About the optimal numerical solution of the Navier-Stokes equations in the "vortex-current" system

Gulomkodirov Komiljon Alisherovich — Head of the Applied Mathematics Department of Termiz State University
kommiljon@mail.ru, phone number: +998976906685

Kholmurzaeva Nigora Abdimuratovna — Assistant at the Department of Automation and Control, Faculty of Mechanics and Mining, Termiz University of Engineering and Agro-Technology
nogora.xolmirzayeva1988@gmail.com, phone number: +998997354756

ABSTRACT

For numerical simulation of the Navier-Stokes equations in the "vortex-current" system, the Peaceman-Reckford scheme is used in combination with an iterative alternating direction scheme with optimal iterative parameters. The efficiency and high accuracy of the applied numerical method are illustrated.

Keywords:

Introduction

To date, there are numerous studies devoted to the numerical simulation of problems of a viscous incompressible fluid based on two-dimensional Navier-Stokes equations in the "vortex-current" system.

Despite this, the question of the effectiveness of the use of certain methods for numerical simulation of the above problem is relevant.

The work [1] proposes a method for solving the Navier-Stokes equations in natural variables. The method is based on the joint solution of the equation of motion and the equation of continuity using a finite difference approximation. The paper [2] proposes a numerical method for solving the Navier-Stokes equations for a viscous incompressible fluid (in physical variables), supplemented by heat conduction equations. When constructing it, an approximate factorization scheme is used with the splitting of the original operators by physical processes in a special way. In [3], a new approach to pressure calculation was proposed when solving the complete Navier-Stokes equations in the “velocity-pressure” variables on structured grids. This method is based on the use of integral forms of the continuity equation and pressure decomposition, on the basis of which an auxiliary problem is formulated.

In [4], the complete system of Navier-Stokes equations in velocity-pressure variables is solved by the numerical method of finite differences for the case of a viscous incompressible fluid. The discretization of the original equations is implemented in staggered grids. In [5], algorithms for the numerical solution of the Navier-Stokes equations are presented using high-performance computing technology, such as multiprocessor systems with distributed memory and graphics accelerators. In [6], the efficiency of an implicit iterative polylinear recurrent method for solving systems of difference elliptic equations that arise in the numerical simulation of two-dimensional flows of a viscous incompressible fluid is analyzed.

The work [7] is devoted to the study of numerical methods for solving the Navier-Stokes equations in "vortex-current" variables. To solve the corresponding linear grid equations, the standard library was used, which contains an efficient parallel LU of matrix decomposition for solving systems of linear equations with sparse matrices.

In this paper, for the numerical solution of the "vortex-current" system of equations, the Pismen-Reckford scheme is used in combination with an iterative alternating direction scheme with optimal iterative parameters.

1. Statement of the problem. In Cartesian coordinates, the system of Navier-Stokes equations in the form of a "vortex-current" is written as follows [8]:

$$\frac{\partial\omega}{\partial t}+\frac{\partial\omega}{\partial x}\frac{\partial\psi}{\partial y}-\frac{\partial\omega}{\partial y}\frac{\partial\psi}{\partial x}=\nu\left(\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2}\right)+Q(t,x,y), \quad (1)$$

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=-\omega, \quad (2)$$

$$\frac{\partial\psi}{\partial y}=u,\ \ \frac{\partial\psi}{\partial x}=-\vartheta,\ \ \omega=\frac{\partial\vartheta}{\partial x}-\frac{\partial u}{\partial y}. \quad (3)$$

Here x, y – spatial coordinates, t – time, u and ϑ – projection of the velocity vector on the coordinate axes, ν – kinematic viscosity coefficient, ψ, ω – current and vortex function, respectively, Q – known function.

For system (1)-(2) in the region $\overline{D}:\{(x,y,t)\in[0,1]\times[0,1]\times[0,T]\}$, we set the following boundary conditions:

$$\psi\big|_{x=0}=0,\quad \psi\big|_{x=1}=0,\quad \left.\frac{\partial\psi}{\partial x}\right|_{x=0}=0,\quad \left.\frac{\partial\psi}{\partial x}\right|_{x=1}=0,\quad 0\le y\le 1, \quad (4)$$

$$\psi\big|_{y=0}=0,\quad \psi\big|_{y=1}=0,\quad \left.\frac{\partial\psi}{\partial y}\right|_{x=0}=0,\quad \left.\frac{\partial\psi}{\partial y}\right|_{x=1}=0,\quad 0\le x\le 1, \quad (5)$$

initial conditions at $t=0$ have the following form:

$$\psi(0,x,y)=0,\quad \omega(0,x,y)=0. \quad (6)$$

2. Finite-difference approximation. In the region $\overline{D}$, we introduce a uniform grid along the spatial coordinates x, y

$$\overline{\Omega}_h=\left\{x_i=ih,\ \ y_j=jh,\ \ 0\le i,j\le N,\ \ h=\frac{1}{N}\right\}$$

and time grid t

$$\overline{\Omega}_\tau=\{t_k=k\tau,\ \ k=0,1,\ldots M\},\ \text{где}\ \tau=T/M.$$

The system of differential equations (1), (2) is approximated on the $\overline{\Omega}=\overline{\Omega}_h\times\overline{\Omega}_\tau$ difference grid.

