the natural ones for the boundary conditions, in most papers the boundary conditions were chosen as periodic boundary conditions. The Kuramoto–Sivashinsky equation describes one of the simplest nonlinear systems that exhibit turbulence Kuramoto [1]. It has been used to study various reaction-diffusion problems and, in particular, it is used to model the thermal mechanism of flame propagation or combustion waves G. I. Sivashinsky [2]. Another version of the Kuramoto – Sivashinsky equation is used as a model of the formation process reliefs of various configurations on the surface of semiconductors under the influence of flow ions. This physical and technological process has found wide application in modern nanoelectronics Bradley, Harper[3]. In addition, the Kuramoto-Sivashinsky equation has found wide application in the theory of

surface self-organization during ion irradiation, electrochemical etching of semiconductors and metals, as well as wave fluid, flows, diffusion chaos, long waves at the interface between two viscous liquids. The problem of numerical solution of partial differential equations is an area of intensive research.

To develop an understanding associated with a nonlinear phenomenon with the Kuramoto-Sivashinsky equation, either an exact or numerical solution is required. Since an exact solution to the equation is rarely possible due to its non-linearity, numerical methods play an essential role in its equations. Therefore, the development of an efficient and accurate numerical method is of practical importance and attracts the attention of many researchers. For example, the Kuramoto-Sivashinsky equation was studied numerically in [3].

Since the exact solution to the equation is rarely possible due to its non-linearity, numerical methods play an essential role in the solution of the problem. Many powerful methods have been created and successfully developed to find the exact solution to the Kuramoto-Sivashinsky equation. One of such methods to resolve the problem is the exponential time difference (ETD) scheme was introduced in [7] and modified by the authors of the work [8]. The basics of the ETD scheme are exactly integrating the linear parts of a differential equation and approximating the non-linear terms by a polynomial, which is then exactly integrated [9-14].

We use Runge-Kutta's fourth-order exponential time difference (ETDRK4) method to obtain an efficient numerical solution for the Kuramoto-Sivashinsky. We solve the diagonal problem of the Kuramoto-Sivashinsky equation using the Runge-Kutta 4 ETD method and present the results using the Matlab program.

Methodology

Our 1D problem can be written as [8]

$$u_t = -uu_x - u_{xx} - u_{xxxx}, \qquad x \in [0, 32\pi]. \tag{1}$$

The last equation contains both second and fourth-order derivatives, which means it's a complex behavior. The second-order term acts as an energy source and has a destabilizing effect, and the nonlinear term transfers energy from low to high wavenumbers, whereas the fourth-order term has a stabilizing effect. We use the initial condition

$$u(x,\ 0) = \cos\left(\frac{x}{16}\right)\left(1+\sin\left(\frac{x}{16}\right)\right). \tag{2}$$

As the equation is periodic, we discretize the spatial part using a Fourier spectral method. Transforming to Fourier space gives,

$$\hat{u}_t = -\frac{ik}{2}\hat{u}^2 + (k^2 - k^4)\hat{u}, \tag{4}$$

and, in the standard form, we have

$$(L\hat{u})(k) = (k^2 - k^4)\hat{u}(k),$$

$$N(\hat{u}, t) = N(\hat{u}) = -\frac{ik}{2}\left(F\left(\frac{1}{F(\hat{u})}\right)^2\right) \tag{5}$$

where F denotes the discrete Fourier transform [8, 10]. We solve the problem in Fourier space and use ETDRK4 time-stepping for t=100 to t = 200. The time evolution for the Kuramoto-Sivashinsky equation is presented in Figures 1 and 2.

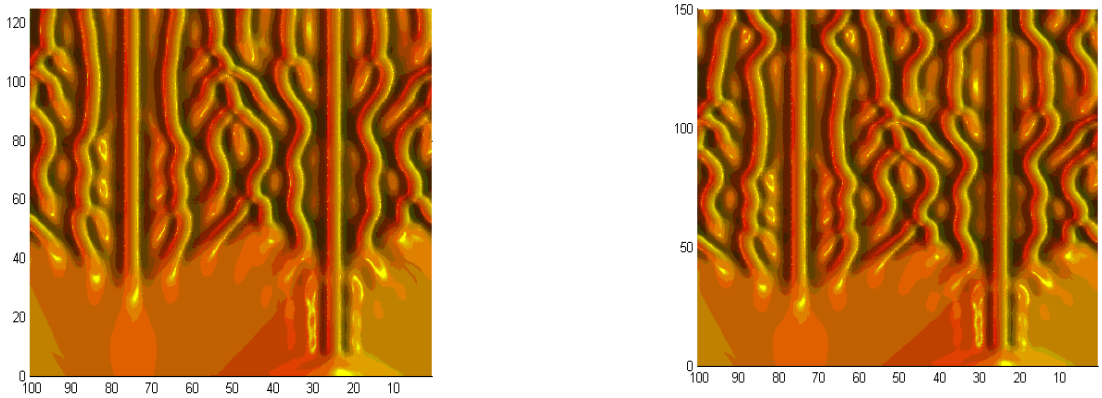

Fig. 1. Time evolution for the Kuramoto-Sivashinsky equation for t=125and 150.

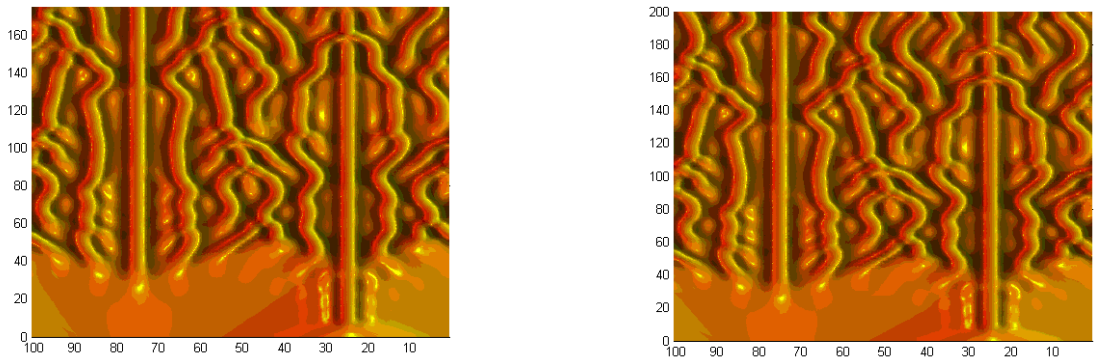

Fig. 2. Time evolution for the Kuramoto-Sivashinsky equation for t=175and 200.

The obtained numerical results show that using Runge-Kutta's fourth-order exponential time difference (ETDRK4) method for a nonlinear term with spectral discretization for time integration one can have better results for large time steps.

Conclusion

We have applied Runge-Kutta's fourth-order exponential time difference (ETDRK4) method to the Kuramoto-Sivashinsky equation. For the simulation tests, we chose periodic initial conditions and applied Fourier spectral approximation for the spatial discretization. It was shown that this method is preferable since it requires less iteration for its implementation and has a low error. To implement the computational procedure, the Matlab computer algebra system was used. Analysis of the obtained results confirmed the effectiveness of the method.

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