



# Frechet Reliability Estimation of (3+1) Cascade Model

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## ABSTRACT

In this paper presents the  $R$  reliability mathematical formula of (3+1) Frechet Cascade model. The reliability of the model is expressed by Frechet random variables, which are stress and strength distributions. The reliability model was estimated by seven dissimilar methods (ML, Mo, LS, WLS, Rg, Pr and Pi) and simulation was performed using MATLAB 2012 program to compare the results of the reliability model estimates using the MSE criterion, the results indicated that the best estimator among the seven estimators was ML and Pi.

## Keywords:

Standby redundancy, Parameter, Frechet distribution, Unit, distributed Identically

## 1. Introduction

Many researches have been performed on reliability estimation  $R = p(X > Y)$  in the field of strength and stress models. The Cascade is a special kind of stress-strength model.

Cascade redundancy is a hierarchical standby redundancy in which a standby unit with different stress substitutes for a system. When a system unit fails, it is replaced by a standby unit and the stress changed  $k$  Times the previous stress [12]. In a previous study Karam and Khaleel (2019) presented a study of (2+1) Cascade model, which the model consists of two main components and one redundancy standby. In this paper, we assumed that the (3+1) of Cascade with  $(U_1, U_2, U_3 \text{ and } U_4)$  Unit, in assumed that the (3+1) of Cascade with Units, in which three units  $U_1, U_2, U_3 \text{ and } U_4$  are work and the unit  $U_4$  is a standby unit. Assume that  $X_1, X_2, X_3, X_4$  denote the unit strengths

$(U_1, U_2, U_3 \text{ and } U_4)$  respectively and  $Y_1, Y_2, Y_3, Y_4$  indicated the enforcement of stress. Here, if the active unit  $U_1$  is a failure then the standby component  $U_4$  is activated, where  $X_4 = mX_1$  and  $Y_4 = kY_1$ , if the active unit  $U_2$  is a failure then the standby component  $U_4$  is activated, where  $X_4 = mX_2$  and  $Y_4 = kY_2$  and if the active unit  $U_3$  is a failure then the standby component  $U_4$  is activated, where  $X_4 = mX_3$  and  $Y_4 = kY_3$ . Where "k" and "m" denote the stress and strength attenuation factors respectively, such that  $0 < m < 1$  and  $k > 1$ . Reddy (2016) [20] presents of  $R = p(X > Y)$  by discussing model stress - strength of a cascade, assuming all the parameters are independent and following Weibull stress-strength distribution in one parameter and calculating first four cascade reliability for different stress-strength values. Mutkekar and Munoli (2016)[17], (1+1) exponential distribution cascade model is derived with the common effect of the force and stress reduction

factors . Kumar and Vaish (2017) [15] , discussed that Gompertz distribution is stress and that strength is power distribution parameters . Karam and Khaleel (2018) [12] derived a special (2+1) stress-strength reliability cascade model for the distribution of Weibull . Khaleel and Karam (2019) [13] discussed the reliability of the (2+1) cascade inverse distribution Weibull model , reliability can be found when reverse Weibull random variables with unknown parameters scale and known shape parameter are distributed with

strength-stress and used six different estimations method to estimate reliability . Karam and Khaleel (2019) [11] , expression for model confidence is found when strength and stress distribution are generalized in reversed Rayleigh random variable Rayleigh , derived from mathematical formulas for Reliability to Special (2+1) . Khaleel (2021)[14] , (3+1) exponential distribution cascade model is derived with the common effect of the force and stress reduction factors

**2. The mathematical formula :**

Suppose , for the four units ( three basic and one redundant standby ) , the random strength-stress variables of the four units  $j =$

1,2,3,4 each independently and identically distributed Frechet of the parameter scale  $\beta_i$  ,  $i=1,2,3,4$  and scale  $\mu_j$  ,  $j =1,2,3,4$  .

**2.1 . Frechet Distribution :[6],[22],[1]**

$$f(x, \alpha, \beta) = \alpha\beta x^{-(\alpha+1)} e^{-\beta x^{-\alpha}} \quad , x \geq 0 \quad , \alpha > 0, \beta > 0 \quad \dots \dots \dots (1)$$

$$R(x) = 1 - e^{-\beta x^{-\alpha}} \quad \dots \dots \dots (2)$$

$$h(x) = \frac{\alpha\beta x^{-(\alpha+1)} e^{-\beta x^{-\alpha}}}{1 - e^{-\beta x^{-\alpha}}} \quad \dots \dots \dots (3)$$

$$E(x) = \beta^{-\frac{1}{\alpha}} \Gamma\left(1 - \frac{1}{\alpha}\right) \quad \dots \dots \dots (4)$$

$$V(x) = \beta^{-\frac{2}{\alpha}} \left[ \left( \Gamma\left(1 - \frac{1}{\alpha}\right) \right)^2 - \Gamma\left(1 - \frac{2}{\alpha}\right) \right] \quad \dots \dots \dots (5)$$

The Cumulative distribution function of  $Fr(\alpha, \beta)$  is :

$$F(x) = e^{-\beta x^{-\alpha}} \quad , x \geq 0, \alpha > 0, \beta > 0 \quad \dots \dots \dots (6)$$

The Cumulative distribution function of  $Fr(\alpha, \mu)$  is :

$$G(y) = e^{-\mu y^{-\alpha}} \quad , y \geq 0, \alpha > 0, \mu > 0 \quad \dots \dots \dots (7)$$

**2.2. Reliability Model for Frechet Distribution ( $R_{Fr}$ ) :**

Let  $X_i \sim Fr(\alpha, \beta_i); i = 1,2,3,4$  and  $Y_j \sim Fr(\alpha, \mu_j); j = 1,2,3,4$  be strength and stress random variables of the three components ( three components are basic and

one is standby )with unknown scale parameters  $\beta_i, \mu_j$  and common known shape parameter  $\alpha$  , where  $X_i$  and  $Y_j$  are independently and identically distributed Frechet random variables .

**The reliability function for (3+1) cascade model is :**

$$R = P[X_1 \geq Y_1, X_2 \geq Y_2, X_3 \geq Y_3]$$

$$+ P[X_1 < Y_1, X_2 \geq Y_2, X_3 \geq Y_3, X_4 \geq Y_4]$$

$$+ P[X_1 \geq Y_1, X_2 < Y_2, X_3 \geq Y_3, X_4 \geq Y_4]$$

$$+ P[X_1 \geq Y_1, X_2 \geq Y_2, X_3 < Y_3, X_4 \geq Y_4]$$

$$R = R_1 + R_2 + R_3 + R_4 \quad \dots \dots \dots (8)$$

$$\begin{aligned} P_1 &= P(X_1 \geq Y_1) = \int_{y_1} [1 - F_1(y_1)]f(y_1) dy_1 \\ &= \int_0^{\infty} (1 - e^{-\beta_1 y_1^{-\alpha}}) \alpha \mu_1 y_1^{-(\alpha+1)} e^{-\mu_1 y_1^{-\alpha}} dy_1 \\ &= \int_0^{\infty} \alpha \mu_1 y_1^{-(\alpha+1)} e^{-\mu_1 y_1^{-\alpha}} - \alpha \mu_1 y_1^{-(\alpha+1)} e^{-(\mu_1 + \beta_1) y_1^{-\alpha}} dy_1 \\ &= \int_0^{\infty} \alpha \mu_1 y_1^{-(\alpha+1)} e^{-\mu_1 y_1^{-\alpha}} - \alpha \frac{(\mu_1 + \beta_1)}{(\mu_1 + \beta_1)} \mu_1 y_1^{-(\alpha+1)} e^{-(\mu_1 + \beta_1) y_1^{-\alpha}} dy_1 \\ &= \left[ e^{-\mu_1 y_1^{-\alpha}} - \frac{\mu_1}{\mu_1 + \beta_1} e^{-(\mu_1 + \beta_1) y_1^{-\alpha}} \right]_0^{\infty} \\ &= \left( 1 - \frac{\mu_1}{\mu_1 + \beta_1} \right) - (0 - 0) = \frac{\beta_1}{\mu_1 + \beta_1} \quad \dots \dots \dots (9) \end{aligned}$$

$$\begin{aligned} P_2 &= P(X_2 \geq Y_2) = \int_{y_2} [1 - F_2(y_2)]f(y_2) dy_2 \\ &= \int_0^{\infty} (1 - e^{-\beta_2 y_2^{-\alpha}}) \alpha \mu_2 y_2^{-(\alpha+1)} e^{-\mu_2 y_2^{-\alpha}} dy_2 \\ &= \int_0^{\infty} \alpha \mu_2 y_2^{-(\alpha+1)} e^{-\mu_2 y_2^{-\alpha}} - \alpha \mu_2 y_2^{-(\alpha+1)} e^{-(\mu_2 + \beta_2) y_2^{-\alpha}} dy_2 \\ &= \int_0^{\infty} \alpha \mu_2 y_2^{-(\alpha+1)} e^{-\mu_2 y_2^{-\alpha}} - \alpha \frac{(\mu_2 + \beta_2)}{(\mu_2 + \beta_2)} \mu_2 y_2^{-(\alpha+1)} e^{-(\mu_2 + \beta_2) y_2^{-\alpha}} dy_2 \\ &= \left[ e^{-\mu_2 y_2^{-\alpha}} - \frac{\mu_2}{\mu_2 + \beta_2} e^{-(\mu_2 + \beta_2) y_2^{-\alpha}} \right]_0^{\infty} \\ &= \left( 1 - \frac{\mu_2}{\mu_2 + \beta_2} \right) - (0 - 0) = \frac{\beta_2}{\mu_2 + \beta_2} \quad \dots \dots \dots (10) \end{aligned}$$

$$\begin{aligned} P_3 &= P(X_3 \geq Y_3) = \int_{y_3} [1 - F_3(y_3)]f(y_3) dy_3 \\ &= \int_0^{\infty} (1 - e^{-\beta_3 y_3^{-\alpha}}) \alpha \mu_3 y_3^{-(\alpha+1)} e^{-\mu_3 y_3^{-\alpha}} dy_3 \\ &= \int_0^{\infty} \alpha \mu_3 y_3^{-(\alpha+1)} e^{-\mu_3 y_3^{-\alpha}} - \alpha \mu_3 y_3^{-(\alpha+1)} e^{-(\mu_3 + \beta_3) y_3^{-\alpha}} dy_3 \\ &= \int_0^{\infty} \alpha \mu_3 y_3^{-(\alpha+1)} e^{-\mu_3 y_3^{-\alpha}} - \alpha \frac{(\mu_3 + \beta_3)}{(\mu_3 + \beta_3)} \mu_3 y_3^{-(\alpha+1)} e^{-(\mu_3 + \beta_3) y_3^{-\alpha}} dy_3 \\ &= \left[ e^{-\mu_3 y_3^{-\alpha}} - \frac{\mu_3}{\mu_3 + \beta_3} e^{-(\mu_3 + \beta_3) y_3^{-\alpha}} \right]_0^{\infty} \\ &= \left( 1 - \frac{\mu_3}{\mu_3 + \beta_3} \right) - (0 - 0) = \frac{\beta_3}{\mu_3 + \beta_3} \quad \dots \dots \dots (11) \end{aligned}$$

$$\begin{aligned} P_{11} &= P[X_1 < Y_1, X_4 \geq Y_4] \\ &= \int_{y_1} F_1(y_1) \left[ 1 - F_1\left(\frac{k}{m} y_1\right) \right] f(y_1) dy_1 \\ &= \int_0^{\infty} (e^{-\beta_1 y_1^{-\alpha}}) \left[ 1 - e^{-\beta_1 \left(\frac{k}{m} y_1\right)^{-\alpha}} \right] \alpha \mu_1 y_1^{-(\alpha+1)} e^{-\mu_1 y_1^{-\alpha}} dy_1 \end{aligned}$$

$$\begin{aligned}
 &= \int_0^\infty \alpha \mu_1 y_1^{-(\alpha+1)} e^{-(\mu_1+\beta_1)y_1^{-\alpha}} - \alpha \mu_1 y_1^{-(\alpha+1)} e^{-\left[\mu_1+\left(1+\left(\frac{k}{m}\right)^{-\alpha}\right)\beta_1\right]y_1^{-\alpha}} dy_1 \\
 &= \int_0^\infty \alpha \frac{(\mu_1 + \beta_1)}{(\mu_1 + \beta_1)} \mu_1 y_1^{-(\alpha+1)} e^{-(\mu_1+\beta_1)y_1^{-\alpha}} \\
 &\quad - \alpha \frac{\left[\mu_1+\left(1+\left(\frac{k}{m}\right)^{-\alpha}\right)\beta_1\right]}{\left[\mu_1+\left(1+\left(\frac{k}{m}\right)^{-\alpha}\right)\beta_1\right]} \mu_1 y_1^{-(\alpha+1)} e^{-\left[\mu_1+\left(1+\left(\frac{k}{m}\right)^{-\alpha}\right)\beta_1\right]y_1^{-\alpha}} dy_1 \\
 &= \frac{\mu_1}{\mu_1 + \beta_1} e^{-(\mu_1+\beta_1)y_1^{-\alpha}} - \frac{\mu_1}{\left[\mu_1+\left(1+\left(\frac{k}{m}\right)^{-\alpha}\right)\beta_1\right]} e^{-\left[\mu_1+\left(1+\left(\frac{k}{m}\right)^{-\alpha}\right)\beta_1\right]y_1^{-\alpha}} \Bigg|_0^\infty \\
 &= \left( \frac{\mu_1}{\mu_1 + \beta_1} - \frac{\mu_1}{\left[\mu_1+\left(1+\left(\frac{k}{m}\right)^{-\alpha}\right)\beta_1\right]} \right) - (0 - 0) \\
 &= \frac{\mu_1 \beta_1 \left(\frac{k}{m}\right)^{-\alpha}}{(\mu_1 + \beta_1) \left[\mu_1+\left(1+\left(\frac{k}{m}\right)^{-\alpha}\right)\beta_1\right]} \dots \dots \dots (12)
 \end{aligned}$$