For the numerical solution of the vortex equation (1), we use the method of alternating directions (the Peaceman-Reckford scheme) [9]. It is known that this scheme, as an implicit scheme, is absolutely stable. The Peaceman-Reckford scheme has an approximation error $\theta(\tau^2+|h|^2)$, where $|h|^2=h_1^2+h_2^2$, h_1, h_2 is the grid steps in x and y respectively.

Equation (1) is approximated by the Peaceman-Reckford scheme. In this scheme, the transition from layer k to layer $k+1$ is carried out in two stages. At the first stage, intermediate values $\omega_{i,j}^{k+1/2}$ are determined from the following system of equations:

$$\begin{gathered}\frac{\omega_{i,j}^{k+1/2}-\omega_{i,j}^{k}}{0,5\tau}+\frac{\omega_{i+1,j}^{k+1/2}-\omega_{i-1,j}^{k+1/2}}{2h}\frac{\psi_{i,j+1}^{k}-\psi_{i,j-1}^{k}}{2h}-\\-\frac{\omega_{i,j+1}^{k}-\omega_{i,j-1}^{k}}{2h}\frac{\psi_{i+1,j}^{k}-\psi_{i-1,j}^{k}}{2h}=\frac{\nu}{h^2}\left(\omega_{i+1,j}^{k+1/2}-2\omega_{i,j}^{k+1/2}+\omega_{i-1,j}^{k+1/2}\right)+\\+\frac{\nu}{h^2}\left(\omega_{i,j+1}^{k}-2\omega_{i,j}^{k}+\omega_{i,j-1}^{k}\right)+Q\left(t_{k+1/2},x_i,y_i\right),\\ i,j=1,2,\ldots,N-1\ \ k=0,1,\ldots,M-1,\end{gathered}\tag{7}$$

and at the second stage, using the found values of $\omega_{i,j}^{k+1/2}$, the values of $\omega_{i,j}^{k+1}$ are determined from the system of equations:

$$\begin{gathered}\frac{\omega_{i,j}^{k+1}-\omega_{i,j}^{k+1/2}}{0,5\tau}+\frac{\omega_{i+1,j}^{k+1/2}-\omega_{i-1,j}^{k+1/2}}{2h}\frac{\psi_{i,j+1}^{k}-\psi_{i,j-1}^{k}}{2h}-\\-\frac{\omega_{i,j+1}^{k+1}-\omega_{i,j-1}^{k+1}}{2h}\frac{\psi_{i+1,j}^{k}-\psi_{i-1,j}^{k}}{2h}=\frac{\nu}{h^2}\left(\omega_{i+1,j}^{k+1/2}-2\omega_{i,j}^{k+1/2}+\omega_{i-1,j}^{k+1/2}\right)+\\+\frac{\nu}{h^2}\left(\omega_{i,j+1}^{k+1}-2\omega_{i,j}^{k+1}+\omega_{i,j-1}^{k+1}\right)+Q(t_{k+1},x_i,y_i),\\ i,j=1,2,\ldots,N-1\ \ k=0,1,\ldots,M-1.\end{gathered}\tag{8}$$

Equation (7), reduced to standard form

$$\begin{gathered}\bar{A}_i\omega_{i-1,j}^{k+1/2}-\bar{C}_i\omega_{i,j}^{k+1/2}+\bar{B}_i\omega_{i+1,j}^{k+1/2}=-\bar{F}_{i,j}^{k},\\ i,j=1,2,\ldots,N-1\ \ k=0,1,\ldots,M-1.\end{gathered}\tag{9}$$

where

$$\bar{A}_i=0,5\tau\left[\frac{\nu}{h^2}-\frac{1}{4h}\left(\psi_{i,j+1}^{k}-\psi_{i,j-1}^{k}\right)\right],\quad \bar{B}_i=0,5\tau\left[\frac{\nu}{h^2}+\frac{1}{4h}\left(\psi_{i,j+1}^{k}-\psi_{i,j-1}^{k}\right)\right],$$

$$\bar{C}_i=1+\frac{\tau\nu}{h^2},\qquad \bar{F}_{i,j}^{k}=\omega_{i,j}^{k}+\frac{0,5\tau\nu}{h^2}\left(\omega_{i,j+1}^{k}-2\omega_{i,j}^{k}+\omega_{i,j-1}^{k}\right)+$$

$$+\frac{0,5\tau}{4h}\left(\omega_{i,j+1}^{k}-\omega_{i,j-1}^{k}\right)\left(\psi_{i+1,j}^{k}-\psi_{i-1,j}^{k}\right)+Q\left(t_{k+1/2},x_i,y_i\right).$$

Difference equation (9) is solved by the sweep method, while to determine the values of $\omega_{ij}^{k+1/2}$ at all nodes of the difference grid, $O\left(N^2\right)$ arithmetic operations are spent [10].

