Distribution And Density Functions Of Continuous Random Variables, Their Properties

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ABSTRACT
This article provides information about the distribution and density functions of continuous random variables and their properties. The article also presents results that directly follow from the properties of distribution and density functions. In addition, the article also describes in detail how to find the mathematical expectation using the distribution and density functions of continuous random variables.

Keywords: probability of an event, function, interval, continuous random variables, distribution function, density function, mathematical expectation.

It is known that the values of continuous random variables completely fill an interval (a, b). For example, the distance to a target is a continuous random variable, and its possible values completely fill an interval. Obviously, the distribution law of such a random variable cannot be expressed in a table.

If the random variable is continuous, the concept of the distribution function is introduced and the random variable is studied through it. Suppose that ξ is a continuous random variable, and its possible values consist of the interval (a, b). Let us consider the event that, taking some real number x, this random variable takes values less than x. We denote this event as $\xi < x$. Its probability

$$P\{\xi < x\}$$

depends on the given x, that is, it is a function of x.

Definition. This function

$$F(x) = P\{\xi < x\}$$

is called *the distribution function* of the random variable ξ.

We present the properties of the distribution function $F(x)$:

Property 1. $0 \leq F(x) \leq 1$ for the distribution function $F(x)$.

This property follows from the definition of the distribution function and the fact that $0 \leq P(A) \leq 1$ for the probability of an event.

Property 2. $F(x)$ is an increasing function, that is, for any x_1, x_2 satisfying the inequality $x_1 < x_2$, $F(x_1) \leq F(x_2)$.

Clearly, the event $\{\xi < x_2\}$ is equal to the sum of the events $\{\xi < x_2\}$ and $\{x_1 \leq \xi < x_2\}$, and according to the addition theorem for their probabilities,

$$P\{\xi < x_2\} = P\{\xi < x_1\} + P\{x_1 \leq \xi < x_2\}.$$

Now, considering that

$$F(x_2) = P\{\xi < x_2\}, F(x_1) = P\{\xi < x_1\}, P\{x_1 \leq \xi \leq x_2\} \geq 0,$$

we find from the next equality that $P\{x_1 \leq \xi < x_2\} = F(x_2) - F(x_1) \geq 0$, that is, $F(x_1) \leq F(x_2)$.

Result. The probability that a random variable ξ falls in the half-interval $[a, b)$ is $P\{a \leq \xi < b\} = F(b) - F(a)$.

If the distribution function $F(x)$ of a random variable ξ is continuous, the random variable is called continuous.

Property 3. The probability of a continuous random variable taking a given value is zero:

$$P\{\xi = x_1\} = 0.$$

In the above relation, we get $x_2 = x_1 + \Delta x$. Then, from the equation $P\{x_1 \leq \xi < x_1 + \Delta x\} = F(x_1 + \Delta x) - F(x_1)$, it follows that $P\{\xi = x_1\} = 0$ at $\Delta x \to 0$.

Property 4. The probability that a continuous random variable falls into the intervals (a, b), $[a, b]$, $[a, b)$, $(a, b]$ is the same:

$$P\{a < \xi < b\} = P\{a \leq \xi \leq b\} = P\{a \leq \xi < b\} = P\{a < \xi \leq b\}.$$

Property 5. If all possible values of the random variable ξ belong to the interval (a, b), then

$F(x) = 0$ when $x \leq a$,

and $F(x) = 1$ when $x \geq b$.

Suppose $x_1 \leq a$. In this case, the event $\{\xi < x_1\}$ is an impossible event, and

$P\{\xi < x_1\} = 0$, i.e. $F(x) = 0$.

Suppose $x_2 \geq b$. In this case, $\{\xi < x_2\}$ is an inevitable event, and

$P\{\xi < x_2\} = 1$, i.e. $F(x) = 1$.

Result. If the values that the continuous random variable takes are located on the number axis, then

$$F(-\infty) = \lim_{x \to -\infty} F(x) = 0, F(+\infty) = \lim_{x \to +\infty} F(x) = 1.$$

Suppose that the distribution function $F(x)$ of a random variable ξ is a differentiable function.

Definition. The derivative of the function $F(x)$ is called the probability density of a random variable ξ and is denoted as $p(x)$:

$$p(x) = F'(x).$$

Now we give the properties of the probability density of a random variable.

Property 1. For an arbitrary x, $p(x) \geq 0$, and

$$P\{x_1 < \xi < x_2\} = \int_{x_1}^{x_2} p(x)dx.$$

It is known that $F(x)$ is an increasing function. Then $F'(x) \geq 0$, and since $p(x) = F'(x)$ it follows that $p(x) \geq 0$.

Clearly, according to the Newton-Leibnitz formula,

$$\int_{x_1}^{x_2} F'(x)dx = F(x_2) - F(x_1).$$

Meanwhile, according to property 4 of the distribution function,

$$P\{x_1 < \xi < x_2\} = F(x_2) - F(x_1).$$

From the last equalities, it follows that

$$P\{x_1 < \xi < x_2\} = \int_{x_1}^{x_2} p(x)dx.$$

Property 2. For the probability density of a random variable,

$$\int_{-\infty}^{+\infty} p(x)dx = 1.$$

Using the definition of the non-specific integral and the properties of the function $F(x)$, we find:

$$\int_{-\infty}^{+\infty} p(x)dx = \lim_{\substack{v \to -\infty \\ u \to +\infty}} \int_{v}^{u} p(x)dx = \lim_{\substack{v \to -\infty \\ u \to +\infty}} [F(u) - F(v)] = $$
$$= F(+\infty) - F(-\infty) = 1 - 0 = 1.$$

Property 3. The distribution function of a random variable is $F(x)$ with probability density $p(x)$, then

$$F(x) = \int_{-\infty}^{x} p(x)dx.$$

Now we give the definition of the mathematical expectation of a continuous random variable.

Let $p(x)$ be the density function of the random variable ξ.

Definition. The mathematical expectation of a continuous random variable ξ is called the integral (if this integral is an absolute approximation)

$$M\xi = \int_{-\infty}^{\infty} x\, p(x)dx.$$

Example 1. Find the mathematical expectation of a normally distributed random variable with parameters $\xi \sim (a, \sigma)$.

Solution: According to the formula

$$M\xi = \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-a)^2}{2\sigma^2}} dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-a) e^{-\frac{(x-a)^2}{2\sigma^2}} dx + \frac{a}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-a)}{2\sigma^2}} dx$$

substituting $z=\frac{x-a}{\sigma}$ and considering that the integral of an odd function over a symmetric interval with respect to zero is zero, we get

$$M\xi=\frac{a}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{z^2}{2}}dz=\frac{a}{\sqrt{2\pi}}\cdot\sqrt{2\pi}=a.$$

So, the mathematical expectation of the random variable $\xi\sim(a,\sigma)$ is equal to the parameter a.

Example 2. Find the mathematical expectation of a uniformly distributed random variable ξ on the interval $[a,b]$.

Solution: We know that,

$$f(x)=\begin{cases}0, & if\ x\le a,\\ \frac{1}{b-a}, & if\ \ a<x\le b,\\ 0, & if\ x>b.\end{cases}$$

According to the formula

$$M\xi=\int_a^b x\cdot\frac{1}{b-a}dx=\frac{b+a}{2}.$$

If the random variable ξ is given by the distribution function $F(x)$, its mathematical expectation

$$M\xi=\int_{-\infty}^{\infty}xdF\,(x)$$

is determined by the equation.

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