$$\begin{aligned}
 P_{22} &= P[X_2 < Y_2, X_4 \geq Y_4] \\
 &= \int_{y_2} F_2(y_2) \left[1 - F_2\left(\frac{k}{m}y_2\right)\right] f(y_2) dy_2 \\
 &= \int_0^\infty (e^{-\beta_2 y_2^{-\alpha}}) \left[1 - e^{-\beta_2 \left(\frac{k}{m}y_2\right)^{-\alpha}}\right] \alpha \mu_2 y_2^{-(\alpha+1)} e^{-\mu_2 y_2^{-\alpha}} dy_2 \\
 &= \int_0^\infty \alpha \mu_2 y_2^{-(\alpha+1)} e^{-(\mu_2+\beta_2)y_2^{-\alpha}} - \alpha \mu_2 y_2^{-(\alpha+1)} e^{-\left[\mu_2+\left(1+\left(\frac{k}{m}\right)^{-\alpha}\right)\beta_2\right]y_2^{-\alpha}} dy_2 \\
 &= \int_0^\infty \alpha \frac{(\mu_2 + \beta_2)}{(\mu_2 + \beta_2)} \mu_2 y_2^{-(\alpha+1)} e^{-(\mu_2+\beta_2)y_2^{-\alpha}} \\
 &\quad - \alpha \frac{\left[\mu_2+\left(1+\left(\frac{k}{m}\right)^{-\alpha}\right)\beta_2\right]}{\left[\mu_2+\left(1+\left(\frac{k}{m}\right)^{-\alpha}\right)\beta_2\right]} \mu_2 y_2^{-(\alpha+1)} e^{-\left[\mu_2+\left(1+\left(\frac{k}{m}\right)^{-\alpha}\right)\beta_2\right]y_2^{-\alpha}} dy_2 \\
 &= \frac{\mu_2}{\mu_2 + \beta_2} e^{-(\mu_2+\beta_2)y_2^{-\alpha}} - \frac{\mu_2}{\left[\mu_2+\left(1+\left(\frac{k}{m}\right)^{-\alpha}\right)\beta_2\right]} e^{-\left[\mu_2+\left(1+\left(\frac{k}{m}\right)^{-\alpha}\right)\beta_2\right]y_2^{-\alpha}} \Bigg|_0^\infty \\
 &= \left( \frac{\mu_2}{\mu_2 + \beta_2} - \frac{\mu_2}{\left[\mu_2+\left(1+\left(\frac{k}{m}\right)^{-\alpha}\right)\beta_2\right]} \right) - (0 - 0) \\
 &= \frac{\mu_2 \beta_2 \left(\frac{k}{m}\right)^{-\alpha}}{(\mu_2 + \beta_2) \left[\mu_2+\left(1+\left(\frac{k}{m}\right)^{-\alpha}\right)\beta_2\right]} \dots \dots \dots (13)
 \end{aligned}$$

$$P_{33} = P[X_3 < Y_3, X_4 \geq Y_4]$$

$$\begin{aligned}
 &= \int_{y_3} F_3(y_3) \left[ 1 - F_3\left(\frac{k}{m}y_3\right) \right] f(y_3) dy_3 \\
 &= \int_0^\infty (e^{-\beta_3 y_3^{-\alpha}}) \left[ 1 - e^{-\beta_3 \left(\frac{k}{m}y_3\right)^{-\alpha}} \right] \alpha \mu_3 y_3^{-(\alpha+1)} e^{-\mu_3 y_3^{-\alpha}} dy_3 \\
 &= \int_0^\infty \alpha \mu_3 y_3^{-(\alpha+1)} e^{-(\mu_3 + \beta_3) y_3^{-\alpha}} - \alpha \mu_3 y_3^{-(\alpha+1)} e^{-\left[\mu_3 + \left(1 + \left(\frac{k}{m}\right)^{-\alpha}\right) \beta_3\right] y_3^{-\alpha}} dy_3 \\
 &= \int_0^\infty \alpha \frac{(\mu_3 + \beta_3)}{(\mu_3 + \beta_3)} \mu_3 y_3^{-(\alpha+1)} e^{-(\mu_3 + \beta_3) y_3^{-\alpha}} \\
 &\quad - \alpha \frac{\left[\mu_3 + \left(1 + \left(\frac{k}{m}\right)^{-\alpha}\right) \beta_3\right]}{\left[\mu_3 + \left(1 + \left(\frac{k}{m}\right)^{-\alpha}\right) \beta_3\right]} \mu_3 y_3^{-(\alpha+1)} e^{-\left[\mu_3 + \left(1 + \left(\frac{k}{m}\right)^{-\alpha}\right) \beta_3\right] y_3^{-\alpha}} dy_3 \\
 &= \frac{\mu_3}{\mu_3 + \beta_3} e^{-(\mu_3 + \beta_3) y_3^{-\alpha}} - \frac{\mu_3}{\left[\mu_3 + \left(1 + \left(\frac{k}{m}\right)^{-\alpha}\right) \beta_3\right]} e^{-\left[\mu_3 + \left(1 + \left(\frac{k}{m}\right)^{-\alpha}\right) \beta_3\right] y_3^{-\alpha}} \Bigg|_0^\infty \\
 &= \left( \frac{\mu_3}{\mu_3 + \beta_3} - \frac{\mu_3}{\left[\mu_3 + \left(1 + \left(\frac{k}{m}\right)^{-\alpha}\right) \beta_3\right]} \right) - (0 - 0) \\
 &= \frac{\mu_3 \beta_3 \left(\frac{k}{m}\right)^{-\alpha}}{(\mu_3 + \beta_3) \left[\mu_3 + \left(1 + \left(\frac{k}{m}\right)^{-\alpha}\right) \beta_3\right]} \dots \dots \dots (14)
 \end{aligned}$$

$$\begin{aligned}
 R_1 &= P[X_1 \geq Y_1, X_2 \geq Y_2, X_3 \geq Y_3] \\
 &= P[X_1 \geq Y_1] P[X_2 \geq Y_2] P[X_3 \geq Y_3] \\
 R_1 &= P_1 P_2 P_3 \\
 \rightarrow R_1 &= \left[ \frac{\beta_1}{\beta_1 + \mu_1} \right] \left[ \frac{\beta_2}{\beta_2 + \mu_2} \right] \left[ \frac{\beta_3}{\beta_3 + \mu_3} \right] \dots \dots \dots (15)
 \end{aligned}$$

$$\begin{aligned}
 R_2 &= P[X_1 < Y_1, X_2 \geq Y_2, X_3 \geq Y_3, X_4 \geq Y_4] \\
 &= P[X_1 < Y_1, X_2 \geq Y_2, X_3 \geq Y_3, mX_1 \geq kY_1] \\
 &= P[X_1 < Y_1, mX_1 \geq kY_1] P[X_2 \geq Y_2] P[X_3 \geq Y_3] \\
 R_2 &= P_{11} P_2 P_3 \\
 &= \left[ \frac{\mu_1 \beta_1 \left(\frac{k}{m}\right)^{-\alpha}}{(\beta_1 + \mu_1) \left(\beta_1 \left(1 + \left(\frac{k}{m}\right)^{-\alpha}\right) + \mu_1\right)} \right] \left[ \frac{\beta_2}{\beta_2 + \mu_2} \right] \left[ \frac{\beta_3}{\beta_3 + \mu_3} \right] \dots \dots \dots (16)
 \end{aligned}$$

$$\begin{aligned}
 R_3 &= P[X_1 \geq Y_1, X_2 < Y_2, X_3 \geq Y_3, X_4 \geq Y_4] \\
 &= P[X_1 \geq Y_1, X_2 < Y_2, X_3 \geq Y_3, mX_2 \geq kY_2] \\
 &= P[X_1 \geq Y_1] P[X_2 < Y_2, mX_2 \geq kY_2] P[X_3 \geq Y_3] \\
 R_3 &= P_1 P_{22} P_3 \\
 &= \left[ \frac{\beta_1}{\beta_1 + \mu_1} \right] \left[ \frac{\mu_2 \beta_2 \left(\frac{k}{m}\right)^{-\alpha}}{(\beta_2 + \mu_2) \left(\beta_2 \left(1 + \left(\frac{k}{m}\right)^{-\alpha}\right) + \mu_2\right)} \right] \left[ \frac{\beta_3}{\beta_3 + \mu_3} \right] \dots \dots \dots (17)
 \end{aligned}$$

$$\begin{aligned}
 R_4 &= P[X_1 \geq Y_1, X_2 \geq Y_2, X_3 < Y_3, X_4 \geq Y_4] \\
 &= P[X_1 \geq Y_1, X_2 \geq Y_2, X_3 < Y_3, mX_3 \geq kY_3] \\
 &= P[X_1 \geq Y_1] P[X_2 \geq Y_2] P[X_3 < Y_3, mX_3 \geq kY_3] \\
 R_4 &= P_1 P_2 P_{33}
 \end{aligned}$$

$$= \left[ \frac{\beta_1}{\beta_1 + \mu_1} \right] \left[ \frac{\beta_2}{\beta_2 + \mu_2} \right] \left[ \frac{\mu_3 \beta_3 \left(\frac{k}{m}\right)^{-\alpha}}{(\beta_3 + \mu_3) \left(\beta_3 \left(1 + \left(\frac{k}{m}\right)^{-\alpha}\right) + \mu_3\right)} \right] \dots \dots \dots (18)$$

$$R = R_1 + R_2 + R_3 + R_4$$

$$= \left[ \frac{\beta_1}{\beta_1 + \mu_1} \right] \left[ \frac{\beta_2}{\beta_2 + \mu_2} \right] \left[ \frac{\beta_3}{\beta_3 + \mu_3} \right]$$

$$+ \left[ \frac{\mu_1 \beta_1 \left(\frac{k}{m}\right)^{-\alpha}}{(\beta_1 + \mu_1) \left(\beta_1 \left(1 + \left(\frac{k}{m}\right)^{-\alpha}\right) + \mu_1\right)} \right] \left[ \frac{\beta_2}{\beta_2 + \mu_2} \right] \left[ \frac{\beta_3}{\beta_3 + \mu_3} \right]$$

$$+ \left[ \frac{\beta_1}{\beta_1 + \mu_1} \right] \left[ \frac{\mu_2 \beta_2 \left(\frac{k}{m}\right)^{-\alpha}}{(\beta_2 + \mu_2) \left(\beta_2 \left(1 + \left(\frac{k}{m}\right)^{-\alpha}\right) + \mu_2\right)} \right] \left[ \frac{\beta_3}{\beta_3 + \mu_3} \right]$$

$$+ \left[ \frac{\beta_1}{\beta_1 + \mu_1} \right] \left[ \frac{\beta_2}{\beta_2 + \mu_2} \right] \left[ \frac{\mu_3 \beta_3 \left(\frac{k}{m}\right)^{-\alpha}}{(\beta_3 + \mu_3) \left(\beta_3 \left(1 + \left(\frac{k}{m}\right)^{-\alpha}\right) + \mu_3\right)} \right] \dots \dots \dots (19)$$

**3.Parameters Estimation of Frechet distribution .**

**3.1 Maximum Likelihood Estimation Method (ML) :**

Making the maximum likelihood was one of most important developments in 20<sup>th</sup> century statistics . In (1922) Fisher introduced the method of maximum likelihood . He first presented the numerical procedure in (1912) , but in (1922) the maximum likelihood method gave estimates satisfying the criteria of efficiency and sufficiency and there were two

forms for sometimes Fisher based the likelihood on the distribution of the entire sample , sometimes on the distribution of a specific statistic .[3]

Suppose that a random sample  $X_1, X_2, X_3, \dots, X_n$  have  $Fr(\alpha, \beta)$  distribution with sample size n , where  $\beta$  is unknown scale parameter and  $\alpha$  is known shape parameter , then the likelihood function "L", the joint probability function with the general form , can be written as follows :[7]

$$L(X_1, X_2, \dots, X_n, \alpha, \beta) = f(X_1, \alpha, \beta) f(X_2, \alpha, \beta) \dots f(X_n, \alpha, \beta) = \prod_{i=1}^n f(X_i, \alpha, \beta)$$

