Nonlocal Problem for a Fourth-Order Equation

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ABSTRACT

This article considers a nonlocal problem for a fourth-order equation in a quadrilateral domain. Using the Galerkin method, the existence and uniqueness of a generalized soliton in a Sobolev space is proven under certain conditions on the coefficients and the right-hand side of the equation.

Keywords: nonlocal problem, quadrangular domain, generalized solution, fourth-order equation.

Introduction

The works [1-3] marked the beginning of a new stade in the formulation of well-posed problems for third- and fourth-order equations. Nowadays, due to their physical interpretations, nonlocal problems represent a rapidly developing area within the theory of differential equations [4-7]. A comprehensive bibliography on the subject can be found, for example, in the monograph [4].

Let us consider a fourth-order equation in the domain $D=\{(x,t):\ 0<x<1,\ \ 0<t<T\}$:

$$Lu \equiv u_{tttt} + u_{xxtt} + a(x,t)u = f(x,t) \tag{1}$$

where, $a(x,t),\ f(x,t)$ – the functions are given.

Nonlocal problem. Find a solution to equation (1) in the domain D satisfying the following conditions:

$$u\big|_{x=0} = u\big|_{x=1} = 0\,, \tag{2}$$

$$u\big|_{t=0} = u\big|_{t=T},\ u_t\big|_{t=0} = u_t\big|_{t=T},\ u_{tt}\big|_{t=0} = u_{tt}\big|_{t=T},\quad u_{ttt}\big|_{t=0} = u_{ttt}\big|_{t=T}\,. \tag{3}$$

Definition 1. Let denote the space $H(D)$ of functions obtained by closing the set of functions in $C^4(D)$, satisfying conditions (2), (3) with respect to the norm

$$\|u\|^2_{H(D)} = \int_D (u^2_{tttt} + u^2_{xxtt} + u^2_{ttt} + u^2_{xtt} + u^2_{xxt} + u^2_{tt} + u^2_{xt} + u^2_x + u^2_t + u^2)dD$$

Definition 2. A function $u \in H(D)$ will be called a generalized solution to problem (1)-(3) if it satisfies equation (1) almost everywhere (a.e) in D.

Theorem. Let $a(x,t),\ a_t(x,t), a_{tt}(x,t),\ f(x,t) \in L_2(D)$ and

$$a(x,0)=a(x,T),\ a_t(x,0)=a_t(x,T),\ a(x,t)\geq\delta>0. \quad (4)$$

Then there exists a unique solution to problem (1) – (3) in the space $H(D)$.

Proof. We seek a solution to problem (1) – (3) using the Galerkin method [8]:

$$u_m(x,t)=\sum_{i=1}^{m} g_{im}(t)\varphi_i(x),$$

where functions $\varphi_i(x)$ are solutions to the problem

$$\begin{cases} -\Delta\varphi_i(x)=\lambda\varphi_i(x), \quad x\in(0,1),\ (i=1,2,\ldots), \\ \varphi_i(0)=\varphi_i(1), \end{cases} \quad (5)$$

and the coefficients $g_{im}(t)$ are found from the solution of the problem for a system of ordinary differential equations

$$\int_0^1 (u_{mtttt}+u_{mxxtt}+au_m)\varphi_i dx=\int_0^1 f_m\varphi_i dx, \quad (6)$$

$$g_{im}(0)=g_{im}(T),\ g_{imt}(0)=g_{imt}(T),\ g_{imtt}(0)=g_{imtt}(T),\ \ g_{imttt}(0)=g_{imttt}(T) \quad (7)$$

The standard existence theorems for the system of ordinary differential equations ensure the solvability of problems (6), (7). The domain is assumed to be sufficiently smooth in position, so that $\varphi_i(x)\in W_2^2(0,1)$. We obtain estimates m for Galerkin approximations that are uniform in. C, C_1, C_2, C_3, C_4 we will denote by various positive constants that do not depend on m.

Lemma 1. Let the conditions of the theorem be satisfied, then to solve the problem (6), (7) the assessment is correct

$$\int_D (u_{mtt}^2+u_{mxt}^2+u_m^2)dD\leq C_1\int_D f_m^2 dD \quad (8)$$

Proof. Multiply (6) by $g_{im}(t)$ and sum over i, then we get

$$\int_0^1 (u_{mtttt}+u_{mxxtt}+au_m)u_m dx=\int_0^1 fu_m dx.$$

Now we integrate the resulting equality over t from 0 to T, we get

$$\int_D (u_{mtttt}+u_{mxxtt}+au_m)u_m dD=\int_D fu_m dD$$

Further, using integration by parts, in view of (2), (3), the conditions theorems and inequalities Cauchy, we obtain the estimate (8).