After all $\omega_{ij}^{k+1/2}$ are found, the difference equation (8) is solved, bringing it to the standard form:

$$A_j\omega_{i,j-1}^{k+1}-C_j\omega_{i,j}^{k+1}+B_j\omega_{i,j+1}^{k+1}=-F_{i,j}^{k+1/2},\qquad (10)$$
$$i,j=1,2,\ldots,N-1\;\; k=0,1,\ldots,M-1.$$

where

$$A_j=0.5\tau\left[\frac{\nu}{h^2}-\frac{1}{4h}\left(\psi_{i,j+1}^{k}-\psi_{i,j-1}^{k}\right)\right],\quad B_j=0.5\tau\left[\frac{\nu}{h^2}+\frac{1}{4h}\left(\psi_{i,j+1}^{k}-\psi_{i,j-1}^{k}\right)\right],$$
$$C_j=1+\frac{\tau\nu}{h^2},\quad F_{i,j}^{k+1/2}=\omega_{i,j}^{k+1/2}+\frac{0,5\tau\nu}{h^2}\left(\omega_{i+1,j}^{k+1/2}-2\omega_{i,j}^{k+1/2}+\omega_{i-1,j}^{k+1/2}\right)+$$
$$+\frac{0,5\tau}{4h}\left(\omega_{i+1,j}^{k+1/2}-\omega_{i-1,j}^{k+1/2}\right)\left(\psi_{i,j+1}^{k}-\psi_{i,j-1}^{k}\right)+Q\left(t_{k+1},x_i,y_j\right).$$

To find all ω_{ij}^{k+1} by equation (10) by the sweep method, $O\left(N^2\right)$ arithmetic operations are required

To determine the values of the vortex at the boundary nodes of the grid, the Woods conditions are used [11]:

$$\omega_{i,0}^{s+1}+\frac{\omega_{i,1}^{s+1}}{2}=\frac{3\left(\psi_{i,0}^{s}-\psi_{i,1}^{s}\right)}{h^2},\qquad \omega_{i,N}^{s+1}+\frac{\omega_{i,N-1}^{s+1}}{2}=\frac{3\left(\psi_{i,N}^{s}-\psi_{i,N-1}^{s}\right)}{h^2},\qquad (11)$$

$$\omega_{0,j}^{s+1}+\frac{\omega_{1,j}^{s+1}}{2}=\frac{3\left(\psi_{0,j}^{s}-\psi_{1,j}^{s}\right)}{h^2},\qquad \omega_{N,j}^{s+1}+\frac{\omega_{N-1,j}^{s+1}}{2}=\frac{3\left(\psi_{N,j}^{s}-\psi_{N-1,j}^{s}\right)}{h^2}.\qquad (12)$$

The systems of difference equations (9), (11) and (10), (12) are solved by the sweep method.

Let us present an algorithm for solving the boundary value problem (9), (11) by the sweep method:

$$\omega_{i,j}^{k+1/2}=\bar{\alpha}_{i+1}\omega_{i+1,j}^{k+1/2}+\bar{\beta}_{i+1},\ .\qquad (13)$$
$$i=N-1,N-2,\ldots,1,0,\ (0<j<N),\ k=0,1,\ldots,M-1$$

$$\bar{\alpha}_{i+1}=\frac{\bar{B}_i}{\bar{C}_i-\bar{A}_i\bar{\alpha}_i},\quad \bar{\beta}_{i+1}=\frac{\bar{A}_i\bar{\beta}_i+\bar{F}_{ij}^{k}}{\bar{C}_i-\bar{A}_i\bar{\alpha}_i},\qquad (14)$$
$$i=1,2,\ldots,N-1,\ (0<j<N),\ k=0,1,\ldots,M-1.$$

$$\omega_{0,j}^{k+1/2}=\bar{\alpha}_1\omega_{1,j}^{k+1/2}+\bar{\beta}_1,\qquad \omega_{0,j}^{k+1/2}=-\frac{1}{2}\omega_{1,j}^{k+1/2}+\frac{3\left(\psi_{0j}^{k}-\psi_{1j}^{k}\right)}{h^2},$$
$$\bar{\alpha}_1=-0,5,\quad \bar{\beta}_1=\frac{3\left(\psi_{0j}^{k}-\psi_{1j}^{k}\right)}{h^2},\quad \left(0<j<N\right),\qquad (15)$$

$$\omega_{N,j}^{k+1/2}+\frac{\omega_{N-1,j}^{k+1/2}}{2}=\frac{3\left(\psi_{N,j}^{k}-\psi_{N-1,j}^{k}\right)}{h^2},\qquad (16)$$

$$\omega_{N-1,j}^{k+1/2}=\bar{\alpha}_N\omega_{N,j}^{k+1/2}+\bar{\beta}_N,\qquad (17)$$