Then likelihood function using equation (1) will be as :

$$L(X_1, X_2, \dots, X_n, \alpha, \beta) = \prod_{i=1}^n [\alpha \beta x^{-(\alpha+1)} e^{-\beta x^{-\alpha}}]$$

$$L(X_1, X_2, \dots, X_n, \alpha, \beta) = \alpha^n \beta^n \left( \prod_{i=1}^n X_i \right)^{-(\alpha+1)} e^{-\beta \sum_{i=1}^n X_i^{-\alpha}} \dots \dots \dots (20)$$

Then natural logarithm function for equation (20) can be written as ;

$$\ln L = \ln \left[ \alpha^n \beta^n \left( \prod_{i=1}^n X_i \right)^{-(\alpha+1)} e^{-\beta \sum_{i=1}^n X_i^{-\alpha}} \right]$$

$$lnL = nln\alpha + nln\beta - (\alpha + 1) \sum_{i=1}^n lnX_i - \beta \sum_{i=1}^n X_i^{-\alpha} \dots \dots \dots (21)$$

To minimize , natural logarithm in equation (21) , must compute the great endings by taking partial derivative with respect to unknown scale parameter  $\beta$  , then will get as:

$$\frac{\partial lnL}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^n X_i^{-\alpha} \dots \dots \dots (22)$$

Equating partial derivative to zero , thus the right-hand side of (22) will be :

$$\rightarrow \frac{n}{\hat{\beta}} - \sum_{i=1}^n X_i^{-\alpha} = 0 \dots \dots \dots (23)$$

The maximum likelihood estimator for  $\beta$  is given by :

$$\rightarrow \hat{\beta}_{(ML)} = \frac{n}{\sum_{i=1}^n X_i^{-\alpha}} \dots \dots \dots (24)$$

In the same way above , let  $Y_1, Y_2, Y_3, \dots, Y_m$  a random sample have  $Fr(\alpha, \mu)$  distribution with the sample size  $m$  , then the maximum likelihood estimator of unknown scale parameter  $\mu$  ; says  $\hat{\mu}_{(ML)}$  ; is :

$$\hat{\mu}_{(ML)} = \frac{m}{\sum_{j=1}^m Y_j^{-\alpha}} \dots \dots \dots (25)$$

Now , suppose that

$$X_1 \sim Fr(\alpha, \beta_1) , X_2 \sim Fr(\alpha, \beta_2) , X_3 \sim Fr(\alpha, \beta_3) \text{ and } X_4 \sim Fr(\alpha, \beta_4)$$

are strengths r.v.'s with the samples sizes  $n_1, n_2, n_3$  and  $n_4$  respectively , where  $(\beta_1, \beta_2, \beta_3, \beta_4)$  are the unknown scale parameters and suppose that  $Y_1 \sim Fr(\alpha, \mu_1) , Y_2 \sim Fr(\alpha, \mu_2) , Y_3 \sim Fr(\alpha, \mu_3)$  and  $Y_4 \sim Fr(\alpha, \mu_4)$  are the stresses r.v.'s with samples sizes  $m_1, m_2, m_3$  and  $m_4$  respectively , where  $(\mu_1, \mu_2, \mu_3, \mu_4)$  are unknown scale parameters . By using the same way , the maximum likelihood estimators  $(\beta_1, \beta_2, \beta_3, \beta_4)$  and  $(\mu_1, \mu_2, \mu_3, \mu_4)$  are :

$$\hat{\beta}_{\delta(ML)} = \frac{n_{\delta}}{\sum_{i_{\delta}=1}^{n_{\delta}} X_{\delta i_{\delta}}^{-\alpha}} , \delta = 1,2,3,4 \dots \dots \dots (26)$$

and

$$\hat{\mu}_{\delta(ML)} = \frac{m_{\delta}}{\sum_{j_{\delta}=1}^{m_{\delta}} Y_{\delta j_{\delta}}^{-\alpha}} , \delta = 1,2,3,4 \dots \dots \dots (27)$$

Substituting (26) and (27) in (19) , the maximum likelihood estimator for reliability  $R$  ;  $\hat{R}_{(ML)}$  ; invariability will be as :

$$\begin{aligned} \hat{R}_{Fr(ML)} &= \hat{R}_{1(ML)} + \hat{R}_{2(ML)} + \hat{R}_{3(ML)} + \hat{R}_{4(ML)} \\ &= \left[ \frac{\hat{\beta}_{1(ML)}}{\hat{\beta}_{1(ML)} + \hat{\mu}_{1(ML)}} \right] \left[ \frac{\hat{\beta}_{2(ML)}}{\hat{\beta}_{2(ML)} + \hat{\mu}_{2(ML)}} \right] \left[ \frac{\hat{\beta}_{3(ML)}}{\hat{\beta}_{3(ML)} + \hat{\mu}_{3(ML)}} \right] \end{aligned}$$

$$\begin{aligned}
 & + \left[ \frac{\hat{\mu}_{1(ML)} \hat{\beta}_{1(ML)} \left(\frac{k}{m}\right)^{-\alpha}}{(\hat{\beta}_{1(ML)} + \hat{\mu}_{1(ML)}) \left(\hat{\beta}_{1(ML)} \left(1 + \left(\frac{k}{m}\right)^{-\alpha}\right) + \hat{\mu}_{1(ML)}\right)} \right] \left[ \frac{\hat{\beta}_{2(ML)}}{\hat{\beta}_{2(ML)} + \hat{\mu}_{2(ML)}} \right] \left[ \frac{\hat{\beta}_{3(ML)}}{\hat{\beta}_{3(ML)} + \hat{\mu}_{3(ML)}} \right] \\
 & + \left[ \frac{\hat{\beta}_{1(ML)}}{\hat{\beta}_{1(ML)} + \hat{\mu}_{1(ML)}} \right] \left[ \frac{\hat{\mu}_{2(ML)} \hat{\beta}_{2(ML)} \left(\frac{k}{m}\right)^{-\alpha}}{(\hat{\beta}_{2(ML)} + \hat{\mu}_{2(ML)}) \left(\hat{\beta}_{2(ML)} \left(1 + \left(\frac{k}{m}\right)^{-\alpha}\right) + \hat{\mu}_{2(ML)}\right)} \right] \left[ \frac{\hat{\beta}_{3(ML)}}{\hat{\beta}_{3(ML)} + \hat{\mu}_{3(ML)}} \right] \\
 & + \left[ \frac{\hat{\beta}_{1(ML)}}{\hat{\beta}_{1(ML)} + \hat{\mu}_{1(ML)}} \right] \left[ \frac{\hat{\beta}_{2(ML)}}{\hat{\beta}_{2(ML)} + \hat{\mu}_{2(ML)}} \right] \left[ \frac{\hat{\mu}_{3(ML)} \hat{\beta}_{3(ML)} \left(\frac{k}{m}\right)^{-\alpha}}{(\hat{\beta}_{3(ML)} + \hat{\mu}_{3(ML)}) \left(\hat{\beta}_{3(ML)} \left(1 + \left(\frac{k}{m}\right)^{-\alpha}\right) + \hat{\mu}_{3(ML)}\right)} \right]
 \end{aligned}$$

... .. (28)

**3.2 Moments Estimation Method (Mo):**

Karl Pearson in (1894) introduced a formal approach to the statistical estimation through his method of moments (Mo) estimation . He quite unceremoniously suggested a method that simply equal the first five sample moments to the respective population counterparts . It was not simple to solve five highly the nonlinear equations . Therefore , he took an analytical approach of removing one parameter in all step . After considerable algebra Pearson found a ninth  $E(X) = \beta^{\frac{1}{\alpha}} \Gamma\left(1 - \frac{1}{\alpha}\right)$

degree polynomial equation in unknown one . Then after solving the equation and by reiterated back substitutions , Pearson found the solutions to five parameters in the terms of the five first sample moments , and was beginning of the moments method (Mo) estimation [5] . To derive method of the moments estimator parameters of FrD , assume that  $x_i, i = 1,2,3, \dots, n$  random sample have  $Fr(\alpha, \beta)$  distribution with the sample size  $n$  , first step the mean population of  $Fr(\alpha, \beta)$  , obtain by equation (4):[10]

... .. (29)

The second step equating mean sample with corresponding the mean population , then will get as :

$$\frac{\sum_{i=1}^n X_i}{n} = \beta^{\frac{1}{\alpha}} \Gamma\left(1 - \frac{1}{\alpha}\right) \quad \dots \dots \dots (30)$$

Then the moment estimator of  $\beta$  says  $\hat{\beta}_{(Mo)}$  is :

$$\hat{\beta}_{(Mo)} = \left[ \frac{\bar{X}}{\Gamma\left(1 - \frac{1}{\alpha}\right)} \right]^\alpha \quad \dots \dots \dots (31)$$

In the same manner , the moments estimator of unknown scale parameter  $\mu$  ; says  $\hat{\mu}_{(Mo)}$  ; is :

$$\hat{\mu}_{(Mo)} = \left[ \frac{\bar{Y}}{\Gamma\left(1 - \frac{1}{\alpha}\right)} \right]^\alpha \quad \dots \dots \dots (32)$$

Now , by using the same technique , the moments estimators of the unknown scale parameters  $(\beta_1, \beta_2, \beta_3)$  and  $(\mu_1, \mu_2, \mu_3)$  are :



$$\hat{\beta}_{\delta(MO)} = \left[ \frac{\bar{X}_{\delta}}{\Gamma(1-\frac{1}{\alpha})} \right]^{\alpha}, \delta = 1,2,3,4 \dots \dots \dots (33)$$

And

$$\hat{\mu}_{\delta(MO)} = \left[ \frac{\bar{Y}_{\delta}}{\Gamma(1-\frac{1}{\alpha})} \right]^{\alpha}, \delta = 1,2,3,4 \dots \dots \dots (34)$$

Substitution ( 33 ) and (34) in ( 19 ) , the moments estimator for reliability  $R_{Fr}$  ; says  $\hat{R}_{Fr(MO)}$  ; approximately will be as :

$$\begin{aligned} \hat{R}_{Fr(MO)} &= \hat{R}_{1(MO)} + \hat{R}_{2(MO)} + \hat{R}_{3(MO)} + \hat{R}_{4(MO)} \\ &= \left[ \frac{\hat{\beta}_{1(MO)}}{\hat{\beta}_{1(MO)} + \hat{\mu}_{1(MO)}} \right] \left[ \frac{\hat{\beta}_{2(MO)}}{\hat{\beta}_{2(MO)} + \hat{\mu}_{2(MO)}} \right] \left[ \frac{\hat{\beta}_{3(MO)}}{\hat{\beta}_{3(MO)} + \hat{\mu}_{3(MO)}} \right] \\ &+ \left[ \frac{\hat{\mu}_{1(MO)}\hat{\beta}_{1(MO)}\left(\frac{k}{m}\right)^{-\alpha}}{(\hat{\beta}_{1(MO)} + \hat{\mu}_{1(MO)})\left(\hat{\beta}_{1(MO)}\left(1 + \left(\frac{k}{m}\right)^{-\alpha}\right) + \hat{\mu}_{1(MO)}\right)} \right] \left[ \frac{\hat{\beta}_{2(MO)}}{\hat{\beta}_{2(MO)} + \hat{\mu}_{2(MO)}} \right] \left[ \frac{\hat{\beta}_{3(MO)}}{\hat{\beta}_{3(MO)} + \hat{\mu}_{3(MO)}} \right] \\ &+ \left[ \frac{\hat{\beta}_{1(MO)}}{\hat{\beta}_{1(MO)} + \hat{\mu}_{1(MO)}} \right] \left[ \frac{\hat{\mu}_{2(MO)}\hat{\beta}_{2(MO)}\left(\frac{k}{m}\right)^{-\alpha}}{(\hat{\beta}_{2(MO)} + \hat{\mu}_{2(MO)})\left(\hat{\beta}_{2(MO)}\left(1 + \left(\frac{k}{m}\right)^{-\alpha}\right) + \hat{\mu}_{2(MO)}\right)} \right] \left[ \frac{\hat{\beta}_{3(MO)}}{\hat{\beta}_{3(MO)} + \hat{\mu}_{3(MO)}} \right] \\ &+ \left[ \frac{\hat{\beta}_{1(MO)}}{\hat{\beta}_{1(MO)} + \hat{\mu}_{1(MO)}} \right] \left[ \frac{\hat{\beta}_{2(MO)}}{\hat{\beta}_{2(MO)} + \hat{\mu}_{2(MO)}} \right] \left[ \frac{\hat{\mu}_{3(MO)}\hat{\beta}_{3(MO)}\left(\frac{k}{m}\right)^{-\alpha}}{(\hat{\beta}_{3(MO)} + \hat{\mu}_{3(MO)})\left(\hat{\beta}_{3(MO)}\left(1 + \left(\frac{k}{m}\right)^{-\alpha}\right) + \hat{\mu}_{3(MO)}\right)} \right] \\ &\dots \dots \dots (35) \end{aligned}$$

