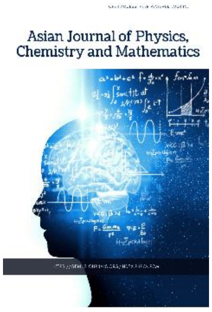

On the properties of generalized solutions of a nonlinear Cauchy problem for a nonlinear parabolic system with absorption or source

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ABSTRACT

This paper deals with the existence and nonexistence of global positive solutions of quasilinear reaction diffusion systems with nonlinear boundary conditions. The properties of generalized solutions of one nonlinear self-similar system of equations are investigated. A condition for global solvability of Fujita type is obtained, generalizing the result of [1]. The asymptotic behavior of the eigenfunction of a nonlinear two-component medium is established. Based on them, numerical calculations were carried out.

Keywords: quasilinear, parabolic equations, existence, nonexistence, self-similar solution, automodel equation, asymptotic behavior.

1. Introduction

This paper is devoted to the study of the global solution of the Cauchy problem for a system of quasilinear parabolic equations with a volume source or absorption:

$$\frac{\partial u}{\partial t}=\nabla\left(|x|^n u^{\sigma_1}\frac{\partial u}{\partial x}\right)+\varepsilon v^p, t>0, x\in R^N$$
$$\frac{\partial v}{\partial t}=\nabla\left(|x|^n v^{\sigma_1}\nabla v\right)+\varepsilon u^q$$
(1)

Where $\varepsilon=\pm 1,$ $\sigma_i, p, q\,(i=1,2)$ are positive real numbers.

Equation (1) describes many physical processes: the conductance processes in a two–component nonlinear medium with the sources or absorption; the filtering in nonlinear two – phase liquid and gas subject to the laws of polytropy and so on. Members εu^p, εv^q accord to having sources or absorption whose powers are equal to εu^p, εv^q.

The functions u and v may be discussed in the same way as the temperatures of two interacting with each other components of some fuel mixture [1].

The properties of unbounded solutions of Cauchy problem for system (1) under the conditions are the following:

$$u(0,x)=u_0(x)\ge 0,\quad v(0,x)=v_0(x)\ge 0,\quad \in R^N$$
(2)

System (1) for $\sigma_i=0,\ (i=1,2)$ was considered in [1] as well as conditions for the global solvability of problem (1), (2) with $\gamma<N/2$ where $\gamma=\max\left(\frac{q+1}{pq-1},\frac{p+1}{pq-1}\right)$

The following restrictions are imposed on the functions $u(x,t)$, $v(x,t)$:

$$u(0,t)=0,\ u(x,t)\to 0,\ u^{\sigma_1}\frac{\partial u}{\partial x}\to 0,\ t\ge 0$$
$$v(0,t)=0,\ v(x,t)\to 0,\ v^{\sigma_1}\frac{\partial v}{\partial x}\to 0,\ t\ge 0$$
(3)

And $u_0(x)\ne 0,\ v_0(x)\ne 0$ is a non-trivial, non-negative, bounded and sufficiently smooth functions. In this paper, in the case $\varepsilon = +1$, in particular, the following condition for the global solvability of problem (1), (2) is obtained on the basis of the analysis of self-similar solutions using the method of nonlinear prlitting [3].

$$\max\left(\frac{q+1}{pq-1}, \frac{p+1}{pq-1}\right) < \frac{N}{2+\sigma_i N} \quad (i=1,2)$$

which generalizes the result of [1].

Problem (1), (2) also models other processes, for example, the process of heat propagation, liquid and gas filtration, diffusion in a nonlinear medium with a coefficient with a double nonlinearity.

Problem (1), (2) has been studied by many authors [14-16] . In particular, the original properties of solutions to this problem were established in [14], the paper [15], devoted to the study of solutions with destruction, and the asymptotic behavior of the self-similar solution was studied in [16]. A large number of works are devoted to the study of various qualitative properties of solutions to problem (1), (2) for the case of one equation (see [17] and the literature there, [19-28]).

Other than that at first Fujita considered the following initial value problem [28]:

$$u_t = \Delta u + u^p \qquad (x,t) \in \mathbb{R}^N \times (0,T)$$

$$u(x,0) = u_0(x) \ge 0 \quad x \in \mathbb{R}^N \qquad (4)$$

(Δ denotes the N-dimensional Laplace operator)

$u_0(x)$ is a bounded nonnegative continuous function, $u_0(x) \ne 0$ where p > 1 and $T \le \infty$ (the length of the existence interval).

