Boundary Value Problem For One Class Of Third-Order Equations Of Mixed-Composite Type In A Quadrangular Region

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ABSTRACT

A boundary value problem for a third-order equation of mixed-composite type in a quadrangular domain is considered. Using the Galerkin method, under certain conditions on the coefficients and the right-hand side of the equation, the existence of a weakly generalized solution in the Sobolev space is proved. Under the same conditions, the uniqueness of the generalized solution is proved.

Keywords: boundary value problem, mixed-composite type equation, third-order equation, generalized solution.

Introduction. Boundary value problems for non-classical partial differential equations arise in many applied problems, in particular, in various sections of mechanics, physics, in the description of dispersion and transport processes, in geometry and population genetics, hydrodynamics, as well as in many other areas.

The formulation of boundary value problems for mixed-composite equations of the second and higher orders, the study of the existence and uniqueness of their generalized solution in various spaces have been carried out in many scientific publications, including the works [1–14]. In this paper, the authors continue their series of works [1], [2] on the formulation and study of boundary value problems for mixed-composite and composite type equations.

Statement of the boundary value problem. Let us define the domain $Q=\{(x,t):-1\le x\le 1,\ 0\le t\le T\}$ and consider the equation

$$Lu\equiv k(t)u_{ttt}+\mu(x)u_{xxx}+a(x,t)u_{xx}+b(x,t)u_{tt}=f(x,t), \qquad (1)$$

where $(t-\frac{T}{2})k(t)>0$ at $t\ne\frac{T}{2}$, $k(\frac{T}{2})=0$, and also $x\mu(x)>0$ at $x\ne 0$, $\mu(0)=0$.

The problem is to find in the region Q a solution of the equation (1) that satisfies the boundary condition

$$u\big|_{\partial Q}=0. \qquad (2)$$

This formulation differs from the previously considered formulation [2], in which the boundary value problem

$$Lu \equiv \frac{\partial}{\partial x}\Big(tu_{xx} + u_{yy} + u_{tt} + au_x + bu\Big) + cu_y + du_t + ju = f(x,y,t),$$

$$u\Big|_{\partial D} = 0,\ u_x\Big|_{x=0} = 0 \text{ (at } t>0\text{)}$$

was investigated. We will assume that the coefficients of equation (1) are infinitely differentiable functions.

Method of solving the problem. To solve the problem (1), (2) we use the Galerkin method to obtain a priori estimates and, using these estimates, prove that the solution of this problem exists and is unique in space $W_2^2(Q)$.

Definition 1. Let us denote by the $W_2^2(Q)$ space of functions obtained by closing the functions from the space $C^\infty(Q)$, satisfying condition (2), according to the norm [15]

$$\|u\|_{W_2^2(Q)} = \int_Q (u_{tt}^2 + u_{xx}^2 + u_{xt}^2 + u_x^2 + u_t^2 + u^2)dQ.$$

Definition 2. A function $u \in W_2^2(Q)$ will be called a weak generalized solution of problem (1), (2) if the identity

$$\int_Q (-ku_{tt}v_t - k_t u_{tt}v - \mu u_{xx}v_x - \mu_x u_{xx}v + au_{xx}v + bu_{tt}v)dQ = \int_Q fvdQ. \qquad (3)$$

holds for all $v \in C_0^\infty(Q)$.

For the given problem we formulate the following statement.

Theorem. Suppose the conditions

$$a(x,t) - \frac{3}{2}|\mu_x| \ge \delta > 0, \quad b(x,t) - \frac{3}{2}|b_t| \ge \delta_1 > 0, \qquad (4)$$

are met. Then for any function $f(x,t)$, such that, $f \in L_2(Q)$, there is a unique solution of the problem (1), (2) in $W_2^2(Q)$.

