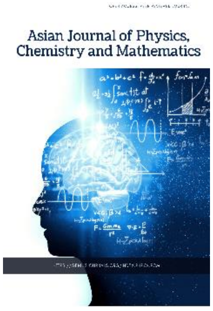

A Study of Approximated Solutions of the Optimal-Control Problems Governed by Initial Value Problems using Basic Functions

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ABSTRACT

The problem of determining a control law for a given system is addressed by optimal control(OC). Using fundamental functions, a computational method for addressing optimum control problems (OCPs). This technique relies on the use of fundamental functions to approximate the problem's solution. In fact, some of the coefficients in the optimal formula are determined by initial or boundary conditions(BCs), while others are approximated using the minimizing calculation. On the other hand, an OCP transforms into a numerically solvable optimization problem. The approximate value of the performance index is calculated using this numerical method. Approximate results for several test conditions are provided to demonstrate the method's efficiency and applicability.

Keywords: Optimal-control(OC); Duffing-oscillator; Approximation, ODEs; BVPs; Basic-functions.

1 Introduction

In most design problems, a control design problem should deal with the "best" or "optimal" design. The concept of an optimal controller is one that meets all of the design criteria while minimizing the performance index. For a continuous-time linear dynamical system(DS), the elements of the optimum control problem can really be described [1]. An approach for modeling and investigating events that modify through time and space is DS theory. OCPs play a major role in a variety of fields, like economics, engineering, and finance. The subject of control theory is a subfield of the theory of optimization that deals with lowering costs or maximizing payouts [2]. Finding an optimal control of open-loop in which denoted by $u^*(\tau)$ or even the feedback of optimal control that denoted by $u^*(\tau, x)$ which satisfies the DS which optimizes in some sense performance index is an apparent objective. Direct and indirect approaches are the two most common ways for addressing OCPs today. This problem is converted into another problem via an indirect way. The original OC problem can transformed into a boundary value problem(BVP), which can be solved numerically or analytically utilizing classical or modern methods for solving differential equations (DEs)[2-3]. Because the analytical solutions for OCPs are not always available, finding numerical solutions for handling OCPs is the suitable and most reasonable way to deal with OCPs, and it has proven to be an appealing study for mathematical scientists.

Numerous mathematical computational approaches and efficient methods have been used to solve the OCPs in recent years [2-4]. On the other hand, the indirect methods provide the optimal solutions by directly minimization of the performance index subject to limitations. In reality, the OCPs can be converted into a problem optimization. The three different ways: control parameterizations, control-state parameterizations, and state parameterizations are used in the technique of parameterizations can be used to implement the direct techniques. The control-state parameterizations and control parameterizations have been used extensively to solve general OCPs. [5-6] proposed numerical methods for solving unconstrained and constrained OCPs, and then, they extended their ideas to nonlinear OCPs with control inequality constraints, terminal state constraints, and simple state variable limitations. [7] described a method for solving nonlinear constrained OCPs using numerical methods. Using a new Chebyshev spectral approach, [8] has proposed a numerical method for solving OCPs and the controlled-Duffing-oscillator(CDO). [9] showed how to solve the CDO using a spectral technique. [10] presented a numerical algorithm for solving the CDO, in which the control and state variables have been approximated by Chebyshev series, and [11] presented a method for solving OCPs and the CDO, however, the algorithm of the method of solution is based on state parametrization, where the variable state can be considered as a linear combination of Chebyshev polynomials with unknown coefficients, and later, extended state parametrization. This paper is designed as the following sections, the first one is the introduction of the article . Preliminaries is introduced in Section 2. The proposed estimated approach is presented in sections three and four which contains the implementations. Finally, there is a conclusion to the paper.

2 Preliminaries

In this section, we have introduced the background which related with the problem of this
study.