**3.3 Least Squares Estimation Method (LS) :**

The German mathematician Carl Friedrich Gauss had inspected the least squares as early in (1794) , he did not publish the method until (1809) . This estimation method is very popular for the model fitting , especially in linear and non-linear regression . The method of least square estimator scan is produced by

$$S = \sum_{i=1}^n (\hat{F}(X_i) - F(X_i))^2 \dots \dots \dots (36)$$

Suppose that  $X_1, X_2, X_3, \dots, X_n$  be a random sample have  $Fr(\alpha, \beta)$  distribution with the sample size  $n$  . The procedure attempts to minimize the following function with respect to  $\alpha$  and  $\beta$  will get as :

$$S(\alpha, \beta) = \sum_{i=1}^n (\hat{F}(X_i) - (e^{-\beta x_i^{-\alpha}}))^2 \dots \dots \dots (37)$$

To obtain the formula of  $F(X_i)$  ; use the equation (6) :

the minimizing sum of squares error between value and its expected value . [2] The least squares method is a combination of the parametric (F) and the non-parametric ( $\hat{F}$ )

Distribution functions . The minimizing following equation : [8]

$$F(X_i) = e^{-\beta x_i^{-\alpha}}$$

$$-\ln F(X_i) = \beta X_i^{-\alpha} \quad \dots \dots \dots (38)$$

On the other hand, since  $\hat{F}(X_i)$  is unknown, it better to use  $\hat{F}(X_{(i)})$  as follows  $\hat{F}(X_{(i)}) = P_i$  and  $P_i$  is the plotting position Where

$$P_i = \frac{i}{n+1}; i = 1, 2, \dots, n \quad \dots \dots \dots (39)$$

Here  $X_{(i)}$  is the  $i$ :th order statistics of the random sample of the size  $n$  from FrD.

Hence for the FrD, to obtain the LS estimates  $\hat{\beta}$  of the parameter  $\beta$  can be define following the function from equation (37):

$$S(\alpha, \beta) = \sum_{i=1}^n (q_i - \beta X_{(i)}^{-\alpha})^2$$

$$\text{Where } q_i = -\ln \hat{F}(X_{(i)}) = -\ln p_i \quad \dots \dots \dots (40)$$

By taking the derivative equation (40) with respect to the parameter  $\beta$  and equating result to the zero:

$$\frac{\partial S(\alpha, \beta)}{\partial \beta} = \sum_{i=1}^n 2(q_i - \beta X_{(i)}^{-\alpha})(-X_{(i)}^{-\alpha})$$

$$\rightarrow -\sum_{i=1}^n q_i X_{(i)}^{-\alpha} + \hat{\beta} \sum_{i=1}^n X_{(i)}^{-2\alpha} = 0 \quad \dots \dots \dots (41)$$

Then the least squares estimator of  $\beta$ ; says  $\hat{\beta}_{(LS)}$ , will get as:

$$\hat{\beta}_{(LS)} = \frac{\sum_{i=1}^n q_i X_{(i)}^{-\alpha}}{\sum_{i=1}^n X_{(i)}^{-2\alpha}} \quad \dots \dots \dots (42)$$

In the same way, the least squares estimator of unknown parameter  $\mu$ ; says  $\hat{\mu}_{(LS)}$ ; is:

$$\hat{\mu}_{(LS)} = \frac{\sum_{j=1}^m q_j Y_{(j)}^{-\alpha}}{\sum_{j=1}^m Y_{(j)}^{-2\alpha}}$$

Where  $\hat{G}(Y_{(j)}) = \frac{j}{m+1}$ ;  $j = 1, 2, \dots, m$  and

$$q_j = -\ln \hat{G}(Y_{(j)}) = -\ln P_j \quad \dots \dots \dots (43)$$

Now, by using the same way, the last squares estimator of the unknown scale parameters  $(\beta_1, \beta_2, \beta_3)$  and  $(\mu_1, \mu_2, \mu_3)$  are:

$$\hat{\beta}_{\delta(LS)} = \frac{\sum_{i_{\delta}=1}^{n_{\delta}} q_{i_{\delta}} X_{\delta(i_{\delta})}^{-\alpha}}{\sum_{i_{\delta}=1}^{n_{\delta}} X_{\delta(i_{\delta})}^{-2\alpha}}, \delta = 1, 2, 3, 4 \quad \dots \dots \dots (44)$$

and

$$\hat{\mu}_{\delta(LS)} = \frac{\sum_{j_{\delta}=1}^{m_{\delta}} q_{j_{\delta}} Y_{\delta(j_{\delta})}^{-\alpha}}{\sum_{j_{\delta}=1}^{m_{\delta}} Y_{\delta(j_{\delta})}^{-2\alpha}}, \delta = 1, 2, 3, 4 \quad \dots \dots \dots (45)$$

Substitution (44) and (45) in (19), the last squares estimator for reliability  $R_{Fr}$  says  $\hat{R}_{Fr(LS)}$ ; approximately will be as :

$$\begin{aligned} \hat{R}_{Fr(LS)} &= \hat{R}_{1(LS)} + \hat{R}_{2(LS)} + \hat{R}_{3(LS)} + \hat{R}_{4(LS)} \\ &= \left[ \frac{\hat{\beta}_{1(LS)}}{\hat{\beta}_{1(LS)} + \hat{\mu}_{1(LS)}} \right] \left[ \frac{\hat{\beta}_{2(LS)}}{\hat{\beta}_{2(LS)} + \hat{\mu}_{2(LS)}} \right] \left[ \frac{\hat{\beta}_{3(LS)}}{\hat{\beta}_{3(LS)} + \hat{\mu}_{3(LS)}} \right] \\ &+ \left[ \frac{\hat{\mu}_{1(LS)} \hat{\beta}_{1(LS)} \left(\frac{k}{m}\right)^{-\alpha}}{(\hat{\beta}_{1(LS)} + \hat{\mu}_{1(LS)}) \left(\hat{\beta}_{1(LS)} \left(1 + \left(\frac{k}{m}\right)^{-\alpha}\right) + \hat{\mu}_{1(LS)}\right)} \right] \left[ \frac{\hat{\beta}_{2(LS)}}{\hat{\beta}_{2(LS)} + \hat{\mu}_{2(LS)}} \right] \left[ \frac{\hat{\beta}_{3(LS)}}{\hat{\beta}_{3(LS)} + \hat{\mu}_{3(LS)}} \right] \\ &+ \left[ \frac{\hat{\beta}_{1(LS)}}{\hat{\beta}_{1(LS)} + \hat{\mu}_{1(LS)}} \right] \left[ \frac{\hat{\mu}_{2(LS)} \hat{\beta}_{2(LS)} \left(\frac{k}{m}\right)^{-\alpha}}{(\hat{\beta}_{2(LS)} + \hat{\mu}_{2(LS)}) \left(\hat{\beta}_{2(LS)} \left(1 + \left(\frac{k}{m}\right)^{-\alpha}\right) + \hat{\mu}_{2(LS)}\right)} \right] \left[ \frac{\hat{\beta}_{3(LS)}}{\hat{\beta}_{3(LS)} + \hat{\mu}_{3(LS)}} \right] \\ &+ \left[ \frac{\hat{\beta}_{1(LS)}}{\hat{\beta}_{1(LS)} + \hat{\mu}_{1(LS)}} \right] \left[ \frac{\hat{\beta}_{2(LS)}}{\hat{\beta}_{2(LS)} + \hat{\mu}_{2(LS)}} \right] \left[ \frac{\hat{\mu}_{3(LS)} \hat{\beta}_{3(LS)} \left(\frac{k}{m}\right)^{-\alpha}}{(\hat{\beta}_{3(LS)} + \hat{\mu}_{3(LS)}) \left(\hat{\beta}_{3(LS)} \left(1 + \left(\frac{k}{m}\right)^{-\alpha}\right) + \hat{\mu}_{3(LS)}\right)} \right] \\ &\dots \dots \dots (46) \end{aligned}$$

**3.4 Weighted Least Squares Estimation**

**Method (WLS):**

The method weighted least squares extend the method least squares procedure to case where the sample data have different variance . By other words , some the samples have more error or less influence than others . This method reflects the behavior of random errors in the model and it can be used with the

$$Q = \sum_{i=1}^n W_i (\hat{F}(X_i) - F(X_i))^2 \dots \dots \dots (47)$$

Where  $W_i = \frac{1}{var[F(X_{(i)})]} = \frac{(n+1)^2(n+2)}{i(n-i+1)}$  ,  $i = 1, 2, \dots, n$  ..... (48)

Let a random sample  $(X_1, X_2, X_4, \dots, X_n)$  size  $n$  take from have  $Fr(\alpha, \beta)$  distribution . The procedure attempts to minimize the following function with respect to  $\alpha$  and  $\beta$  will get as :

$$Q(\alpha, \beta) = \sum_{i=1}^n W_i (\hat{F}(X_i) - (e^{-\beta x^{-\alpha}}))^2 \dots \dots \dots (49)$$

As steps in equations (38) and (40) will get as :

$$Q(\alpha, \beta) = \sum_{i=1}^n W_i (q_i - \beta X_{(i)}^{-\alpha})^2 \dots \dots \dots (50)$$

By taking partial derivative to the equation (50) with respect to  $\beta$  , and equating result to the zero we obtain :

functions that are either linear or nonlinear in parameters . It works by incorporating extra nonnegative weights or constants associated with all data point into the fitting criterion . The size of weight shows the precision of the information contained in associated observation [16]. The method of weighted last squares can be used in minimizing the following equation :[4]

$$\frac{\partial Q(\alpha, \beta)}{\partial \beta} = \sum_{i=1}^n 2W_i(q_i - \beta X_{(i)}^{-\alpha})(-X_{(i)}^{-\alpha})$$

$$\rightarrow -\sum_{i=1}^n W_i q_i X_{(i)}^{-\alpha} + \hat{\beta} \sum_{i=1}^n W_i X_{(i)}^{-2\alpha} = 0 \dots \dots \dots (51)$$

The weighted least square estimator of  $\beta$  ; says  $\hat{\beta}_{(WLS)}$  :

$$\hat{\beta}_{(WLS)} = \frac{\sum_{i=1}^n W_i q_i X_{(i)}^{-\alpha}}{\sum_{i=1}^n W_i X_{(i)}^{-2\alpha}} \dots \dots \dots (52)$$

In the same technique , the weighted least squares estimator of unknown scale parameter  $\mu$  ; says  $\hat{\mu}_{(WLS)}$  ; is :

$$\hat{\mu}_{(WLS)} = \frac{\sum_{j=1}^m W_j q_j Y_{(j)}^{-\alpha}}{\sum_{j=1}^m W_j Y_{(j)}^{-2\alpha}} \dots \dots \dots (53)$$

Where  $W_j = \frac{1}{Var[G(Y_{(j)})]} = \frac{(m+1)^2(m+2)}{j(m-j+1)}$

$$, j = 1, 2, \dots, m \dots \dots \dots (54)$$

Now , by using the same way , the weighted least squares estimators of the unknown scale parameters  $(\beta_1, \beta_2, \beta_3)$  and  $(\mu_1, \mu_2, \mu_3)$  are :

$$\hat{\beta}_{\delta(WLS)} = \frac{\sum_{i_{\delta}=1}^{n_{\delta}} W_{i_{\delta}} q_{i_{\delta}} X_{i_{\delta}}^{-\alpha}}{\sum_{i_{\delta}=1}^{n_{\delta}} W_{i_{\delta}} X_{i_{\delta}}^{-2\alpha}} , \delta = 1, 2, 3, 4 \dots \dots \dots (55)$$

and

$$\hat{\mu}_{\delta(WLS)} = \frac{\sum_{j_{\delta}=1}^{m_{\delta}} W_{j_{\delta}} q_{j_{\delta}} Y_{j_{\delta}}^{-\alpha}}{\sum_{j_{\delta}=1}^{m_{\delta}} W_{j_{\delta}} Y_{j_{\delta}}^{-2\alpha}} , \delta = 1, 2, 3, 4 \dots \dots \dots (56)$$