He proved the following result concerning nonnegative solutions:

If $1 < p < 1 + \frac{2}{N}$ then all nonnegative solutions blow up in finite time with any nontrivial initial values.

If $p > 1 + \frac{2}{N}$ then global, nontrivial nonnegative solutions exist with sufficiently small initial values.

The case $p = 1 + \frac{2}{N}$ belongs to case (a) but this was proved later. The number $p = 1 + \frac{2}{N}$ is called the critical blow up exponent. Case (a) is called the global nonexistence case (the blow up case), while case (b) is called the global existence case ($T = \infty$). Whenever case (a) happens in a single equation of the initial value problem, there exists a finite number T ($0 < T < \infty$) such that $\lim_{t \to T^-} \sup\left(\sup_{x \in \mathbb{R}^N} u(x,t)\right) = \infty$

M. Escobedo and M. A. Herrero [5] studied the initial value problem for the system (5) :

$$\begin{cases} u_t = \Delta u + u^{p_2} \\ v_t = \Delta v + v^{p_2} \end{cases}$$

$$(x,t) \in \mathbb{R}^N \times (0,T) \qquad (5)$$

$$u(x,0) = u_0(x) \ge 0 \quad v(x,0) = v_0(x) \ge 0, \quad x \in \mathbb{R}^N$$

$p_1 > 0, \quad p_2 > 0$

H. A. Levine [29] studied nonnegative solutions of the initial boundary value problem for the system:

$$u_t = \Delta u + u^{p_2}, \quad v_t = \Delta v + v^{p_2}$$

$$u = v = 0 \qquad (x,t) \in D \times (0,T) \qquad (6)$$

$$u(x,0) = u_0(x) \ge 0, \quad v(x,0) = v_0(x) \ge 0 \qquad x \in D$$

, $p_1, p_2 \ge 1$ with $p_1, p_2 > 1$

where D is a cone or the exterior of a bounded domain.

A cone with vertex at the origin is a set of the form

$$\left\{ x \in \mathbb{R}^N \,||x| = r,\ x = \left(r, \underset{\sim}{\theta}\right),\ \underset{\sim}{\theta} \in \Omega \subset S^{N-1} \right\}$$

where Ω is a region on the unit sphere $S^{N-1} = \left\{x \in \mathbb{R}^N \,||x| = 1\right\}$. Then D is a cone with vertex at the origin if and only if we can write $D = (0,\infty) \times \Omega$.

In this work the main attention is given to argumentation of the algorithm of nonlinear splitting for system (1), to constructing automodel solutions, to the existence the solution of the system of automodel equations and their asymptotic representation [8].

We also study the properties of solutions with a finite propagation velocity, perturbation, asymptotic behavior of with the solution to compact supports and in the case of fast diffusion [6].

Such a widespread use of quasilinear parabolic equations is explained primarily by the fact that they are derived from fundamental conservation laws (energy, mass, the number of particles, etc.). Therefore, a situation is possible when two physical processes that at first glance have nothing in common (for example, thermal conductivity in semiconductors and the process of magnetic field propagation in a

medium with finite conductivity) are described by the same nonlinear diffusion equation, only with different numerical parameters [5].

The asymptotic properties of self-similar solutions of the system of equations (1)

Let us analyze in the area of $Q_T=[0,T]\in R^N$ the system of equations (1). Equation (1) with u=0, v=0 degenerates to the first-order equation. In the domain of degeneration of the problem (1),(2) may not have the classic solution. That is why we discuss the generalized solution having the property of continuity of solution. The Cauchy problem and boundary value problems for the system of equations (1) in one-dimensional and multidimensional cases with different initial and boundary conditions were analyzed by many authors [1-5].

Let us construct a self-similar system of equations for (1) based on the method of nonlinear splitting [3]. To do this, we first solve the system of ordinary differential equations (without the main part of system (1)).

$$\frac{d\bar{u}}{dt}=-\bar{v}^{p},\qquad \frac{d\bar{v}}{dt}=-\bar{u}^{q}\quad \text{т.е.}\quad \bar{u}(t)=A(T+t)^{\alpha};\ \bar{v}(t)=B(T+t)^{\beta}\qquad (7)$$

$$\alpha=-\frac{p+1}{pq-1},\ \beta=-\frac{q+1}{pq-1}$$

where T>0 is constant (the time of existence of the solution at ε = -1).