2.1 The Optimal Control System

In general, we define an OC system as a state variable $x^*(\tau)$ that optimizes the performance index in some way.

$$\text{minimize } J\big(t_0, x(\tau_0); u(\tau)\big) =$$

$$\text{minimize } \int_{\tau_0}^{t_1} \mathrm{L}(\tau, x(\tau), \dot{x}(\tau), \ddot{x}(\tau), \dddot{x}(\tau), \ldots, \dot{x}^{(n)}(\tau), u(\tau))d\tau, \qquad (1)$$

subject to the following ordinary-differential equation(ODE)

In this section, we deal with the OC with the problem of finding a control law for a given system

$$f\left(\tau, x(\tau), \dot{x}(\tau), \ddot{x}(\tau), \dddot{x}(\tau), \ddddot{x}(\tau), \ldots, \dot{x}^{(n)}(\tau), u(\tau)\right) = 0,, \tau \in I \qquad (2)$$

Where, f is a continuously differentiable real function and $f: I \times E \times E \times E \times U \to \Re^n$. Also, the time interval $I = [\tau_0, \tau_1]$ and the control , $u(\tau): I \to \Re^n$ and $x(\tau): I \to \Re^m$ is used for the state variable, with the following BCs

$$(3)\, x^{(k_1)}(\tau_0) = x_{k_1}; x^{(k_2)}(\tau_1) = x_{k_2},$$

For $k_1 = 1,2, \ldots, n_1$ and $k_2 = 1,2, \ldots, n_2$ .where , $n_1 + n_2 = n$ and $x(\tau_0)$, $x(\tau_1)$ are the initial and ending states of in $\Re^n$; resp., That might be fixed or unrestricted-control $u^*(\tau)$ is *called an OC and the state variable* $x^*(\tau)$. an optimal trajectory(OT) . Also, $L: I \times E^n \times U \to \Re^n$ in all three parameters, is considered to be a continuously differentiable function. A Lagrange problem is a problem of optimization with a performance index like in Equation (2). The Bolza-Mayer problems are two other optimization problems that are similar. An energy or fuel function can really be significant in OCPs.

2.1.1 Optimal Control with Second-Order ODEs

In this subsection, we deal firstly with the OC of the problem of determine a control law for a given system of second-order ODEs .

$$f\big(t, x(\tau), \dot{x}(\tau), \ddot{x}(\tau), u(\tau)\big) = 0, \qquad t \in I \qquad (4)$$

where, $f: I \times E \times E \times E \times U \to \Re^n$ is a real differentiable continuously function. Also, $I =$

$[\tau_0, \tau_1]$ is the time interval and the function $u(\tau): I \to \Re^n$ for the control and $x(\tau): I \to \Re^m$ for the state variable is used. As the control function is changed, the solution to the DE can be changed. The objective is to find at least a piecewise continuous control u^* and the associated state variable $x^*(\tau)$ that optimizes in some sense the performance index

$$minimize\ J(\tau, x(\tau); u(\tau)) = minimize \int_{t_0}^{t_1} L(t, x(\tau), \dot{x}(\tau), \ddot{x}(\tau), u(\tau))\, d\tau, \quad (5)$$

subject to the conditions in Equation (1) with the following BCs:

$x(\tau_0) = x_0$ and x_1 $x(\tau_1) = x_1$ where, x_0 and x_1 are the initial and final state in $\Re^n$. resp. Which are fixed or free points. The control u^* is called an OC and state variable x^*. an OT. However, in all three arguments, $L: I \times E \times \to \Re^n$ is assumed to be a differentiable continuously function. A Lagrange problem is optimization with a performance index as shown in Equation (2). The Bolza and Mayer problems [2] are two other optimization problems that are similar. L can be an energy or fuel function in OCPs, as shown below [28]:

$$L(\tau, x(\tau), u(\tau)) = \frac{1}{2}(x^2(\tau) + u^2(\tau)) \quad (6)$$

$$L(\tau, x(\tau), u(\tau)) = |x(\tau)| + |u(\tau)| \quad (7)$$

J can be a multi-purpose or multi-objective functional in general, such as minimizing fuel dissipation or maximizing utility.

3. The Proposed Approximated Method

[3] and [12] have been used the approach of state parameterization to convert it to a nonlinear optimization problem and distinguish polynomial coefficients of degree at most n in the following form

$$x(t) = \sum_{i=0}^{n} a_i \Omega_i(\tau) \quad (8)$$

For the optimal solution.