Proof. We will find the solution of the problem (1), (2) using the Galerkin method

$u_m(x,t) = \sum_{i=1}^m j_i(t)\varphi_i(x)$, where the basis functions $\varphi_i(x)$ are solutions of the problem $\varphi_i'' = -\lambda_i\varphi_i, \quad \varphi_i(-1) = \varphi_i(1) = 0,$ and the coefficients $j_i(t)$ are found as solutions of the following boundary value problems for a system of ordinary differential equations

$$(ku_{mttt},\varphi_i)_0 + (\mu u_{mxxx},\varphi_i)_0 + (au_{mxx},\varphi_i)_0 + (bu_{mtt},\varphi_i)_0 = (f,\varphi_i)_0, \qquad (5)$$

$$j_i(0) = j_i(T) = 0, \qquad i = 1,2,\ldots,m, \qquad (6)$$

where, for example, $(f,\varphi_i)_0 = \int_{-1}^1 f(x)\varphi_i(x)dx$, i.e. the scalar product in $L_2(-1,1)$.

The solvability of problem (5)-(6) for fixed m follows from the general theory of ordinary differential equations. Due to the boundary conditions (2), it is easy to see that the solution satisfies the estimate

$$\int_Q u_m^2 dQ \le C\int_Q u_{mx}^2 dQ. \qquad (7)$$

Let us derive m-uniform estimates for Galerkin approximations. To do this, we multiply (5) by $-j_i(t)$ and, summing over i, we obtain

$$(ku_{mttt},-u_m)_0 + (\mu u_{mxxx},-u_m)_0 + (au_{mxx},-u_m)_0 + (bu_{mtt},-u_m)_0 = (f,-u_m)_0. \qquad (8)$$

From here, integrating over t, by the method of integration by parts, in view of (2), (4), after some transformations we arrive at the inequality

$$\int_Q (u_{mt}^2 + u_{mx}^2 + u_m^2)dQ \le C. \qquad (9)$$

Next, consider the following equality

$$(u_{mttt}, u_{mxx})_0 + (\mu u_{mxxx}, u_{mxx})_0 + (au_{mxx}, u_{mxx})_0 + (bu_{mtt}, u_{mxx})_0 = (f, u_{mxx})_0. \qquad (10)$$

From identity (10), by virtue of (2), (4) and estimate (9), integrating over t, and integrating by parts, after simple transformations, the following estimate follows

$$\int_Q (u_{mxx}^2 + u_{mxt}^2)dQ \le C. \qquad (11)$$

Now, consider the following equality

$$(u_{mttt}, u_{mtt})_0 + (\mu u_{mxxx}, u_{mtt})_0 + (au_{mxx}, u_{mtt})_0 + (bu_{mtt}, u_{mtt})_0 = (f, u_{mtt})_0. \qquad (12)$$

From identity (12), by virtue of (2), (4) and estimates (9), (11), integrating over t and integrating by parts, after simple transformations, the following estimate follows

$$\int_Q (u_{mtt}^2 + u_{mxt}^2)dQ \le C. \qquad (13)$$

From estimates (9), (11), (13) it follows that the sequence of approximate solutions in space $W_2^2(Q)$, is limited; it is possible to select a subsequence $\{u_{m_k}(x,t)\}$ and go to the limit at $m_k \to \infty$ in system (5). It is easy to verify that the limit function belongs to the space $W_2^2(Q)$ and satisfies identity (3). Since the system $\{\varphi_i(x)\}$ is dense in $L_2(-1, 1)$.

Now we will prove that the solution of the problem (1), (2) is unique. If functions u and v are two solutions of the problem (1), (2), then the function $w = u - v$ satisfies the equation $kw_{ttt} + \mu(x)w_{xxx} + a(x,t)w_{xx} + b(x,t)w_{tt} = 0.$ Let's consider the integral $\int_Q (kw_{ttt} + \mu(x)w_{xxx} + a(x,t)w_{xx} + b(x,t)w_{tt})wdQ = 0$ and integrating by parts, by virtue of (2) we obtain $\int_Q (w_t^2 + w_x^2 + w^2)dQ \le 0$. From where it follows that $w = 0$ in Q. The theorem is proved.

Conclusion. This work is devoted to the study of the solvability of a boundary value problem for a third-order equation of mixed-composite type in a quadrangular region. The paper proves new theorems of existence and uniqueness of the solution of the boundary value problem (1), (2) in space $W_2^2(Q)$, which allows us to expand the range of solvable problems in the theory of non-classical equations of mathematical physics.

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