Numerical Algorithm for A Computational Experiment for An Applied Optimization Problem in Systems with Distributed Parameters

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ABSTRACT
The paper considers the optimization problem that takes place in the optimal control of oil production, where the state of the system is described by equations of the elliptic type. A two-stage algorithm for solving the problem with the proof of theorems on qualitative estimates of iterative processes is presented.

Keywords: optimization, optimal control, oil production, state, system, equation, elliptic type, algorithm, solutions, functional, minimum, computational experiment, iterative process, difference analogue, convergence.

Introduction

The problem statement of the modern theory of control of systems with distributed parameters has great potential and is associated with a real physical basis, and the role of mathematical modeling is very large in this [1,2].

The practical application of the problems of analysis and synthesis of systems with distributed parameters is directly related to the applied problems of development and additional development of mineral deposits (in particular, oil and gas fields). At the development stage of these fields, the problems of analysis and synthesis are solved in order to determine the qualitative and quantitative indicators of development associated with hydrostatic and hydrodynamic processes. At the same time, the proposed mathematical model and the created numerical algorithm for the computational experiment are a useful mathematical research tool [3].

Oil (gas) reservoirs and wells located in it (production and injection) in a single hydrodynamic connection is a multi-connected system with distributed parameters. The conditions of the productive formation and the wells located in it are constantly changing in time, as an object of control. Thus, the “reservoir-well” system can be represented as a technical control system.

The state of the system under consideration is described by a partial differential equation, and the parameters change in space and time.

The considered problem of optimal control of systems with distributed parameters is described by equations of elliptic type, and linear restrictions are placed on the state functions.

Formulation of the problem. Let Ω bounded connected area n- dimensional $(n = 2,3)$ space R^n with border Γ, $x = (x_1, \dots, x_n)$ - point in this space.

It is required to find the minimum of the functional

$$I = \int\limits_{\Omega} \left[\sum_{i=1}^{n} b(x) \left(\frac{\partial p(x)}{\partial x_i} \right)^2 + a(x) p^2(x) \right] dx - 2 \int\limits_{\Omega} C(x) p(x) dx \qquad (1)$$

under the conditions that

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[b(x) \frac{\partial p(x)}{\partial x_i} \right] - a(x) p(x) = g(x),\ x \in \Omega, \qquad (2)$$

$$p(x) = 0,\ x \in \Gamma;\ p(x) \ge f(x),$$
$$q(x) \ge d(x),\ x \in \Omega. \qquad (3)$$

Such a problem arises in the optimization of oil production. Wherein $p(x)$ - reservoir pressure, $q(x)$ - fluid flow rate, functional (1) - internal energy of the oil reservoir [1,4,5].

A special case of this problem is the minimization of the functional (1) under the conditions

$$p(x) = 0,\ x \in \Gamma,\ p(x) \ge f(x),\ x \in \Omega. \qquad (4)$$

finds application in the theory of elasticity and underground hydraulics. Examples include the problem of the equilibrium of a membrane [1] and the problem with a free boundary associated with the flow of a liquid through porous media [3].

The existence of a solution to problem (1) - (4) necessary and sufficient optimality conditions have been studied in many works, for example, in [4-8].

Carrying out computational experiments is directly related to the creation of numerical algorithms and a set of programs for the implementation on a computer of the problems of analysis and synthesis to be solved.

Let in space R^n a grid with a step h_i on i – th coordinate $(i = 1, 2, \dots, n)$. We call a grid node internal if it belongs to the region $\bar{\Omega} = \Omega \cup \Gamma$ along with all neighboring nodes. The set of such nodes will be denoted S. We call a grid node boundary if it belongs to Ω, but at least one of its nodes does not belong Ω. We denote the set of boundary nodes γ. Let under given control $q(x), x \in S$, state of the system $p(x),\ x \in S \cup \gamma,$ can be found from the solution of a system of linear algebraic equations, which is a finite-difference analogue of equation (2)