Lemma 1 is proven.

Lemma 2. Let the conditions of the theorem be satisfied, then to solve the problem (6), (7) the assessment is correct

$$\int_D (u_{mxtt}^2+u_{mxxt}^2+u_{mx}^2)dD\leq C_2\int_D f^2 dD \quad (9)$$

Proof. Thanks to (5), we can replace in (6) φ_i by $-\Delta\varphi_i$; further, multiplying (6) by $g_{im}(t)$ and summing over i, as we get

$$\int_0^1 (u_{mtttt}+u_{mxxtt}+au_m)u_{mxx}dx=\int_0^1 fu_{mxx}dx.$$

Integrating this equality over t from 0 to T, we obtain

$$-\int_D (u_{mtttt}+u_{mxxtt}+au_m)u_{mxx}dD=-\int_D fu_{mxx}dD$$

The obtained equalities using integration by parts, in view of (2), (3), (8), the conditions of the theorem and the Cauchy inequality, we obtain the estimate (9).

Lemma 2 is proven.

Lemma 3. Let the conditions of the theorem be satisfied, then to solve the problem (6), (7) the assessment is correct

$$\int_D (u_{mttt}^2+u_{mxtt}^2+u_{mt}^2)dD\le C_3\int_D f^2dD \tag{10}$$

Proof. Multiply (6) by $g_{imtt}(t)$ and sum over to i, get

$$\int_0^1 (u_{mtttt}+u_{mxxtt}+au_m)u_{mtt}dx=\int_0^1 fu_{mtt}dx.$$

Now integrating the resulting equality over t from 0 to T, we obtain

$$\int_D (u_{mtttt}+u_{mxxtt}+au_m)u_{mtt}dD=\int_D fu_{mtt}dD$$

Next, using integration by parts, in view of (2), (3), (8), the conditions of the theorem and the Cauchy inequality, we obtain estimate (10).

Lemma 4. Let the conditions of the theorem be satisfied, then for the solution of problem (6), (7) the following estimate is true:

$$\int_D (u_{mtttt}^2+u_{mxttt}^2+u_{mtt}^2)dD\le C_4\int_D f^2dD \tag{11}$$

Proof. Multiply (6) by $g_{imtttt}(t)$ and sum over to i, get

$$\int_0^1 (u_{mtttt}+u_{mxxtt}+au_m)u_{mtttt}dx=\int_0^1 fu_{mtttt}dx.$$

Now integrating the resulting equality over t from 0 to T, we obtain

$$\int_D (u_{mtttt}+u_{mxxtt}+au_m)u_{mtttt}dD=\int_D fu_{mtttt}dD$$

Next, using integration by parts, in view of (2), (3), (8), the conditions of the theorem and the Cauchy inequality, we obtain estimate (11).

By virtue of (8), (11) and from equation (6) it follows that

$$u_{mxxtt}\in L_2(Q) \tag{12}$$

From estimates (8) - (12) it follows that the sequence of approximate solutions $\{u_m(x,t)\}$ in the space is bounded $H(Q)$. From these estimates it follows that problem (6), (7) is solvable. From the sequence $\{u_m(x,t)\}$ you can select a subsequence $\{u_{m_k}(x,t)\}$ and pass to the limit at $m_k\to\infty$ in system (6). It is easy to verify that the limit function belongs to the space $H(Q)$ and satisfies equation (1) a.e. in Q. Since the system $\{\varphi_i(x)\}$ is dense in $L_2(\Omega)$.

Let us prove that the solution to problem (1) - (3) is unique.

If $u,v-$there are two solutions to the problem (1) - (3), then it $w=u-v$ satisfies the equation

$$w_{tttt} + w_{xxtt} + a(x,t)w = 0$$

and conditions (2), (3). Similarly, as in Lemma 1, for $w(x,t)$ we obtain

$$\int_Q (w_{tt}^2 + w_{xt}^2 + w^2)dQ \le 0$$

From which it follows that $w = 0$ in Q.

The theorem is proven.

This work establishes new existence and uniqueness results for a nonlocal problem (1) - (3), thereby expanding the scope of solvable boundary value problems in the theory of nonclassical equations of mathematical physics.

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