Substituting (17) into (16) we obtain expressions for determining the value of $\omega_{N,j}^{k+1/2}$ at the boundary node of the grid, i.e. at $i=N$

$$\omega_{N,j}^{k+1/2} = \left[\frac{3\left(\psi_{N,j}^{k} - \psi_{N-1,j}^{k}\right)}{h^2} - 0{,}5\bar{\beta}_N \right] \Big/ \left(1 + 0{,}5\bar{\alpha}_N\right), \qquad (18)$$

Now we present the algorithm for solving the boundary value problem (10), (12) by the sweep method:

$$\omega_{i,j}^{k+1} = \alpha_{j+1}\omega_{i,j+1}^{k+1} + \beta_{j+1}, \quad j = N-1, N-2, \ldots, 1, 0, \quad (0 < i < N), \qquad (19)$$

$$\alpha_{j+1} = \frac{B_j}{C_j - A_j\alpha_j}, \quad \beta_{j+1} = \frac{A_j\beta_j + F_{i,j}^{k+1/2}}{C_j - A_j\alpha_j}, \quad j = 1, 2, \ldots, N-1, \ (0 < i < N), \qquad (20)$$

$$\omega_{i,0}^{k+1} = \alpha_1\omega_{i,1}^{k+1} + \beta_1, \quad \omega_{i,0}^{k+1} = -0{,}5\omega_{i,0}^{k+1} + 3\left(\psi_{i,0}^{k} - \psi_{i,1}^{k}\right)\Big/h^2,$$

$$\alpha_1 = -0{,}5, \quad \beta_1 = 3\left(\psi_{i,0}^{k} - \psi_{i,1}^{k}\right)\Big/h^2, \quad \left(0 < i < N\right), \qquad (21)$$

$$\omega_{i,N}^{k+1} + 0{,}5\omega_{i,N-1}^{k+1} = 3\left(\psi_{i,N}^{k} - \psi_{i,N-1}^{k}\right)\Big/h^2, \qquad (22)$$

$$\omega_{i,N-1}^{k+1} = \alpha_N\omega_{i,N}^{k+1} + \beta_N, \qquad (23)$$

Substituting (23) into (22) we obtain expressions for determining the value of $\omega_{i,N}^{k+1}$ at the boundary node $j = N$:

$$\omega_{i,N}^{k+1} = \left[\frac{3\left(\psi_{i,N}^{k} - \psi_{i,N-1}^{k}\right)}{h^2} - 0{,}5\beta_N \right] \Big/ \left(1 + 0{,}5\alpha_N\right). \qquad (24)$$

Consider the numerical solution of the equation for the current function (2).
Equation (2) is replaced by the following non-stationary equation:

$$\frac{\partial\psi}{\partial t_1} = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \omega, \ 0 \le x, y \le 1. \qquad (25)$$

Equation (25) is considered under the following initial and boundary conditions:

$$\psi\left(t_1, x, y\right)\Big|_{t_1=0} = \psi\left(0, x, y\right), \qquad (26)$$

$$\psi\left(t_1, x, y\right)\Big|_{x=0} = 0, \ \psi\left(t_1, x, y\right)\Big|_{x=1} = 0, \qquad (27)$$

$$\psi\left(t_1, x, y\right)\Big|_{y=0} = 0, \ \psi\left(t_1, x, y\right)\Big|_{y=1} = 0, \qquad (28)$$

Let us approximate equation (25) by a scheme of alternating directions:

$$\frac{\left(\psi_{i,j}\right)^{s+\frac{1}{2}}-\left(\psi_{i,j}\right)^{s}}{\tau_s^{(1)}}=\frac{\left(\psi_{i-1,j}\right)^{s+\frac{1}{2}}-2\left(\psi_{i,j}\right)^{s+\frac{1}{2}}+\left(\psi_{i+1,j}\right)^{s+\frac{1}{2}}}{h_1^2}+$$

$$+\frac{\left(\psi_{i,j-1}\right)^{s}-2\left(\psi_{i,j}\right)^{s}+\left(\psi_{i,j+1}\right)^{s}}{h_2^2}+\omega_{ij}^{k+1}, \qquad (29)$$