$$\sum_{i=1}^{n} (bp_{xi})_{xi} - a(x)\, p\,(x) = q(x),\ x \in S, \qquad (5)$$

provided that on the border of the region

$$p(x) = 0,\ x \in \gamma. \qquad (6)$$

Here indicated

$$p_{xi}(x) = \frac{1}{h_i} \left[p(x + h_i \ell_i) - p(x) \right],$$

where ℓ_i - unit vector with 1 in i - th position,

$$p_{\bar{x}i}(x) = \frac{1}{h_i} \left[p(x) - p(x - h_i \ell_i) \right],$$

$$(bp_{x_i}) = b(x + \frac{1}{2} h_i \ell_i) p_{x_i}(x),$$

$$(bp_{x_i})_{\bar{x}_i} = \frac{1}{h_i} \left[b(x + \frac{1}{2} h_i \ell_i) p_{x_i}(x) - b(x - \frac{1}{2} h_i \ell_i) p_{\bar{x}_i}(x) \right].$$

We pose the problem of determining the state functions $p(x)$, satisfying (5), (6) and under the restrictions

$$p(x) \ge f(x),\ q(x) \ge d(x), x \in S, \qquad (7)$$

minimizing the functional

$$I = \sum_{x \in S \cup \gamma} \left[\sum_{i=1}^{n} (bp_{x_i}) p_{x_i}(x) + a(x) p^2(x) \right] - 2 \sum_{x \in S} c(x) p(x). \qquad (8)$$

Here $a, b, c, d,\ f$ - predefined functions,

$b(x) \geq v > 0, \quad a(x) \geq 0, \; x \in \bar{\Omega}$
(9)

In formula (8) and henceforth, we assume that $p(x) = 0$ at grid nodes in non-area Ω.

In a finite-dimensional space $H(S)$ functions defined in nodes $x \in S$, we introduce the norm and the scalar product by the formulas

$$\|p\| = \left[\sum_{x \in S} p^2(x) \right]^{\frac{1}{2}}, \; (p, \vartheta) = \sum_{x \in S} p(x)\vartheta(x).$$

Outside S functions from $H(S)$ set equal to zero.

In space $H(S)$ define the operator Lp:

$$Lp(x) = -\sum_{i=1}^{n} (bp_{x_i})_{\bar{x}i} \, a(x)\, p(x).$$

This is a self-adjoint operator, i.e. $(Lu, \vartheta) = (u, L\vartheta)$.

Since under conditions (6) the identity [2]

$$-\sum_{x \in S} (bp_{x_i})p_{\bar{x}_i} = \sum_{x \in S \cup \gamma} (b\, p_{x_i})p_{x_i},$$

then the quadratic part of the functional (8) can be written as *(Lp,p).* Under conditions (9) and taking into account the fact that $p(x) = 0$ outside nodes *S*, the inequality is true [1]

$$(Lp, p) \geq v_1 \| p \|, v_1 > 0, \quad (Lp, \, p),$$

therefore *(Lp,p)* – positive definite quadratic form.

Along with problem (5) - (8) (let's call it problem 1), we will consider its special case (problem II), when there is no lower restriction on the function $q(x)$ (and control (5) is dropped from consideration and the problem is reduced to minimizing functional (8) under the conditions

$$p(x) = 0, \, x \in \gamma, \; p(x) \geq f(x), \, x \in S.$$

Problem I is a difference analog of the following variational problem.

The convergence of the solution of difference problem 2 with a change in the grid step to the solution of problem (1) - (4) is substantiated in [9]. It is assumed that the function $f(x)$ quite smooth in Ω and $f(x) = 0$ at $x \in \Gamma$. With minor changes, these studies are also applicable to a more general problem (1) - (3).

In works [1, 10], to solve problem (1) - (4), the methods of local variations and gradient projection were used. In this article, based on the use of properties characteristic of objects described by elliptic type equations, an effective algorithm for solving problem (5) - (8) is constructed.