Substitution (55) and (56) in (19) , the weighted least squares estimator for reliability  $R_{Fr}$  ; says  $\hat{R}_{Fr(WLS)}$  ; approximately will be as :

$$\begin{aligned} \hat{R}_{Fr(WLS)} &= \hat{R}_{1(WLS)} + \hat{R}_{2(WLS)} + \hat{R}_{3(WLS)} + \hat{R}_{4(WLS)} \\ &= \left[ \frac{\hat{\beta}_{1(WLS)}}{\hat{\beta}_{1(WLS)} + \hat{\mu}_{1(WLS)}} \right] \left[ \frac{\hat{\beta}_{2(WLS)}}{\hat{\beta}_{2(WLS)} + \hat{\mu}_{2(WLS)}} \right] \left[ \frac{\hat{\beta}_{3(WLS)}}{\hat{\beta}_{3(WLS)} + \hat{\mu}_{3(WLS)}} \right] \\ &+ \left[ \frac{\hat{\mu}_{1(WLS)} \hat{\beta}_{1(WLS)} \left(\frac{k}{m}\right)^{-\alpha}}{(\hat{\beta}_{1(WLS)} + \hat{\mu}_{1(WLS)}) \left(\hat{\beta}_{1(WLS)} \left(1 + \left(\frac{k}{m}\right)^{-\alpha}\right) + \hat{\mu}_{1(WLS)}\right)} \right] \left[ \frac{\hat{\beta}_{2(WLS)}}{\hat{\beta}_{2(WLS)} + \hat{\mu}_{2(WLS)}} \right] \left[ \frac{\hat{\beta}_{3(WLS)}}{\hat{\beta}_{3(WLS)} + \hat{\mu}_{3(WLS)}} \right] \\ &+ \left[ \frac{\hat{\beta}_{1(WLS)}}{\hat{\beta}_{1(WLS)} + \hat{\mu}_{1(WLS)}} \right] \left[ \frac{\hat{\mu}_{2(WLS)} \hat{\beta}_{2(WLS)} \left(\frac{k}{m}\right)^{-\alpha}}{(\hat{\beta}_{2(WLS)} + \hat{\mu}_{2(WLS)}) \left(\hat{\beta}_{2(WLS)} \left(1 + \left(\frac{k}{m}\right)^{-\alpha}\right) + \hat{\mu}_{2(WLS)}\right)} \right] \left[ \frac{\hat{\beta}_{3(WLS)}}{\hat{\beta}_{3(WLS)} + \hat{\mu}_{3(WLS)}} \right] \end{aligned}$$

$$+ \left[ \frac{\hat{\beta}_{1(WLS)}}{\hat{\beta}_{1(WLS)} + \hat{\mu}_{1(WLS)}} \right] \left[ \frac{\hat{\beta}_{2(WLS)}}{\hat{\beta}_{2(WLS)} + \hat{\mu}_{2(WLS)}} \right] \left[ \frac{\hat{\mu}_{3(WLS)} \hat{\beta}_{3(WLS)} \left(\frac{k}{m}\right)^{-\alpha}}{(\hat{\beta}_{3(WLS)} + \hat{\mu}_{3(WLS)}) \left(\hat{\beta}_{3(WLS)} \left(1 + \left(\frac{k}{m}\right)^{-\alpha}\right) + \hat{\mu}_{3(WLS)}\right)} \right]$$

... .. (57)

**3.5 Regression Estimation Method (Rg):**

Regression is one of the important procedures that uses supplementary information to construct estimators with a good efficiency . Regression is conceptually the simple method for examining functional relations among  $z_i = a + b\mu_i + e_i$

variables . The relations is expressed in form of an equation or the model connecting the response variable "Y" and one "X" or more expository variables . The simple true relations can be approximated by the standard regression equation :[18]  
 ... .. (58)

Where  $(z_i)$  is the dependent variable ,  $(u_i)$  is the independent variable and  $(e_i)$  is error random variable independent .

Assume that  $X_1, X_2, \dots, X_n$  random samples have  $Fr(\alpha, \beta)$  with the sample size  $n$  .

Taking the natural logarithm to CDF [19] , obtain by equation (6) :

$$F(X_i) = e^{-\beta X_i^{-\alpha}}$$

$$(F(X_i))^{-1} = e^{\beta X_i^{-\alpha}}$$

$$\ln (F(X_i))^{-1} = \beta X_i^{-\alpha}$$

Estimating  $F(X_{(i)})$  by  $P_i$  in equation (39)

$$\ln (P_i)^{-1} = \beta X_{(i)}^{-\alpha} \dots \dots \dots (59)$$

Comparing the equation (59) with equation (58),we get :

$$z_i = \ln (P_i)^{-1} , a = 0 , b = \beta , u_i = X_{(i)}^{-\alpha}$$

Where ;  $i = 1, 2, \dots, n$  ... .. (60)

Where  $b$  can be estimated by the minimizing summation of the squared error with respect to  $b$  ,then we get :

$$\hat{b} = \frac{n \sum_{i=1}^n z_i u_i - \sum_{i=1}^n z_i \sum_{i=1}^n u_i}{n \sum_{i=1}^n (u_i)^2 - (\sum_{i=1}^n u_i)^2} \dots \dots \dots (61)$$

By substitution (60) in (61) , the estimator for  $\beta$  ; says  $\hat{\beta}_{(Rg)}$  ; is :

$$\hat{\beta}_{(Rg)} = \frac{n \sum_{i=1}^n X_{(i)}^{-\alpha} \ln(P_i)^{-1} - \sum_{i=1}^n X_{(i)}^{-\alpha} \sum_{i=1}^n \ln (P_i)^{-1}}{n \sum_{i=1}^n [X_{(i)}^{-\alpha}]^2 - [\sum_{i=1}^n X_{(i)}^{-\alpha}]^2} \dots \dots \dots (62)$$

In the same way , the regression estimator of unknown scale parameter  $\mu$  ; says  $\hat{\mu}_{(Rg)}$  ; is :

$$\hat{\mu}_{(Rg)} = \frac{m \sum_{j=1}^m Y_{(j)}^{-\alpha} \ln (P_j)^{-1} - \sum_{j=1}^m Y_{(j)}^{-\alpha} \sum_{j=1}^m \ln (P_j)^{-1}}{m \sum_{j=1}^m [Y_{(j)}^{-\alpha}]^2 - [\sum_{j=1}^m Y_{(j)}^{-\alpha}]^2} \dots \dots \dots (63)$$

As in equation (60) where  $z_j = \ln (P_j)^{-1}$

$$, a = 0, b = \mu, u_j = y_{(j)}^{-\alpha}; j = 1, 2, \dots, m$$

Now, by using the same way above, the regression estimators of the unknown scale parameters  $(\beta_1, \beta_2, \beta_3)$  and  $(\mu_1, \mu_2, \mu_3)$  are:

$$\hat{\beta}_{\delta(Rg)} = \frac{n_{\delta} \sum_{i_{\delta}=1}^{n_{\delta}} X_{\delta(i_{\delta})}^{-\alpha} \ln (P_{i_{\delta}})^{-1} - \sum_{i_{\delta}=1}^{n_{\delta}} X_{\delta(i_{\delta})}^{-\alpha} \sum_{i_{\delta}=1}^{n_{\delta}} \ln (P_{i_{\delta}})^{-1}}{n_{\delta} \sum_{i_{\delta}=1}^{n_{\delta}} [X_{\delta(i_{\delta})}^{-\alpha}]^2 - [\sum_{i_{\delta}=1}^{n_{\delta}} X_{\delta(i_{\delta})}^{-\alpha}]^2}$$

... .. (64)

and

$$\hat{\mu}_{\delta(Rg)} = \frac{m_{\delta} \sum_{j_{\delta}=1}^{m_{\delta}} Y_{\delta(j_{\delta})}^{-\alpha} \ln (P_{j_{\delta}})^{-1} - \sum_{j_{\delta}=1}^{m_{\delta}} Y_{\delta(j_{\delta})}^{-\alpha} \sum_{j_{\delta}=1}^{m_{\delta}} \ln (P_{j_{\delta}})^{-1}}{m_{\delta} \sum_{j_{\delta}=1}^{m_{\delta}} [Y_{\delta(j_{\delta})}^{-\alpha}]^2 - [\sum_{j_{\delta}=1}^{m_{\delta}} Y_{\delta(j_{\delta})}^{-\alpha}]^2}$$

... .. (65)

Substitution (64) and (65) in (19), the regression estimator for reliability  $R_{Fr}$ ; says  $\hat{R}_{Fr(Rg)}$ ; approximately will be as:

$$\begin{aligned} \hat{R}_{Fr(Rg)} &= \hat{R}_{1(Rg)} + \hat{R}_{2(Rg)} + \hat{R}_{3(Rg)} + \hat{R}_{4(Rg)} \\ &= \left[ \frac{\hat{\beta}_{1(Rg)}}{\hat{\beta}_{1(Rg)} + \hat{\mu}_{1(Rg)}} \right] \left[ \frac{\hat{\beta}_{2(Rg)}}{\hat{\beta}_{2(Rg)} + \hat{\mu}_{2(Rg)}} \right] \left[ \frac{\hat{\beta}_{3(Rg)}}{\hat{\beta}_{3(Rg)} + \hat{\mu}_{3(Rg)}} \right] \\ &+ \left[ \frac{\hat{\mu}_{1(Rg)} \hat{\beta}_{1(Rg)} \left(\frac{k}{m}\right)^{-\alpha}}{(\hat{\beta}_{1(Rg)} + \hat{\mu}_{1(Rg)}) \left(\hat{\beta}_{1(Rg)} \left(1 + \left(\frac{k}{m}\right)^{-\alpha}\right) + \hat{\mu}_{1(Rg)}\right)} \right] \left[ \frac{\hat{\beta}_{2(Rg)}}{\hat{\beta}_{2(Rg)} + \hat{\mu}_{2(Rg)}} \right] \left[ \frac{\hat{\beta}_{3(Rg)}}{\hat{\beta}_{3(Rg)} + \hat{\mu}_{3(Rg)}} \right] \\ &+ \left[ \frac{\hat{\beta}_{1(Rg)}}{\hat{\beta}_{1(Rg)} + \hat{\mu}_{1(Rg)}} \right] \left[ \frac{\hat{\mu}_{2(Rg)} \hat{\beta}_{2(Rg)} \left(\frac{k}{m}\right)^{-\alpha}}{(\hat{\beta}_{2(Rg)} + \hat{\mu}_{2(Rg)}) \left(\hat{\beta}_{2(Rg)} \left(1 + \left(\frac{k}{m}\right)^{-\alpha}\right) + \hat{\mu}_{2(Rg)}\right)} \right] \left[ \frac{\hat{\beta}_{3(Rg)}}{\hat{\beta}_{3(Rg)} + \hat{\mu}_{3(Rg)}} \right] \\ &+ \left[ \frac{\hat{\beta}_{1(Rg)}}{\hat{\beta}_{1(Rg)} + \hat{\mu}_{1(Rg)}} \right] \left[ \frac{\hat{\beta}_{2(Rg)}}{\hat{\beta}_{2(Rg)} + \hat{\mu}_{2(Rg)}} \right] \left[ \frac{\hat{\mu}_{3(Rg)} \hat{\beta}_{3(Rg)} \left(\frac{k}{m}\right)^{-\alpha}}{(\hat{\beta}_{3(Rg)} + \hat{\mu}_{3(Rg)}) \left(\hat{\beta}_{3(Rg)} \left(1 + \left(\frac{k}{m}\right)^{-\alpha}\right) + \hat{\mu}_{3(Rg)}\right)} \right] \end{aligned}$$

... .. (66)

**3.6 Percentile Estimation Method (Pr):**

The method was originally discovered by Kao (1958 - 1959). In case of Fréchet distribution, let a random sample  $X_i; i = 1, 2, 3, \dots, n$  with size n have  $Fr(\alpha, \beta)$ , it is  $F(X_i) = e^{-\beta X_i^{-\alpha}}$

$$\ln F(X_i) = -\beta X_i^{-\alpha}$$

possible to use this method to obtain the estimator unknown scale parameter  $\beta$ , which is obtain from the CDF, defined in equation (6): [9]

$$X_i = \left( \frac{-\ln F(X_i)}{\beta} \right)^{\frac{-1}{\alpha}} \dots \dots \dots (67)$$

If  $P_i ; i = 1, 2, \dots, n$  put the plotting position instead of  $F(X_{(i)}; \alpha, \beta)$ . Can be obtained by minimizing

$$\sum_{i=1}^n \left[ X_{(i)} - \left( \frac{-\ln(p_i)}{\beta} \right)^{\frac{-1}{\alpha}} \right]^2 \dots \dots \dots (68)$$

By taking partial derivative to the (68) with respect to  $\beta$ , and equating the result to zero we obtain :

$$\begin{aligned} & \sum_{i=1}^n 2 \left[ X_{(i)} - (\hat{\beta})^{\frac{1}{\alpha}} (-\ln(p_i))^{\frac{-1}{\alpha}} \right] \left( \frac{-1}{\alpha} (\hat{\beta})^{\left(\frac{1}{\alpha}-1\right)} \right) (-\ln(p_i))^{\frac{-1}{\alpha}} = 0 \\ \rightarrow & \sum_{i=1}^n \left[ X_{(i)} - (\hat{\beta})^{\frac{1}{\alpha}} (-\ln(p_i))^{\frac{-1}{\alpha}} \right] (-\ln(p_i))^{\frac{-1}{\alpha}} = 0 \\ \rightarrow & \sum_{i=1}^n X_{(i)} (-\ln(p_i))^{\frac{-1}{\alpha}} - (\hat{\beta})^{\frac{1}{\alpha}} \sum_{i=1}^n (\ln(p_i))^{\frac{-2}{\alpha}} = 0 \\ \rightarrow & \sum_{i=1}^n X_{(i)} (-\ln(p_i))^{\frac{-1}{\alpha}} = (\hat{\beta})^{\frac{1}{\alpha}} \sum_{i=1}^n (\ln(p_i))^{\frac{-2}{\alpha}} \end{aligned}$$