In this paper, using a base of approximation

$$\Omega = \{\Omega_0(\tau), \Omega_1(\tau), \Omega_2(\tau), \dots, \Omega_n(\tau)\}, \quad (9)$$

The following is an approximate of x(t).

$$x(t) = c_0\Omega_0(\tau) + c_1\Omega_1(\tau) + c_2\Omega_2(\tau) + \cdots + c_n\Omega_n(\tau), \quad (10)$$

If we use the DE in Equation (2) and the BCs in Equation (3), we get an approximation (2).

$$x(t) = c_{i0}\Omega_0(\tau) + c_{i1}\Omega_1(\tau) + c_{i2}\Omega_2(\tau) + \cdots + c_{im}\Omega_n(\tau) + c_{j1}^*\Omega_0(\tau) + c_{j2}^*\Omega_1(\tau) + c_{j3}^*\Omega_2(\tau) + \cdots + c_{ji}^*\Omega_{n-m}(\tau) \quad (11)$$

where $m < n$. To achieve the optimal value, substitute Equation (11) in the minimization problem in Equation (5). As a result, the problem's approximation has been assessed..

3.1 Algorithm of Proposed Method

The least square method for solving DEs [28-29] is one of the most-powerful approximated methods for solving DEs. This type of the methods is based on the approximating of the solution of DE by series of approximation by using a complete sequence of functions which means a combination of sequence of linearly independent functions that have no non-zero function which perpendicular to any function of this sequence of functions. To approximate the solution of OC problem in Equation (1) subject to the ODEs in Equation (2) with the BCs in Equations (3) using the proposed method can applied according to the following steps:

3.1.1 Algorithm

- Consider the approximation's base, as shown in Equation (9).
- Approximate the problem's solution using Equation (10).
- Satisfy the boundary conditions of the research problem to identify some of the approximation parameters, and the approximation problem in Equation ten is simplified to the problem of evaluating the coefficients as in Equation eleven (11).
- As a result, the OC problem in Equation (1) is translated to evaluate the other parameters of the approximation of x(t) by minimizing the functional in Equation (1) to discover the other coefficients a_i for i = 0,1,2,..., n of the approximation by minimizing the functional in Equation (1)

- The least-square method transforms the optimal issue of coefficient evaluation..

$$minimize\, J(\tau, x(\tau), u(\tau)) = minimize\, J(a_0, a_1, \ldots, a_n)$$

Equation (2), in approximation form, gives the best approximation to the optimal problem solution.

- However, in order to determine the coefficients while limiting the functional error $J(t;\, x(t);\, u(t))$, the absolute error(E(x)) should be kept to a minimum, where

$$E(x(\tau)) = \left| J(\tau, x(\tau), u(\tau)) - J(a_0, a_1, \ldots, a_n) \right|$$

4 Implementations

Some examples have been implemented in this part to introduce and test the approximated method.

Example 4.1

Take into consideration the following: OC problem that in some way increases the performance index

$$minimize\, J(\tau_0, x(\tau_0); u(\tau)) = minimize \int_{t_0}^{t_1} u^2(\tau)d\tau \qquad (12)$$

subject to the ODE

$$u(\tau) = \dot{x}(\tau) + \ddot{x}(\tau), \qquad (13)$$

with the boundary conditions(BCs)

$$\dot{x}(0) = x(0) = 0, \text{and}\, , x(2) = \dot{x}(2) - 20 = 28 \qquad (14)$$

Using the approximation base $\Omega(\tau) = \{1, \tau, \tau^2, \tau^3, \tau^4\}$, we have the approximation of $x(t)$ as

$$x(\tau) = c_0 + c_1\tau + c_2\tau^2 + c_3\tau^3 + c_4\tau^4. \qquad (15)$$

If we use the BCs in Equation (14), we obtain the following approximation

$$x(\tau) = \tau^2(-3 + 4c_4 + (5 - 4c_4)\tau + c_4\tau^2). \qquad (16)$$

To get the best value of $c_4 = 0.54$, substitute Equation (16) into the minimization problem in Equation (12). As a result, the problem's approximation is as follows:

$$x(\tau) = \tau^2(-0.84 + 2.84\tau + 0.54\tau^2), \qquad (17)$$

Example 4.2

Take into consideration the following: OC problem that in some way increases the performance index

$$minimize\, J(\tau_0, x(\tau_0)\, ;\, u(\tau)) = minimize \int_{t_0}^{t_1} (\ddot{x}(\tau) + u(\tau))d\tau\, , \qquad (18)$$

subject to the ODE

$$u(\tau) = \dot{x}(\tau) + \ddot{x}(\tau), \qquad (19)$$

with the BCs

$$\dot{x}(0) = x(0) = 0, \text{and}\, , x(2) = \dot{x}(2) - 20 = 28 \qquad (20)$$