Without limitation, we will assume that in the conditions (7) $f(x) = 0$. This can be achieved by replacing in problem (5) – (8) p on $p + \bar{f}$, where $\bar{f}(x) = f(x), \; x \in S$ and $\bar{f}(x) = 0, \; x \in \gamma$ and replacing q on $q - L\bar{f}$. Wherein *c, d* are replaced accordingly with $c - L\bar{f}$ and $d - L\bar{f}$. Taking into account this remark and the notation introduced earlier, we write problem (5) – (8) in the form

$(L\,p, \, p) - 2(c, p) \to \min$
(10)

under conditions

$Lp + q(x) = 0, \; x \in S,$
(11)

$p(x) \geq 0, q(x) \geq d(x), \, x \in S.$
(12)

For the applicability of the proposed algorithm, it is essential that condition (11), (12) be satisfied.

Algorithm for solving the problem. The algorithm given below is constructed according to the scheme proposed in [11,12] for solving optimal control problems in systems described by elliptic-type equations with boundary control and observation.

The algorithm uses the following properties of the operator *L.*

Properties 1. For any function $p, q \in H(S)$ and many $A \subset S$ there are functions p_A, q_A, satisfying equation (11) and conditions

$$q_A(x) = q(x), \, x \in A, \; p_A(x) = 0, \, x \in S / A.$$

Substantiation. Required Features p_A, q_A, can be found in the following way. We solve the system of linear algebraic equations

$Lp_A = -q(x),\ x \in A$ under conditions $p_A(x) = 0,\ x \in S \setminus A$ (such a function exists and is unique) [2], and we set $q_A(x) = q(x), x \in A$ at $x \in S \setminus A,\ q_A(x) = -Lp_A, x \in A$.

Properties 2. Let the functions $p, g \in H(S)$ satisfy equation (11).

Then.

a) $q(x) \le 0,\ x \in A,\ p(x) \ge 0,\ x \in S \setminus A,$ then $p(x) \ge 0,\ x \in S$;

b) if $q(x) \le 0,\ x \in A,\ p(x) = 0,\ x \in S \setminus A,$ then $q(x) \ge 0,\ x \in S \setminus A$.

Substantiation. a) in that $Lp = -q(x) \ge 0,\ x \in A$ and $p(x) \ge 0,\ x \in S \setminus A$ and $p(x) \ge 0, x \in S$; then it follows from the maximum principle for finite-difference analogs of elliptic-type equations [2] that $q(x) \le 0, x \in A$ *b)* in that $p(x) = 0,\ x \in S \setminus A,\ p(x) \ge 0,\ x \in A$ (by property 2a), then at $x \in S \setminus A,$

$$q(x) = \sum_{i=1}^{n} \frac{1}{h_i} \left[\begin{array}{l} b(x + \frac{1}{2} h_i \ell_i) p(x + h_i \ell_i) + \\ +b(x - \frac{1}{2} h_i \ell_i) p(x - h_i \ell_i) \end{array} \right],$$

those $q(x) \ge 0,\ x \in S \setminus A$.

The necessary sufficient optimality conditions for problem (10) – (12) are as follows. Functions $p^0, q^0 \in H(S)$ satisfying (11), (12) are solutions to problem (10) - (12) if and only if there are functions $y^0,\ v^0 \in H(S)$, such that the relations

$$Ly^0 + v^0(x) = 0, \quad (13)$$

$$q^0(x) + v^0(x) + c(x) \le 0, \quad y^0(x) \ge 0, \quad (14)$$

$$p^0(x)\left[q^0(x) + v^0(x) + c(x) \right] = 0,$$

$$y^0(x)\left[q^0(x) - d(x) \right] = 0,\ x \in S.$$

This is the Kuhn-Tucker condition for the considered problem of quadratic programming [10].