$$i=1,2,\ldots,N-1,\;\; j=1,2,\ldots,M-1,\quad s=1,\ldots,ns,\; k=0,1,\ldots,T-1$$

$$h_1=\frac{l_1}{N},\quad h_2=\frac{l_2}{M},$$

$$\frac{\left(\psi_{i,j}\right)^{s+1}-\left(\psi_{i,j}\right)^{s+\frac{1}{2}}}{\tau_s^{(2)}}=\frac{\left(\psi_{i-1,j}\right)^{s+\frac{1}{2}}-2\left(\psi_{i,j}\right)^{s+\frac{1}{2}}+\left(\psi_{i+1,j}\right)^{s+\frac{1}{2}}}{h_1^2}+$$

$$+\frac{\left(\psi_{i,j-1}\right)^{s+1}-2\left(\psi_{i,j}\right)^{s+1}+\left(\psi_{i,j+1}\right)^{s+1}}{h_2^2}+\omega_{ij}^{k+1}, \qquad (30)$$

$$j=1,2,\ldots,M-1,\; i=1,2,\ldots,N-1,\; s=1,\ldots,ns,\; k=0,1,\ldots,T-1,$$

$$h_1=\frac{l_1}{N},\quad h_2=\frac{l_2}{M}.$$

For the numerical solution of equation (25), it is necessary to find the optimal iterative parameters, for which the following constants are preliminarily calculated:

Let the minimum and maximum values of the difference operators $\Lambda_1\psi_{ij}$ and $\Lambda_2\psi_{ij}$ be equal to δ_1,Δ_1 and δ_2,Δ_2, respectively:

$$\delta_1=\frac{4}{h_1^2}\sin^2\frac{\pi h_1}{2l_1},\; \delta_2=\frac{4}{h_2^2}\sin^2\frac{\pi h_2}{2l_2}, \qquad (31)$$

$$\Delta_1=\frac{4}{h_1^2}\cos^2\frac{\pi h_1}{2l_1},\; \Delta_2=\frac{4}{h_2^2}\cos^2\frac{\pi h_2}{2l_2}. \qquad (32)$$

Using these eigenvalues, the following variables are calculated:

$$\xi=\sqrt{\frac{(\Delta_1-\delta_1)(\Delta_2-\delta_2)}{(\Delta_1+\delta_2)(\Delta_2+\delta_1)}},\; \eta=\frac{1-\xi}{1+\xi}, \qquad (33)$$

$$\kappa=\frac{(\Delta_1-\delta_1)\Delta_2}{(\Delta_2+\delta_1)\Delta_1},\quad p=\frac{\kappa-t}{\kappa+t}, \qquad (34)$$

$$r=\frac{\Delta_1-\Delta_2+(\Delta_1+\Delta_2)p}{2\Delta_1\Delta_2},\; q=r+\frac{1-p}{\Delta_1}. \qquad (35)$$

The number of iterations required to find a difference solution with a given accuracy $\varepsilon>0$ is calculated using the following formula:

$$ns\approx\frac{1}{\pi^2}\ln\frac{4}{\varepsilon}\ln\frac{4}{\eta}.. \qquad (36)$$

Then the iterative parameters $\tau_j^{(1)}$, $\tau_j^{(2)}$ necessary for the implementation of the iteration scheme of the variable direction are calculated based on the following formulas:

$$\theta = \frac{1}{16}\eta^2\left(1+\frac{1}{2}\eta^2\right),\ \sigma = \frac{2j-1}{2ns},\ \ j=1,2,\ldots,ns \tag{37}$$

$$\omega_s = \frac{(1+2\theta)\left(1+\theta^{\sigma}\right)}{2\theta^{\frac{\sigma}{2}}\left(1+\theta^{1-\sigma}+\theta^{1+\sigma}\right)}, \tag{38}$$

$$\tau_j^{(1)} = \frac{q\omega_j + r}{1+\omega_j p},\ \ \tau_j^{(2)} = \frac{q\omega_j - r}{1-\omega_j p} \tag{39}$$

We approximate the initial condition (26) as follows:

$$\left(\psi_{i,j}\right)^0 = \psi_{i,j}^0,\ i=0,1,\ldots,N,\ \ j=0,1,\ldots,M. \tag{40}$$

Boundary conditions (27), (28) are approximated as follows:

$$\begin{gathered}\left(\psi_{0,j}\right)^{s+1}=0,\ \left(\psi_{N,j}\right)^{s+1}=0,\ \ j=0,1,\ldots,M,\\ k=0,1,\ldots,N_t-1,\ \ s=0,1,\ldots,ns-1,\end{gathered} \tag{41}$$

$$\begin{gathered}\left(\psi_{i,0}\right)^{s+1}=0,\ \left(\psi_{i,M}\right)^{s+1}=0,\ i=0,1,\ldots,N,\\ k=0,1,\ldots,N_t-1,\ \ s=0,1,\ldots,ns-1,\end{gathered} \tag{42}$$