Using the basis of approximation $\Omega = \{1, \tau, \tau^2, \tau^3, \tau^4\}$, we have the approximation of $x(t)$ as

$$x(\tau) = c_0 + c_1\tau + c_2\tau^2 + c_3\tau^3 + c_4\tau^4. \qquad (21)$$

If we use the BCs in Equation (20), we obtain the following approximation

$$x(\tau) = \tau^2(-3 + 4c_4 + (5 - 4c_4)\tau + c_4\tau^2). \qquad (22)$$

Substitute Equation (22) in minimizing problem in Equation (18) to obtain the optimal-value of $c_5 = 1$, As a result, the problem's approximation is as follows:

$$x(\tau) = \tau^2(1 + \tau + \tau^2). \qquad (23)$$

Exact solution is $x(\tau) = \tau^2 + \tau^3 + \tau^4$.

Example 4.3

Take into account the following: OC problem that improves the performance index in some manner

$$minimize\, J(\tau_0, x(\tau_0); u(\tau)) = \mathrm{minimize} \int_{t_0}^{t_1} \mathrm{u}^2(\tau)\mathrm{d}\tau\, , \qquad (24)$$

subject to the ODE

$$u(\tau) = \dot{x}(\tau) + \ddot{x}(\tau)\, , \qquad (25)$$

with the BCs

$$\dot{x}(0) = x(0) = 0\, , \quad \text{and} \quad x(1) = 0.5\, \dot{x}(1) = 0.5 \qquad (26)$$

Using the basis of approximation $\Omega = \{1, \tau, \tau^2, \tau^3, \tau^4\}$, we have the approximation of $x(t)$ as

$$x(\tau) = c_0 + c_1\tau + c_2\tau^2 + c_3\tau^3 + c_4\tau^4. \quad (27)$$

We get the following approximation if we use the BCs in Equation (26)

$$x(\tau) = \tau^2(4c_4 + (5 - 3c_4)\tau + c_4\tau^2). \quad (28)$$

To determine the optimal value of c_4, substitute Equation (28) in the minimization problem in Equation (24) As a result, the problem's approximation has been written.

Example 4.4

Take into account the following: OC problem that improves the performance index in some manner

$$minimize\ J\big(\tau_0, x(\tau_0);\ u(\tau)\big) = minimize \int_{t_0}^{t_1} L(\tau, x(\tau), \dot{x}(\tau), \ddot{x}(\tau), \dddot{x}(\tau), \ddddot{x}(\tau), \dot{\ddddot{x}}(\tau), u(\tau))d\tau = minimize \int_{t_0}^{t_1} u^2(\tau)d\tau, \quad (31)$$

subject to the ODE

$$u(\tau) = \dot{x}(\tau) + \ddot{x}(\tau) + \dddot{x}(\tau) + \ddddot{x}(\tau) + \dot{\ddddot{x}}(\tau) \quad (32)$$

with the BCs

$$x(0) = \dot{x}(0) = \ddot{x}(0) = \dddot{x}(0) = 0 \text{ and } \quad \dot{x}(1) = 1. \quad (33)$$

Using the approximation base $\Omega = \{1, \tau, \tau^2, \tau^3, \tau^4, \tau^5, \tau^6\}$ and $x(\tau)$ is a close approximation as

$$x(\tau) = c_0 + c_1\tau + c_2\tau^2 + c_3\tau^3 + c_4\tau^4 + c_5\tau^5 + c_6\tau^6. \quad (34)$$

Using the BCs in Equation (33), we could get the approximation of x(t) Substitute Equation (34) in the minimizing problem in Equation (31) to get the optimal value of c_6 showing that the problem's approximation has been evaluated.