The algorithm consists of two stages. At the first stage, there are many grid nodes $S + \gamma$ and features y^0, v^0, satisfying the system of equations (13) and the conditions

$$y^0(x) \ge 0,\ v^0(x) = -(c(x) + d(x)),\ x \in S,$$

$$y^0(x) = 0, v^0(x) \le -(c(x) + d(x)), x \in S \setminus S^0. \quad (15)$$

At the second stage, there are many grid nodes $S + \gamma$ and features p^0, q^0, satisfying the system of equations (11) and the conditions

$$p^0(x) \ge 0,\ q^0(x) = d(x), \quad x \in S^0,$$

$$p^0(x) \ge 0,\ q^0(x) = -(c(x) + v^0(x)), \quad x \in F^0, \quad (16)$$

$$p^0(x) = 0,\ q^0(x) \le -(c(x) + v^0(x)),$$

$$x \in S \mid (S^0 \cup F^0),$$

Given that $d(x) \le 0,\ x \in S$ and applying properties 2,b) on the set $S \setminus F^0$, we get $q^0(x) \ge 0, d(x) \le 0\ x \in S / (S^0 \cup F^0)$, those conditions (12) are satisfied. Conditions (15) and (16) together coincide with conditions (13), (14), therefore $p^0,\ q^0$ optimal solution of problem (10) – (12).

Remark 1. Function $y^0 \in H(S)$, found at the first stage of the procedure is the solution of the following optimal control problem

$$Ly - 2(c + d, y) \to \min,\ y(x) \ge 0,\ x \in S.$$

Indeed, conditions (13), (15) are necessary and sufficient Kuhn-Tucker conditions for such a quadratic programming problem. Thus, the first stage of the algorithm (if you put in it $d(x) = 0$) solves the above problem 2.

Remark 2. Function $\vartheta^0, y^0 \in H(S)$, satisfying conditions (13). (15) are the solution of the linear programming problem [10]

$(L, y) \to \max,\ L\,y + \vartheta(x) = 0, \quad y(x) \ge 0$,

$\vartheta(x) \le -(c(x) + d(x)),\ x \in S$, where $\vartheta(x)$ any given function.

The first stage of the algorithm.

1. Let k=1, S^k - empty set.

2. Let's find functions $y^k, \vartheta^k \in H(S)$, satisfying the system of equations (13) and the conditions

$\vartheta^k(x) = -(c(x) + d(x),\ x \in S^k,\ \ y^k(x) = 0,\ x \in S \setminus S^k$.

By property 1, such functions exist. To determine them, it is necessary to solve the system of linear algebraic equations $Ly^k = c(x) + d(x),\ x \in S^k$, under conditions $y^k(x) = 0,\ x \in S \setminus S^k$ and put $\vartheta^k(x) = -Ly^k$ at $x \in S \setminus S^k$. Then what $y' = v^1 = 0$.

3. Select a set of grid nodes $D^k \subset S \setminus S^k$, in which the inequality $\vartheta^k(x) > -(c(x) + d(x)),\ x \in D^k$.

4. If D^k empty, we assume $S^0 = S^k,\ y^0 = y^k,\ \vartheta^0 = \vartheta^k$ and the first stage is completed.

5. We believe $S^{k+1} = S^k \cup D$, assign k meaning *k+1* and repeat steps 2-4.

As *S*- finite set and $S' \subset S^2 \subset .... \subset S$, then after a finite number of iterations the set D^k will be empty, therefore, the first stage will be completed in a finite number of steps.

Theorem 1. Variables $y^0, \vartheta^0 \in H(S)$, we find at the first stage of the procedure, satisfy conditions (13), (15), and over the iterations of the procedure, the inequalities

$$y^{k+1}(x) \ge y^k(x),\ x \in S,\ \vartheta^{k+1}(x) \ge \vartheta^k(x),\ \ x \in S \setminus S^{k+1}. \quad (17)$$

Substantiation. Let us first prove inequalities (17). These inequalities follow from properties 2 when applied to the function y^k, ϑ^k and take into account that

$$y^{k+1}(x) = y^k(x) = 0,\ x \in S \setminus S^{k+1},$$

$$\vartheta^{k+1}(x) = \vartheta^k(x) = -(c(x) + d(x)),\ \ x \in S^k,$$

$$\vartheta^{k+1}(x) = -(c(x) + d(x)) < \vartheta^k(x),\ x \in S^{k+1} \setminus S^k.$$

Condition (15) follows from the method of constructing the functions ϑ^k, y^k and inequality (17) and the fact that at the last iteration of the first one the set D^k empty.

The second stage of the algorithm.