System (29) is presented in a standard form, convenient for using the sweep method:

$$\begin{gathered}\bar{A}\left(\psi_{i-1}\right)^{s+\frac{1}{2}} - \bar{C}\left(\psi_{i,j}\right)^{s+\frac{1}{2}} + \bar{B}\left(\psi_{i+1,j}\right)^{s+\frac{1}{2}} = -\left(F_{i,j}^{k+1}\right)^{s}\\ i=1,2,\ldots,N-1,\ \ j=1,2,\ldots,M-1,\ s=0,1,\ldots,ns-1,\ k=0,1,\ldots,N_t-1,\end{gathered} \tag{43}$$

here

$$\bar{A}=\bar{B}=\frac{\tau_s^{(1)}}{h_1^2},\ \bar{C}=1+2\bar{A},$$

$$\left(\bar{F}_{i,j}\right)^s = \left(\psi_{i,j}\right)^s + \frac{\tau_s^{(1)}}{h_2^2}\left[\left(\psi_{i,j-1}\right)^s - 2\left(\psi_{i,j}\right)^s + \left(\psi_{i,j+1}\right)^s\right] + \omega_{i,j}^{k+1},$$

after finding $\left(\psi_{i,j}\right)^{s+\frac{1}{2}}$ for all values of the considered indices, equation (30) is written in the following standard form:

$$\begin{gathered}\bar{A}\left(\psi_{i,j-1}\right)^{s+1} - \bar{C}\left(\psi_{i,j}\right)^{s+1} + \bar{B}\left(\psi_{i,j+1}\right)^{s+1} = -\left(F_{i,j}\right)^{s+\frac{1}{2}},\\ j=1,2,\ldots,M-1,\ \ i=1,2,\ldots,N-1,\ \ s=0,1,\ldots,ns-1,\ \ k=0,1,\ldots,N_t-1,\end{gathered} \tag{44}$$

here

$$A=B=\frac{\tau_s^{(1)}}{h_2^2},\ C=1+2A,$$

$$\left(F_{i,j}\right)^{s+\frac{1}{2}} = \left(\psi_{i,j}\right)^{s+\frac{1}{2}} + \frac{\tau_s^{(2)}}{h_1^2}\left[\left(\psi_{i-1,j}\right)^{s+\frac{1}{2}} - 2\left(\psi_{i,j}\right)^{s+\frac{1}{2}} + \left(\psi_{i+1,j}\right)^{s+\frac{1}{2}}\right] + \omega_{i,j}^{k+1},$$

Equation (43) is solved by the sweep method:

$$\bar{\alpha}_{i+1} = \frac{B}{\bar{C} - \bar{A}\bar{\alpha}_i},\ \bar{\beta}_{i+1} = \frac{\bar{A}\bar{\beta} + \left(\bar{F}_{i,j}\right)^s}{\bar{C} - \bar{A}\bar{\alpha}_i},\ i = 1, 2, \ldots, N-1, \tag{45}$$

$$k = 0,1,\ldots,N_t - 1,\ s = 0,1,\ldots,ns-1,\ \bar{\alpha}_1 = 0,\ \bar{\beta}_1 = 0,$$

$$\left(\psi_{N,j}\right)^{s+\frac{1}{2}} = 0,$$

$$\left(\psi_{i,j}\right)^{s+\frac{1}{2}} = \bar{\alpha}_{i+1}\left(\psi_{i+1,j}\right)^{s+\frac{1}{2}} + \bar{\beta}_{i+1}, \tag{46}$$

$$i = N-1,\ldots,1,0,\ k = 0,1,\ldots,N_t-1,\ s = 0,1,\ldots,ns-1,$$

$$\left(\psi_{0,j}\right)^{s+\frac{1}{2}} = \bar{\alpha}_1\left(\psi_{1,j}\right)^{s+\frac{1}{2}} + \bar{\beta}_1,\ \left(\psi_{0,j}\right)^{s+\frac{1}{2}} = 0, \tag{47}$$

then equation (44) is solved by the sweep method:

$$\alpha_{j+1} = \frac{B}{\bar{C} - \bar{A}\bar{\alpha}_j},\ \beta_{j+1} = \frac{A\beta_j + \left(\bar{F}_{i,j}\right)^{s+\frac{1}{2}}}{\bar{C} - \bar{A}\bar{\alpha}_i},\ j = M-1,\ldots,0, \tag{48}$$

$$s = 0,1,\ldots,ns-1,\ k = 0,1,\ldots,N_t-1,\ \alpha_1 = 0,\ \beta_1 = 0,$$

$$\left(\psi_{i,M}\right)^{s+1} = 0$$

$$\left(\psi_{i,j}\right)^{s+1} = \alpha_{i+1}\left(\psi_{i,j+1}\right)^{s+1} + \bar{\beta}_{j+1}, \tag{49}$$

$$j = M-1,\ldots,1,0,\ k = 0,1,\ldots,N_t-1,\ s = 0,1,\ldots,ns-1,$$

$$\left(\psi_{i,0}\right)^{s+1} = \alpha_1\left(\psi_{i,1}\right)^{s+1} + \beta_1,\ \left(\psi_{i,0}\right)^{s+1} = 0, \tag{50}$$

