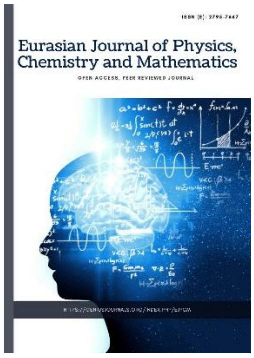

Applications Of The Theory Of Extrema To Practical Problems

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ABSTRACT

This paper explores the applications of the theory of extrema to solving practical problems. A number of applied problems from mechanics, geometry, physics, and engineering are analyzed using methods of mathematical analysis. For each problem, an appropriate mathematical model is constructed, and optimal solutions are obtained by determining the maxima and minima of single-variable functions. Critical points are identified using first- and second-order derivatives. The results demonstrate that the theory of extrema is a universal and effective tool for optimizing real-world processes and plays an important role in integrating theory and practice in teaching mathematical analysis

Keywords: *theory of extrema, mathematical analysis, maximum and minimum, derivative, applied problems, optimization, mathematical modeling*

Introduction. The theory of extrema, which is one of the important branches of modern mathematical analysis, possesses not only theoretical significance but also wide practical applications as an effective tool for solving numerous problems arising in mechanics, physics, engineering, economics, biology, and other natural sciences. In modeling real processes, it is often necessary to determine maximum or minimum values under certain conditions. Such problems are investigated using the methods of differential calculus, and optimal solutions are obtained by finding the extrema of functions.

The practical importance of the theory of extrema is especially evident in problems related to resource saving, determining the most efficient geometric shapes, minimizing energy or material consumption, and optimizing motion parameters. For example, problems such as constructing an object with minimum material consumption for a given volume, determining the shape with the maximum area under given constraints, and finding the optimal values of velocity and acceleration in mechanical motion are directly connected with the theory of extrema.

In this article, this branch of mathematical analysis is illustrated through specific applied problems, and various physical, geometrical, and technical problems are solved using methods for determining the extrema of single-variable functions. For each problem, a mathematical model is constructed, critical points are identified using derivatives, and the type of extremum is justified by means of the second derivative test. In addition, the solutions are explained through graphical illustrations, and the practical significance of the obtained results is clarified.

The main purpose of the article is to systematically demonstrate the practical applications of the theory of extrema and to present this topic to students and teachers in a clear and methodologically well-grounded manner through carefully selected examples. This approach serves to ensure the harmony between theory and practice in teaching mathematical analysis.

Problem Statement. A canal of width a joins another canal of width b at a right angle (Figure 1). The walls of the canals are flat and vertical. What should be the maximum length of a wooden log floating in one of the canals so that it can freely pass into the other canal?

Solution: Figure 1 shows the position of the floating log. The ends of the log touch the sides of the canal, as shown in the figure. It also touches point C. When the length of the log is maximal, it may touch point C.

We determine the length of the log:

$$d = AB = AC + CB = \frac{a}{sin\alpha} + \frac{b}{cos\alpha}$$

where α is the angle of inclination of the log with respect to one of the canal banks. The angle α varies within the interval $(0; \frac{\pi}{2})$.

Since $d > 0$, it attains its minimum value when the square of the expression

$$d = \frac{a}{sin\alpha} + \frac{b}{cos\alpha}$$

attains its minimum value. We find the square of d:

$$d^2 = \left(\frac{a}{sin\alpha} + \frac{b}{cos\alpha}\right)^2 = (a + btg\alpha)^2 \left(1 + \frac{1}{tg^2\alpha}\right).$$

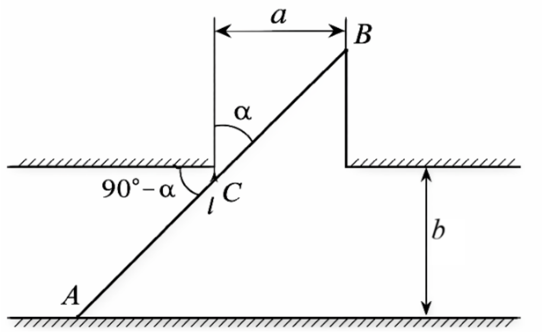

Figure 1.