1. Let k=1, F^k - empty set.

2. Let's find functions $p^k, q^k \in H(S)$, satisfying the system of equations (11) and the conditions

$q^k(x) = d(x),\ x \in S^0$,

$q^k(x) = -(c(x) + v^0(x)),\ x \in F^k$,

$p^k(x) = 0,\ x \in S \setminus (S^0 \cup F^k)$ By property 1, such functions exist; to determine them, it is necessary to solve the system of linear algebraic equations

$$Lp^k = -\tilde{q}^k(x),\ x \in S^0 \cup F^k,$$

where $\tilde{q}^k(x) = d(x),\ x \in S^0$ in $\tilde{q}^k(x) = -(c(x) + v^0(x)), x \in F^k$ under conditions $p^k(x) = 0,\ x \in S \setminus (S^0 \cup F^k)$, and then put $q^k(x) = -Lp^k,\ x \in S \setminus (S^0 \cup F^k)$.

3. Define the set of grid nodes $N^k \in S \setminus (S^0 \cup F^k)$ in which the inequality $q^k(x) > -(c(x) + v^0(x)),\ x \in N^k$.

4. If N^k empty, we assume $F^0 = F^k,\ p^c = p^k, q^0 = q^k$ and the second stage is completed.

5. We believe $F^{k+1} = F^k \cup N^k$, assign k meaning *k+1* and repeat the points

2 – 4. As $S \setminus S^0$ of course and $F' \subset F^2 \subset ... \subset S \setminus S^0$, then the second stage will be completed in a finite number of iterations.

Theorem 2. Function $p^0, q^0 \in H(S)$, found at the second stage of the procedure, satisfy conditions (11), (16) and is a solution to problem (10) - (12). By iterations of the procedure, the relations

$$p^{k+1}(x) \ge p^k(x), x \in S, q^{k+1}(x) \ge q^k(x), x \in S \setminus (S^0 \cup F^{k+1}) \quad (18)$$

Substantiation. Applying property 2 to functions p^k, q^k, constructed in paragraph 2 of the algorithm, we obtain $p^k(x) \geq 0,\ x \in S,\ q^k(x) \geq 0 d(x) \leq 0\ x \in S \setminus (S^0 \cup F^k)$. Applying property 2 to functions $p^{k+1} - p^k,\ q^{k+1} - q^k$ given that

$$p^{k+1}(x) = p^k(x) = 0,\ x \in S \setminus (S^0 \cup F^{k+1}),$$

$$q^{k+1}(x) = q^k(x) = d(x)\ x \in S^0,$$

$$q^{k+1}(x) = q^k(x) = -(c(x) + \vartheta^0(x)),\ x \in F^k,$$

$$q^{k+1}(x) = -(c(x) + \vartheta^0(x)) < q^k(x),\ x \in F^{k+1} \setminus F^k,$$

we obtain inequality (18).

Since at the last iteration of the second stage the set is empty and (18) is satisfied, then conditions (16) are also satisfied, which together with (15) make up the system of necessary and sufficient conditions for the optimality of problem (10) - (12).

Conclusion.

It can be seen from the description of the algorithm that at each iteration of the first and second stages, it is necessary to solve a system of linear algebraic equations of the form (1), which is a difference analog of an elliptic partial differential equation. Efficient algorithms have been developed to solve such systems [2].

The proposed algorithm, due to monotonic convergence in all variables, finiteness, and the possibility of separately finding dual and direct variables, has an advantage (in terms of computation time and the amount of necessary RAM of computers (computers)) over the known methods of quadratic programming [10] and the method of local variations [1,10] . The results of the computational experiment of the numerical implementation of the algorithm on test examples were compared with the methods of Beal and Hildert [10]. Calculations have shown that when approximating the region Ω with a grid with 500 nodes, the time for solving problem (1) - (4) by the Beal, Hildreth method and the proposed algorithm in the work was 17, 20, 5 minutes, respectively.

The algorithm is applicable only for a special class of problems and allows solving problems of large dimensions, which is very important for the optimal control of objects described by elliptic partial differential equations.

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