We supplement equations (29), (30) with the boundary conditions

$$\psi_{i,0}^{s+1} = 0,\quad \psi_{i,N}^{s+1} = 0,\quad i = 0,1,\ldots,N\ \ s = 0,1,\ldots,ns-1, \tag{51}$$

$$\psi_{0,j}^{s+1} = 0,\quad \psi_{N,j}^{s+1} = 0,\quad j = 0,1,\ldots,N\ \ s = 0,1,\ldots,ns-1. \tag{52}$$

3. Results of numerical calculations and conclusions. Let us present the results of numerical calculations for solving the Navier-Stokes equation based on the above methods.

To solve problem (7)-(30), the longitudinal-transverse method (the Peaceman-Rackford scheme) and the iterative method of alternating directions with optimal iterative parameters were used. The grids are selected as follows: $h_x = h_y = 0.05,\ \nu = 1,\ \tau = 0.001$. We will carry out a computational experiment using the trial function method. If the differential problem has an exact stationary solution $\psi(x, y) = \sin^2 \pi x \sin^2 \pi y$, then it is possible to obtain an expression for the function $Q(x, y)$ and $\omega(x, y)$ [11]. From the current equation (2) we obtain formulas for the function $\omega(x, y)$:

$$\omega(x, y) = -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2}. \tag{53}$$

Let's find derivatives

$$\frac{\partial^2 \psi}{\partial x^2} = 2\pi^2 \cos 2\pi x \sin^2 \pi y,\quad \frac{\partial^2 \psi}{\partial y^2} = 2\pi^2 \cos 2\pi y \sin^2 \pi x.$$

Supplying the found derivatives in (53) we obtain the following formulas for the function $\omega(x,y)$

$$\omega(x,y) = 2\pi^2\left(4\sin^2\pi x\sin^2\pi y - \sin^2\pi x - \sin^2\pi y\right).$$

In the stationary case, from the vortex equation (1) for $Q(x,y)$ we have

$$Q(x,y) = -\frac{\partial\omega}{\partial x}\frac{\partial\psi}{\partial y} + \frac{\partial\omega}{\partial y}\frac{\partial\psi}{\partial x} + \nu\left(\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2}\right). \quad (54)$$

Putting (53) into (54) we obtain the following expression for $Q(x,y)$

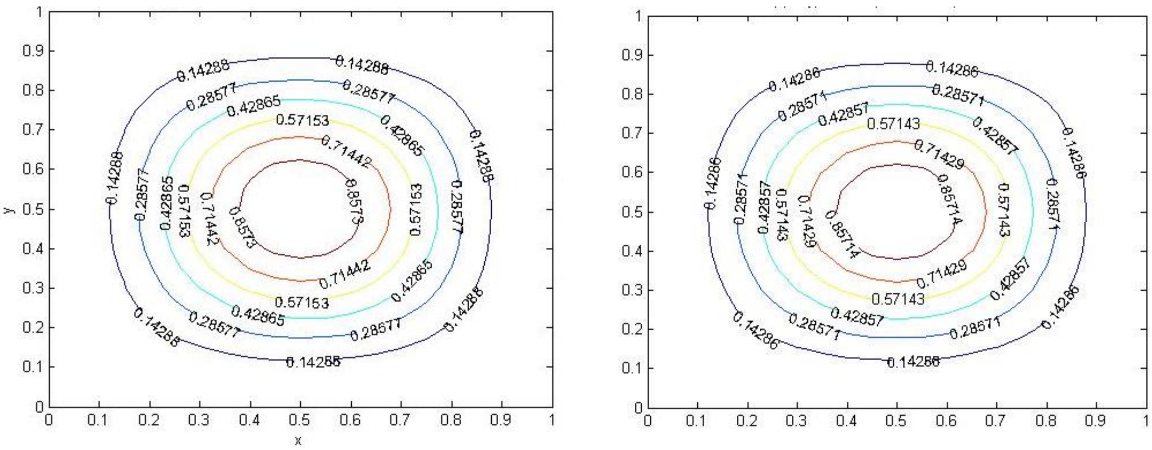
$$Q(x,y) = \left(\frac{\partial^3\psi}{\partial x^3} + \frac{\partial^3\psi}{\partial x\partial y^2}\right)\frac{\partial\psi}{\partial y} - \left(\frac{\partial^3\psi}{\partial x^2\partial y} + \frac{\partial^3\psi}{\partial y^3}\right)\frac{\partial\psi}{\partial x} - \nu\left(\frac{\partial^4\psi}{\partial x^4} + 2\frac{\partial^4\psi}{\partial x^2\partial y^2} + \frac{\partial^4\psi}{\partial y^4}\right). \quad (55)$$