The percentile estimator of  $\beta$ ; says  $\hat{\beta}$  becomes :

$$\hat{\beta}_{(pr)} = \left[ \frac{\sum_{i=1}^n X_{(i)} (-\ln(p_i))^{\frac{-1}{\alpha}}}{\sum_{i=1}^n (\ln(p_i))^{\frac{-2}{\alpha}}} \right]^{\alpha} \dots \dots \dots (69)$$

In the name way above, the percentile estimator of the unknown parameter  $\hat{\beta}$ ; says  $\hat{\mu}$ ; is :

$$\hat{\mu}_{(pr)} = \left[ \frac{\sum_{j=1}^m Y_{(j)} (-\ln(p_j))^{\frac{-1}{\alpha}}}{\sum_{j=1}^m (\ln(p_j))^{\frac{-2}{\alpha}}} \right]^{\alpha} \dots \dots \dots (70)$$

Now, by using the same manner, the percentile estimators of the unknown scale parameter  $(\beta_1, \beta_2, \beta_3)$  and  $(\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3)$  are :

$$\hat{\beta}_{\delta(pr)} = \left[ \frac{\sum_{i_{\delta}=1}^{n_{\delta}} X_{(i_{\delta})} (-\ln(p_{i_{\delta}}))^{\frac{-1}{\alpha}}}{\sum_{i_{\delta}=1}^{n_{\delta}} (\ln(p_{i_{\delta}}))^{\frac{-2}{\alpha}}} \right]^{\alpha}, \delta = 1, 2, 3, 4 \dots \dots \dots (71)$$

and

$$\hat{\mu}_{\delta(pr)} = \left[ \frac{\sum_{j_{\delta}=1}^{m_{\delta}} Y_{(j_{\delta})} (-\ln(p_{j_{\delta}}))^{\frac{-1}{\alpha}}}{\sum_{j_{\delta}=1}^{m_{\delta}} (\ln(p_{j_{\delta}}))^{\frac{-2}{\alpha}}} \right]^{\alpha}, \delta = 1, 2, 3, 4 \dots \dots \dots (72)$$

Substitution (71) and (72) in (19), the percentile estimator for reliability  $R_{Fr}$ ; says  $\hat{R}_{Fr(Rg)}$ ; approximately will be as :

$$\begin{aligned} \hat{R}_{Fr(pr)} &= \hat{R}_{1(pr)} + \hat{R}_{2(pr)} + \hat{R}_{3(pr)} + \hat{R}_{4(pr)} \\ &= \left[ \frac{\hat{\beta}_{1(pr)}}{\hat{\beta}_{1(pr)} + \hat{\mu}_{1(pr)}} \right] \left[ \frac{\hat{\beta}_{2(pr)}}{\hat{\beta}_{2(pr)} + \hat{\mu}_{2(pr)}} \right] \left[ \frac{\hat{\beta}_{3(pr)}}{\hat{\beta}_{3(pr)} + \hat{\mu}_{3(pr)}} \right] \\ &\quad + \left[ \frac{\hat{\mu}_{1(pr)} \hat{\beta}_{1(pr)} \left(\frac{k}{m}\right)^{-\alpha}}{(\hat{\beta}_{1(pr)} + \hat{\mu}_{1(pr)}) \left(\hat{\beta}_{1(pr)} \left(1 + \left(\frac{k}{m}\right)^{-\alpha}\right) + \hat{\mu}_{1(pr)}\right)} \right] \left[ \frac{\hat{\beta}_{2(pr)}}{\hat{\beta}_{2(pr)} + \hat{\mu}_{2(pr)}} \right] \left[ \frac{\hat{\beta}_{3(pr)}}{\hat{\beta}_{3(pr)} + \hat{\mu}_{3(pr)}} \right] \\ &\quad + \left[ \frac{\hat{\beta}_{1(pr)}}{\hat{\beta}_{1(pr)} + \hat{\mu}_{1(pr)}} \right] \left[ \frac{\hat{\mu}_{2(pr)} \hat{\beta}_{2(pr)} \left(\frac{k}{m}\right)^{-\alpha}}{(\hat{\beta}_{2(pr)} + \hat{\mu}_{2(pr)}) \left(\hat{\beta}_{2(pr)} \left(1 + \left(\frac{k}{m}\right)^{-\alpha}\right) + \hat{\mu}_{2(pr)}\right)} \right] \left[ \frac{\hat{\beta}_{3(pr)}}{\hat{\beta}_{3(pr)} + \hat{\mu}_{3(pr)}} \right] \\ &\quad + \left[ \frac{\hat{\beta}_{1(pr)}}{\hat{\beta}_{1(pr)} + \hat{\mu}_{1(pr)}} \right] \left[ \frac{\hat{\beta}_{2(pr)}}{\hat{\beta}_{2(pr)} + \hat{\mu}_{2(pr)}} \right] \left[ \frac{\hat{\mu}_{3(pr)} \hat{\beta}_{3(pr)} \left(\frac{k}{m}\right)^{-\alpha}}{(\hat{\beta}_{3(pr)} + \hat{\mu}_{3(pr)}) \left(\hat{\beta}_{3(pr)} \left(1 + \left(\frac{k}{m}\right)^{-\alpha}\right) + \hat{\mu}_{3(pr)}\right)} \right] \\ &\dots \dots \dots (73) \end{aligned}$$

**3.7 Pitman Method :**

scale parameter and  $X_i > 0$  . if  $\hat{\beta} = g(X_1, X_2, \dots, X_n)$  is the estimator of the scale parameter  $\beta$ , then  $\hat{\beta}$  should be as follows :[21]

$$f(x, \alpha, \beta) = \alpha \beta x^{-(\alpha+1)} e^{-\beta x^{-\alpha}}$$

$$L(X_1, X_2, \dots, X_n) = f(X_1, \alpha, \beta) f(X_2, \alpha, \beta) \dots f(X_n, \alpha, \beta) \dots \dots \dots (74)$$

$$\begin{aligned} &= (\alpha \beta x_1^{-(\alpha+1)} e^{-\beta x_1^{-\alpha}}) (\alpha \beta x_2^{-(\alpha+1)} e^{-\beta x_2^{-\alpha}}) \dots (\alpha \beta x_n^{-(\alpha+1)} e^{-\beta x_n^{-\alpha}}) \\ &= \alpha^n \beta^n \left( \prod_{i=1}^n x_i \right)^{-(\alpha+1)} e^{-\beta \sum_{i=1}^n x_i^{-\alpha}} \dots \dots \dots (75) \end{aligned}$$

$$\hat{\beta} = \frac{\int_0^\infty \frac{1}{\beta^2} L(X_1, X_2, \dots, X_n) d\beta}{\int_0^\infty \frac{1}{\beta^3} L(X_1, X_2, \dots, X_n) d\beta} \dots \dots \dots (76)$$

$$\begin{aligned} &= \frac{\int_0^\infty \frac{1}{\beta^2} \alpha^n \beta^n (\prod_{i=1}^n x_i)^{-(\alpha+1)} e^{-\beta \sum_{i=1}^n x_i^{-\alpha}} d\beta}{\int_0^\infty \frac{1}{\beta^3} \alpha^n \beta^n (\prod_{i=1}^n x_i)^{-(\alpha+1)} e^{-\beta \sum_{i=1}^n x_i^{-\alpha}} d\beta} \\ &= \frac{\alpha^n (\prod_{i=1}^n x_i)^{-(\alpha+1)} \int_0^\infty \beta^{n-2} e^{-\beta \sum_{i=1}^n x_i^{-\alpha}} d\beta}{\alpha^n (\prod_{i=1}^n x_i)^{-(\alpha+1)} \int_0^\infty \beta^{n-3} e^{-\beta \sum_{i=1}^n x_i^{-\alpha}} d\beta} \end{aligned}$$

Let  $X_1, X_2, \dots, X_n$  be a random sample of  $n$  observations from a population whose p.d.f is  $f(x, \alpha, \beta)$ ; where  $\alpha > 0, \beta > 0$  is a



$$= \frac{\int_0^\infty \beta^{n-2} e^{-\beta \sum_{i=1}^n x_i^{-\alpha}} d\beta}{\int_0^\infty \beta^{n-3} e^{-\beta \sum_{i=1}^n x_i^{-\alpha}} d\beta}$$

Let  $u = \beta \sum_{i=1}^n x_i^{-\alpha} \rightarrow du = \sum_{i=1}^n x_i^{-\alpha} d\beta$  ,  $d\beta = \frac{du}{\sum_{i=1}^n x_i^{-\alpha}}$

and  $\beta = \frac{u}{\sum_{i=1}^n x_i^{-\alpha}}$

$$\begin{aligned} \rightarrow \hat{\beta} &= \frac{\int_0^\infty \left(\frac{u}{\sum_{i=1}^n x_i^{-\alpha}}\right)^{n-2} e^{-u} \frac{du}{\sum_{i=1}^n x_i^{-\alpha}}}{\int_0^\infty \left(\frac{u}{\sum_{i=1}^n x_i^{-\alpha}}\right)^{n-3} e^{-u} \frac{du}{\sum_{i=1}^n x_i^{-\alpha}}} \\ &= \frac{\int_0^\infty \frac{u^{n-2}}{(\sum_{i=1}^n x_i^{-\alpha})^{n-1}} e^{-u} du}{\int_0^\infty \frac{u^{n-3}}{(\sum_{i=1}^n x_i^{-\alpha})^{n-2}} e^{-u} du} \\ &= \frac{\frac{1}{(\sum_{i=1}^n x_i^{-\alpha})^{n-1}} \int_0^\infty u^{n-2} e^{-u} du}{\frac{1}{(\sum_{i=1}^n x_i^{-\alpha})^{n-2}} \int_0^\infty u^{n-3} e^{-u} du} \\ &= \left(\frac{1}{\sum_{i=1}^n x_i^{-\alpha}}\right) \frac{\int_0^\infty u^{n-1-1} e^{-u} du}{\int_0^\infty u^{n-2-1} e^{-u} du} \\ &= \left(\frac{1}{\sum_{i=1}^n x_i^{-\alpha}}\right) \frac{\Gamma(n-1)}{\Gamma(n-2)} = \left(\frac{1}{\sum_{i=1}^n x_i^{-\alpha}}\right) \frac{(n-2)!}{(n-3)!} \\ &= \left(\frac{1}{\sum_{i=1}^n x_i^{-\alpha}}\right) \frac{(n-2)(n-3)!}{(n-3)!} = \frac{n-2}{\sum_{i=1}^n x_i^{-\alpha}} \end{aligned}$$

The Pitman estimator of  $\beta$  ; says  $\hat{\beta}$  becomes :

$$\hat{\beta} = \frac{n-2}{\sum_{i=1}^n x_i^{-\alpha}} \dots \dots \dots (77)$$

$\hat{\beta}$  In the name way above , the Pitman estimator of the unknown parameter ; says  $\hat{\mu}$  ; is :

$$\hat{\mu} = \frac{m-2}{\sum_{j=1}^m y_j^{-\alpha}} \dots \dots \dots (78)$$

Now , by using the same manner , the Pitman estimators of the unknown scale parametor  $(\beta_1, \beta_2, \beta_3)$  and  $(\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3)$  are :

$$\hat{\beta}_{\delta(pi)} = \frac{n_\delta - 2}{\sum_{i_\delta=1}^{n_\delta} x_{i_\delta}^{-\alpha}} , \delta = 1,2,3,4 \dots \dots \dots (79)$$

and

$$\hat{\mu}_{\delta(pi)} = \frac{m_\delta - 2}{\sum_{j_\delta=1}^{m_\delta} y_{j_\delta}^{-\alpha}} , \delta = 1,2,3,4 \dots \dots \dots (80)$$