Then we look for solutions to the problem in the form

$$u(t,x)=\bar{u}(t)w(\tau(t),x),\quad v(t,x)=\bar{v}(t)z(\tau(t),x)$$

$$w(\tau(t,x)=f(\xi),\qquad z(\tau(t),x)=\varphi(\xi),\ \xi=\frac{|\eta|}{(\tau(t))^{1/2}};\qquad (8)$$

$$\eta=2/(2-n)]|x|^{2/(2-n)}\ (i=1,2),$$

where $\tau(t)=\tau_1(t)=\tau_2(t)\ \ \tau_1(t)=\int\bar{u}^{\sigma_1}(t)dt,\qquad \tau_2(t)=\int\bar{v}^{\sigma_2}(t)dt$

Then for $\sigma_1\alpha=\sigma_2\beta$ we obtain the system of self-similar equations

$$\xi^{1-s}\frac{d}{d\xi}\left[\xi^{s-1}f^{\sigma_1}\frac{df}{d\xi}\right]+\frac{1}{2}\xi\frac{df}{d\xi}+\frac{\alpha}{1-\sigma_1\alpha}\left[f+\varepsilon\varphi^{p}\right]=0$$

$$\xi^{1-s}\frac{d}{d\xi}\left[\xi^{s-1}\varphi^{\sigma_2}\frac{d\varphi}{d\xi}\right]+\frac{1}{2}\xi\frac{d\varphi}{d\xi}+\frac{\beta}{1-\sigma_2\beta}\left[\varphi+\varepsilon f^{q}\right]=0\qquad (9)$$

$$\xi=\eta(x)/\tau^{12}\ s=2N/(2-n)$$

In accordance with the formulation of the original problem, we will be interested in non-trivial non-negative solutions of the system of equations (6) satisfying the following condition:

$$f'(0)=0,\quad f(\infty)=0,\quad \varphi'(0)=0,\quad \varphi(\infty)=0,\qquad (10)$$

Following [2] this solution of problem (9), (10) is an eigenfunction of a nonlinear two-component medium.

Let us study the asymptotics of solutions to problem (9), (10).

Below we propose a method for studying the asymptotics of the solution to problem (6), (7), based on the transformation of the original system (9)

Case $\sigma_i > 0$, $i = 1,2$, $\varepsilon = -1$. Note that the function:

$$\bar{f}(\xi) = (a - b\xi^2)_+^{\frac{1}{\sigma_1}}$$
$$\bar{\varphi}(\xi) = (a - b\xi^2)_+^{\frac{1}{\sigma_2}},\ (b)_+ = \max(0,b) \tag{11}$$

where

$$a = \frac{\alpha\sigma_1}{\alpha\sigma_1 - N2b(1-\alpha\sigma_1)} = \frac{\beta\sigma_2}{\beta\sigma_2 - N2b(1-\beta\sigma_2)}$$
$$b = \frac{\sigma_1}{(1-\alpha\sigma_1)(2N\sigma_1+4)} = \frac{\sigma_2}{(1-\beta\sigma_2)(2N\sigma_2+4)}$$

is the exact solution to system (1).

Let us show that condition (10) is the only asymptotic behavior of solutions to problem (8), (9) for $\xi = (a/b)^{1/2}$ to be exact .

Theorem 1. Let $p = \frac{\sigma_2(1+\sigma_1)}{\sigma_1}$ $q = \frac{\sigma_1(1+\sigma_2)}{\sigma_2}$, then any solution of system (1) with compact support for has an asymptotic representation.

$$f(\xi) = \bar{f}(\xi)(1+o(1))$$
$$\varphi(\xi) = \bar{\varphi}(\xi)(1+o(1))$$

Proof: We are looking for solutions to the system of equations (2) in the following form

$$f(\xi) = \bar{f}(\xi)w(\eta)$$
$$\varphi(\xi) = \bar{\varphi}(\xi)z(\eta) \tag{12}$$

Where

$$\bar{f}(\xi) = (a - b\xi^2)^{\frac{1}{\sigma_1}}, \qquad \bar{\varphi}(\xi) = (a - b\xi^2)^{\frac{1}{\sigma_2}}, \qquad \eta = -\ln(a - b\xi^2) \tag{13}$$

in addition $\eta \to \infty$ for $\xi = (a/b)^{1/2}$, which allows us to study the asymptotic stability of the solution of problems (2), (4) for $\eta \to \infty$