Example 4.5 The Controlled Linear Oscillator

We'll look at the OC of a linear oscillator that's controlled by the DE.

$$u(\tau) = \ddot{x}(\tau) + w^2x(\tau), \tau \in [-T, 0], \quad (37)$$

T is stated, along with the boundary conditions.

$$x(-T) = x_0\,,\ \dot{x}(-T) = \dot{x}_0\,,\ x(0) = \dot{x}(0) = 0\,. \quad (38)$$

It is desirable to maintain control over the state of this plant in order for the performance index to improve.

$$J = \frac{1}{2}\int_{-T}^{0} u^2(\tau)d\tau, \quad (39)$$

For all acceptable control functions u is minimized (t). When Pontryagins' maximal principle method is applied to this OC problem, the exact analytical answer is as follows:

Using the basis of approximation: $\Omega(\tau) = \{1, \tau, \tau^2, \tau^3, \tau^4\}$, We have a rough estimate o $x(t)$ as

$$x(\tau) = c_0 + c_1\tau + c_2\tau^2 + c_3\tau^3 + c_4\tau^4. \quad (40)$$

We can get the approximation of x(t) if we apply the BCs in Equation (38). Substitute Equation (40) in the minimizing problem in Equation (39) to get the optimal value of c_4, and so the problem's approximation has been evaluated.

5 Conclusion

This paper gives a numerical analysis using the approximated technique to solve OCPs and specific kinds of OCPs. State parameterizations are used in the approximated solutions technique. Using a small number of unknown parameters, it generates an estimated or accurate answer. This strategy is helpful for the classes of OCPs, we focus. In reality, the suggested direct technique has the capability of calculating the state variables as functions of time and continuous control. In addition, the value of the performance index is directly obtained. The numerical method described here is a straightforward way to alter and assess an OC that can be simply applied to various challenges. The quick convergence of this approximation method is one of its most notable advantages. The approximated outcomes of the sample instances show that the suggested approach is a powerful method, which is a critical consideration when selecting a method for engineering applications.

References

1. Bhaya and E. Kaszkurewicz, "Control perspectives on numerical algorithms and matrix problems. SIAM, 2006.

2. Kafash, A. Delavarkhalafi, and S. Karbassi, "A numerical approach for solving opti-mal control problems using the boubaker polynomials expansion scheme," J. Interpolat.Approx. Sci. Comput, vol. 3, pp. 1-18, 2014.
3. H. H. Mehne and A. H. Borzabadi, "A numerical method for solving optimal control problems using state parametrization," Numerical Algorithms, vol. 42, no. 2, pp. 165-169, 2006.
4. S. A. Yousefi, M. Dehghan, and A. Lotfi, "Finding the optimal control of linear systems via he's variational iteration method," International Journal of Computer Mathematics,vol. 87, no. 5, pp. 1042-1050, 2010.
5. H. M. Jaddu, "Numerical methods for solving optimal control problems using cheby-shev polynomials," Thesis, School of Information Science, Japan Advanced Institute of Science and Technology, p. 122, 1998.
6. H. Jaddu, "Direct solution of nonlinear optimal control problems using quasilinearization and chebyshev polynomials," Journal of the Franklin Institute, vol. 339, no. 4-5, pp.479-498, 2002.
7. J. Vlassenbroeck, "A chebyshev polynomial method for optimal control with state con-straints," Automatica, vol. 24, no. 4, pp. 499-506, 1988.
8. T. El-Gindy, H. El-Hawary, M. Salim, and M. El-Kady, "A chebyshev approximation for solving optimal control problems," Computers & Mathematics with Applications,vol. 29, no. 6, pp. 35-45, 1995.
9. G. N. Elnagar and M. Razzaghi, "A chebyshev spectral method for the solution of non-linear optimal control problems," Applied Mathematical Modelling, vol. 21, no. 5, pp.255-260, 1997.
10. M. El-Kady and E. M. Elbarbary, "A chebyshev expansion method for solving nonlinear optimal control problems," Applied Mathematics and Computation, vol. 129, no. 2-3,pp. 171-182, 2002.
11. Kafash, A. Delavarkhalafi, and S.-M. Karbassi, "Application of chebyshev polynomials to derive efficient algorithms for the solution of optimal control problems," Scientia Iranica, vol. 19, no. 3, pp. 795-805, 2012.
12. P. Frick and D. Stech, "Solution of optimal control problems on a parallel machine using the epsilon method," Optimal Control Applications and Methods, vol. 16, no. 1,pp. 1-17, 1995.