Substituting (79) and (80) in (19), the pitman estimator for reliability  $R$ ;  $\hat{R}_{(Pi)}$ ; invariability will be as :

$$\begin{aligned} \hat{R}_{Fr(Pi)} &= \hat{R}_{1(Pi)} + \hat{R}_{2(Pi)} + \hat{R}_{3(Pi)} + \hat{R}_{4(Pi)} \\ &= \left[ \frac{\hat{\beta}_{1(Pi)}}{\hat{\beta}_{1(Pi)} + \hat{\mu}_{1(Pi)}} \right] \left[ \frac{\hat{\beta}_{2(Pi)}}{\hat{\beta}_{2(Pi)} + \hat{\mu}_{2(Pi)}} \right] \left[ \frac{\hat{\beta}_{3(Pi)}}{\hat{\beta}_{3(Pi)} + \hat{\mu}_{3(Pi)}} \right] \\ &\quad + \left[ \frac{\hat{\mu}_{1(Pi)} \hat{\beta}_{1(Pi)} \left(\frac{k}{m}\right)^{-\alpha}}{(\hat{\beta}_{1(Pi)} + \hat{\mu}_{1(Pi)}) \left(\hat{\beta}_{1(Pi)} \left(1 + \left(\frac{k}{m}\right)^{-\alpha}\right) + \hat{\mu}_{1(Pi)}\right)} \right] \left[ \frac{\hat{\beta}_{2(Pi)}}{\hat{\beta}_{2(Pi)} + \hat{\mu}_{2(Pi)}} \right] \left[ \frac{\hat{\beta}_{3(Pi)}}{\hat{\beta}_{3(Pi)} + \hat{\mu}_{3(Pi)}} \right] \\ &\quad + \left[ \frac{\hat{\beta}_{1(Pi)}}{\hat{\beta}_{1(Pi)} + \hat{\mu}_{1(Pi)}} \right] \left[ \frac{\hat{\mu}_{2(Pi)} \hat{\beta}_{2(Pi)} \left(\frac{k}{m}\right)^{-\alpha}}{(\hat{\beta}_{2(Pi)} + \hat{\mu}_{2(Pi)}) \left(\hat{\beta}_{2(Pi)} \left(1 + \left(\frac{k}{m}\right)^{-\alpha}\right) + \hat{\mu}_{2(Pi)}\right)} \right] \left[ \frac{\hat{\beta}_{3(Pi)}}{\hat{\beta}_{3(Pi)} + \hat{\mu}_{3(Pi)}} \right] \\ &\quad + \left[ \frac{\hat{\beta}_{1(Pi)}}{\hat{\beta}_{1(Pi)} + \hat{\mu}_{1(Pi)}} \right] \left[ \frac{\hat{\beta}_{2(Pi)}}{\hat{\beta}_{2(Pi)} + \hat{\mu}_{2(Pi)}} \right] \left[ \frac{\hat{\mu}_{3(Pi)} \hat{\beta}_{3(Pi)} \left(\frac{k}{m}\right)^{-\alpha}}{(\hat{\beta}_{3(Pi)} + \hat{\mu}_{3(Pi)}) \left(\hat{\beta}_{3(Pi)} \left(1 + \left(\frac{k}{m}\right)^{-\alpha}\right) + \hat{\mu}_{3(Pi)}\right)} \right] \\ &\dots \dots \dots (81) \end{aligned}$$

**4. The experimental study :**

We simulate the outputs of all three estimating methods by using MSE . Study of simulation is replicated several times (500) so that the samples of three sizes ( small , moderate and large ) are independently collected .

- 1- The random sample  $(x_{11}, x_{12}, \dots, x_{1r_1}), (x_{21}, x_{22}, \dots, x_{2r_2}),$   
 $(x_{31}, x_{32}, \dots, x_{3r_3}),$  and  $(y_{11}, y_{12}, \dots, y_{1v_1}),$   
 $(y_{21}, y_{22}, \dots, y_{2v_2}), (y_{31}, y_{32}, \dots, y_{3v_3})$

Of sizes  $(r_1, r_2, r_3, v_1, v_2, v_3) = (15, 15, 15, 15, 15, 15)$   
 $, (45, 45, 45, 45, 45, 45)$  and  $(95, 95, 95, 95, 95, 95)$

are generated from Frechet distribution .

- 2- Selected the values of parameters for 6 experiments  $(\beta_1, \beta_2, \beta_3, \mu_1, \mu_2, \mu_3)$  in the table (1) :

Table (1) : Values of parameters and Reliability ,

such that  $\alpha = 0.1$

Experiment	$k$	$m$	$\beta_1$	$\beta_2$	$\beta_3$	$\mu_1$	$\mu_2$	$\mu_3$	$R$

1	1.6	0.3	1.7	1.7	1.7	1.1	1.1	1.1	0.2834
2	1.6	0.3	2	2	2	1.1	1.1	1.1	0.3326
3	1.6	0.3	1.7	1.7	1.7	2	2	2	0.1338
4	1.2	0.8	1.7	1.7	1.7	1.1	1.1	1.1	0.3563
5	1.2	0.8	2	2	2	2	2	2	0.2246
6	1.4	0.9	1.5	1.6	1.7	1.8	1.9	2	0.1781

3-Parameters  $\beta_1, \beta_2, \beta_3, \mu_1, \mu_2, \mu_3$  were estimated (ML, Mo, LS, WLS, Rg, Pr and Pi) in equations : (26),(27),(33),(34),(44),(45),(55),(56),(64),(65),(71),(72),(79) and (80) respectively .

4-  $R$  was estimated in equations : (28),(35),(46),(57),(66),(73) and (81) .

5- Calculate the mean by  $\text{Mean} = \frac{\sum_{i=1}^L R_i}{L}$

6- The last stage is to use the " Mean square Error " to assess the results of the seven estimation methods :

$$MSE(\hat{R}) = \frac{1}{L} \sum_{i=1}^L (\hat{R}_i - R)^2$$

### Simulation Results:

After applying the previous steps of  $R$  for (95,95,95,95,95,95) sample size

$(r_1, r_2, r_3, v_1, v_2, v_3)$ : (15,15,15,15,15,15),

(45,45,45,45,45,45) and

Table (2) : Values MSE and Mean for 6 experiments .

Exp	Simple size	Criterion	ML	MO	LS	WLS	Rg	Pr	Pi	Best
1	(15,15,15,15,15,15)	MSE	0.0037	0.0327	0.0044	0.0053	0.0064	0.0421	0.0037	ML,Pi
		Mean	0.2819	0.2505	0.2828	0.2828	0.2818	0.2403	0.2819	
	(45,45,45,45,45,45)	MSE	0.001	0.026	0.001	0.002	0.0022	0.0363	0.0013	ML,Pi

	45)		3	4	5	4					
		Mean	0.279 3	0.250 3	0.278 3	0.277 5	0.2769	0.2345	0.2793		
	(95,95,95,95,95, 95)	MSE	0.000 6	0.025 3	0.000 7	0.001 4	0.0011	0.0372	0.0006	ML,Pi	
		Mean	0.280 6	0.237 9	0.281 0	0.279 7	0.2807	0.2205	0.2806		
2	(15,15,15,15,15, 15)	MSE	0.003 9	0.037 8	0.004 7	0.005 8	0.0070	0.0494	0.0039	ML,Pi	
		Mean	0.326 9	0.273 0	0.326 7	0.325 7	0.3241	0.2572	0.3269		
	(45,45,45,45,45, 45)	MSE	0.001 4	0.031 0	0.001 7	0.002 9	0.0026	0.0437	0.0014	ML,Pi	
		Mean	0.329 6	0.280 4	0.330 2	0.329 8	0.3294	0.2591	0.3296		
	(95,95,95,95,95, 95)	MSE	0.000 7	0.029 9	0.000 9	0.001 9	0.0014	0.0448	0.0007	ML,Pi	
		Mean	0.332 5	0.283 8	0.331 4	0.329 0	0.3299	0.2598	0.3325		
	3	(15,15,15,15,15, 15)	MSE	0.001 6	0.015 4	0.001 8	0.002 2	0.0027	0.0209	0.0016	ML,Pi
			Mean	0.130 3	0.130 1	0.129 7	0.129 4	0.1291	0.1293	0.1303	
(45,45,45,45,45, 45)		MSE	0.000 5	0.011 3	0.000 7	0.001 1	0.0011	0.0166	0.0005	ML,Pi	
		Mean	0.132 5	0.126 2	0.132 7	0.132 8	0.1328	0.1250	0.1325		
(95,95,95,95,95, 95)		MSE	0.000 3	0.012 0	0.000 3	0.000 8	0.0005	0.0195	0.0003	ML,Pi, LS	
		Mean	0.132 7	0.125 2	0.132 7	0.132 0	0.1327	0.1249	0.1327		
4	(15,15,15,15,15, 15)	MSE	0.004 8	0.039 2	0.005 7	0.006 8	0.0081	0.0506	0.0048	ML,Pi	
		Mean	0.343 8	0.297 5	0.341 8	0.339 7	0.3374	0.2802	0.3438		
	(45,45,45,45,45, 45)	MSE	0.001 4	0.033 1	0.001 8	0.002 9	0.0028	0.0494	0.0014	ML,Pi	
		Mean	0.354	0.308	0.353	0.352	0.3526	0.2839	0.3540		

			0	3	8	2				
	(95,95,95,95,95,95)	MSE	0.0006	0.0304	0.0009	0.0020	0.0015	0.0459	0.0006	ML,Pi
		Mean	0.3547	0.3018	0.3542	0.3525	0.3533	0.2775	0.3547	
5	(15,15,15,15,15,15)	MSE	0.0032	0.0251	0.0036	0.0042	0.0052	0.0322	0.0032	ML,Pi
		Mean	0.2226	0.2021	0.2213	0.2197	0.2183	0.1962	0.2226	
	(45,45,45,45,45,45)	MSE	0.0011	0.0217	0.0014	0.0023	0.0022	0.0312	0.0011	ML,Pi
		Mean	0.2241	0.2016	0.2223	0.2193	0.2203	0.1928	0.2241	
	(95,95,95,95,95,95)	MSE	0.0005	0.0217	0.0006	0.0013	0.0009	0.0323	0.0005	ML,Pi
		Mean	0.2237	0.2005	0.2228	0.2207	0.2215	0.1913	0.2237	
6	(15,15,15,15,15,15)	MSE	0.0025	0.0214	0.0029	0.0035	0.0042	0.0269	0.0025	ML,Pi
		Mean	0.1768	0.1682	0.1773	0.1773	0.1771	0.1629	0.1768	
	(45,45,45,45,45,45)	MSE	0.0008	0.0198	0.0010	0.0017	0.0016	0.0283	0.0008	ML,Pi
		Mean	0.1778	0.1737	0.1771	0.1765	0.1759	0.1717	0.1778	
	(95,95,95,95,95,95)	MSE	0.0004	0.0172	0.0005	0.0011	0.0008	0.0269	0.0004	ML,Pi
		Mean	0.1770	0.1567	0.1774	0.1785	0.1777	0.1523	0.1770	

### Conclusions :

This conclusions according to the simulation study results :

1. We concluded from the table (1) .

I- With increasing value of parameter  $\beta$  , reliability is increasing .

II- With the increasing value of parameter  $\mu$  , reliability is decreasing .

III- With the decreasing value of  $\frac{K}{M}$  , reliability is increasing .