After substituting (12) in to (6) for $w(\eta)$ and $z(\eta)$, we obtain the following system of equations:

$$\frac{d}{d\eta}\left(w^{\sigma_1}\left(w' - \frac{1}{\sigma_1}w\right)\right) + \frac{1}{\sigma_1}w^{\sigma_1}\left(w' - \frac{1}{\sigma_1}w\right) + \frac{1}{2b}\cdot\phi_1(\eta)w^{\sigma_1}\left(w' - \frac{1}{\sigma_1}w\right)$$
$$-\frac{1}{4b}\left(w' - \frac{1}{\sigma_1}w\right) + \frac{\alpha}{1+\alpha\sigma_1}\cdot\frac{1}{4b^2}\phi_1(\eta)w - \frac{\alpha}{1+\alpha\sigma_1}\cdot\frac{1}{4b^2}\phi_2(\eta)z^p = 0$$

(14)

$$\frac{d}{d\eta}\left(z^{\sigma_2}\left(z' - \frac{1}{\sigma_2}z\right)\right) + \frac{1}{\sigma_2}z^{\sigma_2}\left(z' - \frac{1}{\sigma_2}z\right) + \frac{1}{2b}\cdot\phi_1(\eta)z^{\sigma_2}\left(z' - \frac{1}{\sigma_2}z\right)$$
$$-\frac{1}{4b}\left(z' - \frac{1}{\sigma_2}z\right) + \frac{\beta}{1+\beta\sigma_2}\cdot\frac{1}{4b^2}\phi_1(\eta)z - \frac{\beta}{1+\beta\sigma_2}\cdot\frac{1}{4b^2}\phi_2(\eta)w^q = 0$$

Here $\phi_2(\eta)=\dfrac{e^{-\eta\frac{p}{\sigma_2}}}{a-e^{-\eta}}$, $\phi_1(\eta)=\dfrac{e^{-\eta}}{a-e^{-\eta}}$, where η was defined above.

The study of the solution of the last equation is equivalent to the study of those solutions to the system of equations (5), each of which on some interval $[\eta_0,+\infty)$ satisfies the inequality:

$$w(\eta)>0,\qquad w'(\eta)-\frac{1}{\sigma_1}w(\eta)\neq 0,\qquad z(\eta)>0,\qquad z'(\eta)-\frac{1}{\sigma_2}z(\eta)\neq 0$$

Let us show, first of all, that the solution of the system of equations (6) has a finite limit w_0 and z_0 for $\eta\to\infty$. For this, we introduce the following notation

$$v_1(\eta)=w^{\sigma_1}\left(w'-\frac{1}{\sigma_1}w\right)\qquad v_2(\eta)=z^{\sigma_2}\left(z'-\frac{1}{\sigma_1}z\right)$$

Then equation (6), with allowance for (9), has the form

$$\begin{aligned}
v_1'&=-\frac{1}{\sigma_1}v_1-\frac{1}{2b}\cdot\phi_1(\eta)v_1+\frac{1}{4b}w^{-\sigma}v_1-\frac{\alpha}{1+\alpha\sigma_1}\cdot\frac{1}{4b^2}\phi_1(\eta)w+\frac{\alpha}{1+\alpha\sigma_1}\cdot\frac{1}{4b^2}\phi_2(\eta)z^p\\
v_2'&=-\frac{1}{\sigma_2}v_2-\frac{1}{2b}\cdot\phi_1(\eta)v_2+\frac{1}{4b}z^{-\sigma}v_2-\frac{\beta}{1+\beta\sigma_2}\cdot\frac{1}{4b^2}\phi_1(\eta)z+\frac{\beta}{1+\beta\sigma_2}\cdot\frac{1}{4b^2}\phi_2(\eta)w^q
\end{aligned}\tag{15}$$

Consider the helper function

$$\begin{aligned}
\theta_1(\lambda_1,\eta)&=-\frac{1}{\sigma_1}\lambda_1-\frac{1}{2b}\cdot\phi_1(\eta)\lambda_1+\frac{1}{4b}w^{-\sigma_1}\lambda_1-\frac{\alpha}{1+\alpha\sigma_1}\cdot\frac{1}{4b^2}\phi_1(\eta)w+\frac{\alpha}{1+\alpha\sigma_1}\cdot\frac{1}{4b^2}\phi_2(\eta)z^p\\
\theta_2(\lambda_2,\eta)&=-\frac{1}{\sigma_2}\lambda_2-\frac{1}{2b}\cdot\phi_1(\eta)\lambda_2+\frac{1}{4b}z^{-\sigma_2}\lambda_2-\frac{\beta}{1+\beta\sigma_2}\cdot\frac{1}{4b^2}\phi_1(\eta)z+\frac{\beta}{1+\beta\sigma_2}\cdot\frac{1}{4b^2}\phi_2(\eta)w^q
\end{aligned}\tag{16}$$

Where $\lambda_i\quad(i=1,2)$ is a real number.