We put the partial derivatives of $\psi(x,y)$ in (55) and finally obtain the following expression for the function $Q(x,y)$:

$$Q(x,y) = 8\pi^4\left\{8\sigma_1^2\sigma_2^2 - 3\sigma_1^2 - 3\sigma_2^2 + 1 + \sigma_1\sigma_2\sigma_3\sigma_4\left[\sigma_2^2\left(4\sigma_3^2 - 3\right) - \sigma_1^2\left(4\sigma_4^2 - 3\right)\right]\right\},$$

$$\sigma_1 = \sin\pi x,\quad \sigma_2 = \sin\pi y,\quad \sigma_3 = \cos\pi x,\quad \sigma_4 = \cos\pi y.$$

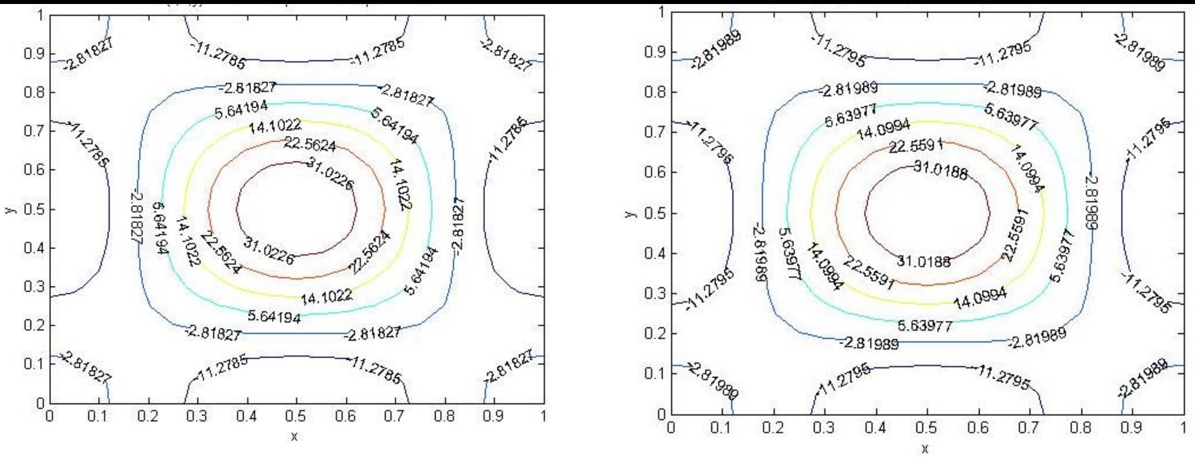
On Figure 1 shows the level lines of the stream function based on exact solution calculations and numerical results.



a) numerical solution b) exact solution

Figure 1. Exact and approximate solutions for the stream function

On Figure 2 shows the vortex level lines obtained from the results of numerical calculations for calculating the exact and approximate solutions.



a) numerical solution b) exact solution

Figure 2. Exact and approximate solution of the vortex equation obtained from the results of numerical calculations.

To solve the stream function, the use of an iterative alternating direction scheme (IADS) with optimal iterative parameters is effective from other iterative methods such as a simple iterative method, the upper relaxation method, the Seidel method [12]. Because $n(\varepsilon)$ the number of iterations required to ensure the accuracy of $\varepsilon > 0$ is greater in other methods than in this method. Comparison of the number of $n(\varepsilon)$ iterations to ensure the accuracy of $\varepsilon > 0$ in different methods is shown in Table 1.

Table 1. The number of $n(\varepsilon)$ iterations to ensure the accuracy of $\varepsilon > 0$ in different methods

ε / Name iteration method	0,001	0,0001	0,00001	0,000001
IADS with optimal parameters	5,437918	6,947587	8,457256	9,966925
Upper relaxation method	87,95487	117,2732	146,5914	175,9097
Simple iterative method	559,9546	746,6061	933,2576	1119,909
Seidel method	279,9773	373,3031	466,6288	559,9546

To solve the vortex equation, the Peaceman-Rackford method was used. Comparison of the calculation results shows that the numerical solutions obtained by the Peaceman-Rackford method are close to the values of the exact solutions.

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