2. We concluded from the table (2) the best estimator for  $R$  is ML for 6 experiments and different sample sizes .

clc  
clear All

```

% Cascade P(X>y)
% 3+1 Frechet, {Ahmed Mezher}
%% parameters real values
n=15 ;k=1.6 ;m=0.3 ;alfa=0.8 ;
meu1=1.1 ;meu2=1.1 ;meu3=1.1 ;beta1=1.7
;beta2=1.7 ;beta3=1.7 ;
%% reliability R real values
g=(k/m)^(-alfa);h=1+g;
P1=(beta1/(meu1+beta1));
P2=(beta2/(meu2+beta2));
P3=(beta3/(meu3+beta3));
P11=(meu1*beta1*g)/((beta1+meu1)*(beta1*
h+meu1));
P22=(meu2*beta2*g)/((beta2+meu2)*(beta2*
h+meu2));
P33=(meu3*beta3*g)/((beta3+meu3)*(beta3*
h+meu3));
R1=P1*P2*P3; R2=P11*P2*P3; R3=P1*P22*P3;
R4=P1*P2*P33;
Rreal=R1+R2+R3+R4;
Par=[n k m alfa g h meu1 meu2 meu3 beta1
beta2 beta3 Rreal]
%% Simulation
sim=0;L=500;
for sim=1:L
Ux1=rand(1,n);
x1=(-1/beta1).*log(Ux1).^(-1/alfa);
Ux2=rand(1,n);
x2=(-1/beta2).*log(Ux2).^(-1/alfa);
Ux3=rand(1,n);
x3=(-1/beta3).*log(Ux3).^(-1/alfa);
Uy1=rand(1,n);
y1=(-1./meu1).*log(Uy1).^(-1./alfa);
Uy2=rand(1,n);
y2=(-1./meu2).*log(Uy2).^(-1/alfa);
Uy3=rand(1,n);
y3=(-1./meu3).*log(Uy3).^(-1/alfa);
%% MLE
beta1ML=n/sum(x1.^-
alfa);beta2ML=n/sum(x2.^-alfa);
beta3ML=n/sum(x3.^-alfa);
meu1ML=n/sum(y1.^-
alfa);meu2ML=n/sum(y2.^-alfa);
meu3ML=n/sum(y3.^-alfa);
P1ML=(beta1ML/(meu1ML+beta1ML));
P2ML=(beta2ML/(meu2ML+beta2ML));
P3ML=(beta3ML/(meu3ML+beta3ML));
P11ML=(meu1ML*beta1ML*g)/((beta1ML+me
u1ML)*(beta1ML*h+meu1ML));

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P22ML=(meu2ML*beta2ML*g)/((beta2ML+me
u2ML)*(beta2ML*h+meu2ML));
P33ML=(meu3ML*beta3ML*g)/((beta3ML+me
u3ML)*(beta3ML*h+meu3ML));
R1ML=P1ML*P2ML*P3ML;
R2ML=P11ML*P2ML*P3ML;
R3ML=P1ML*P22ML*P3ML;
R4ML=P1ML*P2ML*P33ML;
RML(sim)=R1ML+R2ML+R3ML+R4ML;
% ML=[n k m g h alfa meu1ML meu2ML
meu3ML beta1ML beta2ML beta3ML RML]
%% Moments
beta1Mo=(mean(x1)/gamma(1-(1/alfa)))^alfa;
beta2Mo=(mean(x2)/gamma(1-(1/alfa)))^alfa;
beta3Mo=(mean(x3)/gamma(1-(1/alfa)))^alfa;
meu1Mo=(mean(y1)/gamma(1-(1/alfa)))^alfa;
meu2Mo=(mean(y2)/gamma(1-(1/alfa)))^alfa;
meu3Mo=(mean(y3)/gamma(1-(1/alfa)))^alfa;
P1Mo=(beta1Mo/(meu1Mo+beta1Mo));
P2Mo=(beta2Mo/(meu2Mo+beta2Mo));
P3Mo=(beta3Mo/(meu3Mo+beta3Mo));
P11Mo=(meu1Mo.*beta1Mo.*g)/((beta1Mo+m
eu1Mo).*(beta1Mo.*h+meu1Mo));
P22Mo=(meu2Mo.*beta2Mo.*g)/((beta2Mo+m
eu2Mo).*(beta2Mo.*h+meu2Mo));
P33Mo=(meu3Mo.*beta3Mo.*g)/((beta3Mo+m
eu3Mo).*(beta3Mo.*h+meu3Mo));
R1Mo=P1Mo.*P2Mo.*P3Mo;
R2Mo=P11Mo.*P2Mo.*P3Mo;
R3Mo=P1Mo.*P22Mo.*P3Mo;
R4Mo=P1Mo.*P2Mo.*P33Mo;
RMo(sim)=R1Mo+R2Mo+R3Mo+R4Mo;
% Mo=[n k m g h alfa meu1Mo meu2Mo
meu3Mo beta1Mo beta2Mo beta3Mo RMo]
%% Least square
sx1=sort(x1); sx2=sort(x2); sx3=sort(x3);
sy1=sort(y1); sy2=sort(y2); sy3=sort(y3);
i=1:n; p=i./(n+1); q=-log(p);
beta1Ls=sum(q.*sx1.^-alfa)/sum(sx1.^(-
2*alfa)); beta2Ls=sum(q.*sx2.^-
alfa)/sum(sx2.^(-2*alfa));
beta3Ls=sum(q.*sx3.^-alfa)/sum(sx3.^(-
2*alfa)); meu1Ls=sum(q.*sy1.^-
alfa)/sum(sy1.^(-2*alfa));
meu2Ls=sum(q.*sy2.^-alfa)/sum(sy2.^(-
2*alfa)); meu3Ls=sum(q.*sy3.^-
alfa)/sum(sy3.^(-2*alfa));
P1Ls=(beta1Ls/(meu1Ls+beta1Ls));
P2Ls=(beta2Ls/(meu2Ls+beta2Ls));
P3Ls=(beta3Ls/(meu3Ls+beta3Ls));

```

$P11Ls=(meu1Ls*beta1Ls*g)/((beta1Ls+meu1Ls)*(beta1Ls*h+meu1Ls));$   
 $P22Ls=(meu2Ls*beta2Ls*g)/((beta2Ls+meu2Ls)*(beta2Ls*h+meu2Ls));$   
 $P33Ls=(meu3Ls*beta3Ls*g)/((beta3Ls+meu3Ls)*(beta3Ls*h+meu3Ls));$   
 $R1Ls=P1Ls*P2Ls*P3Ls;$   
 $R2Ls=P11Ls*P2Ls*P3Ls;$   
 $R3Ls=P1Ls*P22Ls*P3Ls;$   
 $R4Ls=P1Ls*P2Ls*P33Ls;$   
 $RLs(sim)=R1Ls+R2Ls+R3Ls+R4Ls;$   
 % Par=[n k m alfa g h meu1Ls meu2Ls meu3Ls  
 beta1Ls beta2Ls beta3Ls RLs]  
 %% WLS  
 $w=((n+1)^2*(n+2)./(i.*(n-i+1)));$   
 $beta1wLs=sum(w.*q.*sx1.^-alfa)/sum(w.*sx1.^(-2*alfa));$   
 $beta2wLs=sum(w.*q.*sx2.^-alfa)/sum(w.*sx2.^(-2*alfa));$   
 $beta3wLs=sum(w.*q.*sx3.^-alfa)/sum(w.*sx3.^(-2*alfa));$   
 $meu1wLs=sum(w.*q.*sy1.^-alfa)/sum(w.*sy1.^(-2*alfa));$   
 $meu2wLs=sum(w.*q.*sy2.^-alfa)/sum(w.*sy2.^(-2*alfa));$   
 $meu3wLs=sum(w.*q.*sy3.^-alfa)/sum(w.*sy3.^(-2*alfa));$   
 $P1wLs=(beta1wLs/(meu1wLs+beta1wLs));$   
 $P2wLs=(beta2wLs/(meu2wLs+beta2wLs));$   
 $P3wLs=(beta3wLs/(meu3wLs+beta3wLs));$   
 $P11wLs=(meu1wLs*beta1wLs*g)/((beta1wLs+meu1wLs)*(beta1wLs*h+meu1wLs));$   
 $P22wLs=(meu2wLs*beta2wLs*g)/((beta2wLs+meu2wLs)*(beta2wLs*h+meu2wLs));$   
 $P33wLs=(meu3wLs*beta3wLs*g)/((beta3wLs+meu3wLs)*(beta3wLs*h+meu3wLs));$   
 $R1wLs=P1wLs*P2wLs*P3wLs;$   
 $R2wLs=P11wLs*P2wLs*P3wLs;$   
 $R3wLs=P1wLs*P22wLs*P3wLs;$   
 $R4wLs=P1wLs*P2wLs*P33wLs;$   
 $RwLs(sim)=R1wLs+R2wLs+R3wLs+R4wLs;$   
 % Par=[n k m alfa g h meu1wLs meu2wLs  
 meu3wLs beta1wLs beta2wLs beta3wLs RwLs]  
 %% Regression  
 $Bx1=sx1.^-alfa;$   
 $beta1Rg=(n.*sum(Bx1.*q)-(sum(Bx1)*sum(q)))/(n.*sum(Bx1.^2)-(sum(Bx1).^2));$   
 $Bx2=sx2.^-alfa;$

$beta2Rg=(n.*sum(Bx2.*q)-(sum(Bx2)*sum(q)))/(n.*sum(Bx2.^2)-(sum(Bx2).^2));$   
 $Bx3=sx3.^-alfa;$   
 $beta3Rg=(n.*sum(Bx3.*q)-(sum(Bx3)*sum(q)))/(n.*sum(Bx3.^2)-(sum(Bx3).^2));$   
 $By1=sy1.^-alfa;$   
 $meu1Rg=(n.*sum(By1.*q)-(sum(By1)*sum(q)))/(n.*sum(By1.^2)-(sum(By1).^2));$   
 $By2=sy2.^-alfa;$   
 $meu2Rg=(n.*sum(By2.*q)-(sum(By2)*sum(q)))/(n.*sum(By2.^2)-(sum(By2).^2));$   
 $By3=sy3.^-alfa;$   
 $meu3Rg=(n.*sum(By3.*q)-(sum(By3)*sum(q)))/(n.*sum(By3.^2)-(sum(By3).^2));$   
 $P1Rg=(beta1Rg/(meu1Rg+beta1Rg));$   
 $P2Rg=(beta2Rg/(meu2Rg+beta2Rg));$   
 $P3Rg=(beta3Rg/(meu3Rg+beta3Rg));$   
 $P11Rg=(meu1Rg*beta1Rg*g)/((beta1Rg+meu1Rg)*(beta1Rg*h+meu1Rg));$   
 $P22Rg=(meu2Rg*beta2Rg*g)/((beta2Rg+meu2Rg)*(beta2Rg*h+meu2Rg));$   
 $P33Rg=(meu3Rg*beta3Rg*g)/((beta3Rg+meu3Rg)*(beta3Rg*h+meu3Rg));$   
 $R1Rg=P1Rg*P2Rg*P3Rg;R2Rg=P11Rg*P2Rg*P3Rg;$   
 $R3Rg=P1Rg*P22Rg*P3Rg;R4Rg=P1Rg*P2Rg*P33Rg;$   
 $RRg(sim)=R1Rg+R2Rg+R3Rg+R4Rg;$   
 % Par=[n k m alfa g h meu1Rg meu2Rg  
 meu3Rg beta1Rg beta2Rg beta3Rg RRg]  
 %% Percentile  
 $beta1P=(sum(sx1.*q.^(-1/alfa))/sum(q.^(-2/alfa))).^alfa;$   
 $beta2P=(sum(sx2.*q.^(-1/alfa))/sum(q.^(-2/alfa))).^alfa;$   
 $beta3P=(sum(sx3.*q.^(-1/alfa))/sum(q.^(-2/alfa))).^alfa;$   
 $meu1P=(sum(sy1.*q.^(-1/alfa))/sum(q.^(-2/alfa))).^alfa;$   
 $meu2P=(sum(sy2.*q.^(-1/alfa))/sum(q.^(-2/alfa))).^alfa;$   
 $meu3P=(sum(sy3.*q.^(-1/alfa))/sum(q.^(-2/alfa))).^alfa;$

```

P1P=(beta1P/(meu1P+beta1P));
P2P=(beta2P/(meu2P+beta2P));
P3P=(beta3P/(meu3P+beta3P));
P11P=(meu1P*beta1P*g)/((beta1P+meu1P)*(
beta1P*h+meu1P));
P22P=(meu2P*beta2P*g)/((beta2P+meu2P)*(
beta2P*h+meu2P));
P33P=(meu3P*beta3P*g)/((beta3P+meu3P)*(
beta3P*h+meu3P));
R1P=P1P*P2P*P3P;
R2P=P11P*P2P*P3P;R3P=P1P*P22P*P3P;
R4P=P1P*P2P*P33P;
RP(sim)=R1P+R2P+R3P+R4P;
% Par=[n k m alfa g h meu1P meu2P meu3P
beta1P beta2P beta3P RP]
%%Pitman
beta1Pi=(n-2)/sum(x1.^-alfa);
beta2Pi=(n-2)/sum(x2.^-alfa);
beta3Pi=(n-2)/sum(x3.^-alfa);
meu1Pi=(n-2)/sum(y1.^-alfa);
meu2Pi=(n-2)/sum(y2.^-alfa);
meu3Pi=(n-2)/sum(y3.^-alfa);
P1Pi=(beta1Pi/(meu1Pi+beta1Pi));
P2Pi=(beta2Pi/(meu2Pi+beta2Pi));
P3Pi=(beta3Pi/(meu3Pi+beta3Pi));
P11Pi=(meu1Pi*beta1Pi*g)/((beta1Pi+meu1Pi
)*(beta1Pi*h+meu1Pi));
P22Pi=(meu2Pi*beta2Pi*g)/((beta2Pi+meu2Pi
)*(beta2Pi*h+meu2Pi));
P33Pi=(meu3Pi*beta3Pi*g)/((beta3Pi+meu3Pi
)*(beta3Pi*h+meu3Pi));
R1Pi=P1Pi*P2Pi*P3Pi;
R2Pi=P11Pi*P2Pi*P3Pi;R3Pi=P1Pi*P22Pi*P3Pi;
R4Pi=P1Pi*P2Pi*P33Pi;
RPi(sim)=R1Pi+R2Pi+R3Pi+R4Pi;
% Par=[n k m alfa g h meu1Pi meu2Pi meu3Pi
beta1Pi beta2Pi beta3Pi RPi]
end
MeanRML=mean(RML);MseRML=mean((RML-
Rreal).^2);
MeanRMo=mean(RMo);MseRMo=mean((RMo-
Rreal).^2);
MeanRLs=mean(RLs);MseRLs=mean((RLs-
Rreal).^2);
MeanRwLs=mean(RwLs);MseRwLs=mean((Rw
Ls-Rreal).^2);
MeanRRg=mean(RRg);MseRRg=mean((RRg-
Rreal).^2);
MeanRP=mean(RP);MseRP=mean((RP-
Rreal).^2);

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MeanRPi=mean(RPi);MseRPi=mean((RPi-
Rreal).^2);
Mean=[MeanRML MeanRMo MeanRLs
MeanRwLs MeanRRg MeanRP MeanRPi]
Mse=[MseRML MseRMo MseRLs MseRwLs
MseRRg MseRP MseRPi]

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