An analysis of the solutions of the latter system shows that for $\eta\to\infty$ any solution to the system of equations (9) can be represented in the form $w(\eta)=w_0+o(1),\ z(\eta)=z_0+o(1)$. Then, by transformation (12), we have the required asymptotic representation of the solution to problem (9), (10).

Consider the case of fast diffusion.

Theorem 2. Let $p=\dfrac{\sigma_2(1+\sigma_1)}{\sigma_1}\qquad q=\dfrac{\sigma_1(1+\sigma_2)}{\sigma_2}$. Then the solutions of system (1) disappearing at infinity for $\eta\to\infty$ have an asymptotic representation.

$$f(\xi)=(a+b\xi^2)^{\frac{1}{\sigma_1}}(1+o(1)),\quad \varphi(\xi)=(a+b\xi^2)^{\frac{1}{\sigma_2}}(1+o(1))$$

Theorem 2 is proved similarly to the proof of Theorem 1.

Theorem 3. Let $\dfrac{p+1}{pq-1}<N/(2+\sigma_iN),\quad(i=1,2),\ p>\sigma_1/\sigma_2,\ q>\sigma_2/\sigma_1, pq>1$, then problem (1), (2) is globally solvable for sufficiently small initial data.

The result of [1] follows from this condition for $\sigma_i=0,\quad(i=1,2)$. In this case the condition $p>\sigma_1/\sigma_2,\ q>\sigma_2/\sigma_1, pq>1$ turns condition $p,\ q>1$.

Proof. The proof of Theorem 3 is based on a self-similar analysis of the solution to system (1) by nonlinear splitting of the original system (1). That is why we are looking for a solution to system (1) in the form (8). Then supplying (8) to (1) for *w,z* we obtain the following system

$$\frac{\partial w}{\partial \tau} = \frac{\partial}{\partial x}\left(w^{\sigma_1} \frac{\partial w}{\partial x} \right) + \varepsilon \bar{v}^{\,p} \left(\frac{p+1}{pq-1} w + z^{p} \right),$$
$$\frac{\partial z}{\partial \tau} = \frac{\partial}{\partial x}\left(z^{\sigma_1} \frac{\partial z}{\partial x} \right) + \varepsilon \bar{u}^{\,q} \left(\frac{q+1}{pq-1} z + w^{q} \right) \qquad (14)$$

where $\tau(t)$, $\bar{u}(t)$, $\bar{v}(t)$ are the functions defined above. Then using (4), we obtain the system of self-similar equations (9). Consider the functions $\bar{u}(t,x) = \bar{u}(t)\bar{f}(\xi), \bar{v}(t,x) = \bar{v}(t)\bar{\varphi}(\xi)$, where $\bar{f}(\xi), \bar{\varphi}(\xi)$ are thr functions defined above for *b = 1* by formula (9) if

$$\frac{\alpha}{1-\sigma_1\alpha} < N/2, \ \frac{\beta}{1-\sigma_2\beta} < N/2$$

By virtue of the expressions for α, β these conditions, mean the fulfilment of the condition of Theorem 3. Then according to the principle of comparison of solutions [2] the functions $\bar{u}(t,x), \bar{v}(t,x)$ are the upper solutions of problem (1), (2). Theorem 3 has been proved.

3. Conclusion

As we can see the asymptotic property of the solution to problem (1)-(2) for the critical parameter *p,q* changes its asymptotic property for $t \to \infty$. Proofs of the proposal are based on the principle of comparison solutions. The resulting asymptotic property of the solutions were used as initial approximation. As we now the finite propagation velocity at disturbanxes is a natural effect inherent in nonlinear processes.

It should be noted that for numerical calculations of a nonlinear problem, it is very important to choose a suitable initial approximation, since this ensure convergence with a given accuracy to the solution of the problem with a minimum number of iterations. In the considered nonlinear problem, it is important to establish the value of the numerical parameters, which changes the asymptotic behavior of the solution. Such values of numerical parameters are called critical exponents of the Fujita type [28]. For critical exponents, the effect of a finite

velocity of propagation of disturbances, blow-up solutions, changes in the nature of solutions, ect are observed.

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