If we let $\text{tg}\alpha = x$ $(0 < x < +\infty)$, then we obtain the function whose minimum value is to be determined:

$$f(x) = (a + bx)^2 \left(1 + \frac{1}{x^2}\right), 0 < x < +\infty.$$

We find the derivative of this function:

$$f'(x) = \left((a + bx)^2 \left(1 + \frac{1}{x^2}\right)\right)' = 2b(a + bx)\left(1 + \frac{1}{x^2}\right) + (a + bx)^2\left(-\frac{2}{x^3}\right) =$$

$$= 2(a + bx)\left[b\left(1 + \frac{1}{x^2}\right) - \frac{a + bx}{x^3}\right] = 2(a + bx)\left(b - \frac{a}{x^3}\right).$$

Hence, when $x = \sqrt[3]{\frac{a}{b}}$, we have $f'(x) = 0$. Moreover, if $x > \sqrt[3]{\frac{a}{b}}$, then $f'(x) > 0$, and if $0 < x < \sqrt[3]{\frac{a}{b}}$, then $f'(x) < 0$. Therefore, $f_{\min} = f\left(\sqrt[3]{\frac{a}{b}}\right)$. Thus, d^2 attains its minimum value when $\text{tg}\alpha_0 = \sqrt[3]{\frac{a}{b}}$. Since $\alpha_0 \in \left[0; \frac{\pi}{2}\right]$, we have:

$$sin\alpha_0 = \frac{tg\alpha_0}{\sqrt{1+tg^2\alpha_0}} = \frac{\sqrt[3]{\frac{a}{b}}}{\sqrt{1+\sqrt[3]{\frac{a^2}{b^2}}}} = \frac{\sqrt[3]{a}}{\sqrt[3]{a^2}+\sqrt[3]{b^2}}$$

$$cos\alpha_0 = \frac{1}{\sqrt{1+\sqrt[3]{\frac{a^2}{b^2}}}} = \frac{\sqrt[3]{b}}{\sqrt[3]{a^2}+\sqrt[3]{b^2}}$$

If we denote the maximum length of the log by d_0, then finally we obtain

$$d_0 = \frac{a}{sin\alpha_0} + \frac{b}{cos\alpha_0} = \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}.$$

2. A rectangle is inscribed in an ellipse whose axes are $2a$ and $2b$. What should the dimensions of the rectangle be so that its area is maximal?

Solution: We draw a diagram corresponding to the condition of the problem (Figure 2).

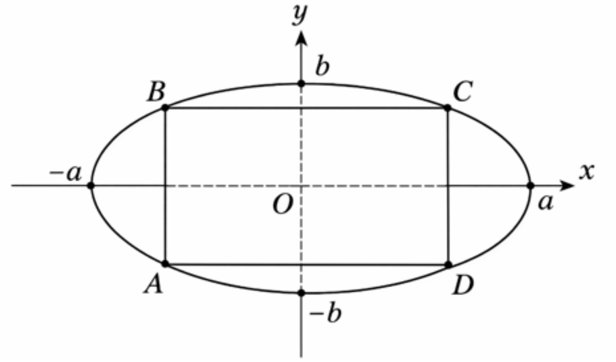

Figure 2.

Let the dimensions of the rectangle be

$$AD = x \text{ and } AB = y.$$

The function whose extremum is to be investigated is the area of the rectangle:

$$S = xy.$$

Using the equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we express y in terms of x:

$$y = \frac{b}{a}\sqrt{a^2 - x^2}.$$

The function whose extremum is to be investigated is

$$S = \frac{b}{a}x\sqrt{a^2 - x^2}.$$

We find the derivative of this function:

$$S'_x = \frac{b}{a}\sqrt{a^2 - x^2} - \frac{b}{a}\frac{x^2}{\sqrt{a^2 - x^2}} = \frac{b}{a} \cdot \frac{a^2 - 2x^2}{\sqrt{a^2 - x^2}}.$$

Equating the derivative to 0, we solve the obtained equation:

$$\frac{b}{a}\frac{a^2 - 2x^2}{\sqrt{a^2 - x^2}} = 0 \;\; \Rightarrow \;\; a^2 - 2x^2 = 0,$$

hence,

$$x = \frac{a}{\sqrt{2}}.$$

We determine the sign of the derivative to the left and to the right of the point $x = \frac{a}{\sqrt{2}}$. When $x = \frac{a}{\sqrt{2}}$, we have:

$$S'_x = \frac{b}{a}\frac{a^2 - \frac{a^2}{2}}{\sqrt{a^2 - \frac{a^2}{4}}} = \frac{b}{a}\frac{\frac{a^2}{2}}{\frac{\sqrt{3}}{2}a} = \frac{b}{\sqrt{3}} > 0.$$

When $x = \frac{4}{3}a$, we have

$$S'_x = \frac{b}{a}\cdot\frac{a^2 - \left(\frac{4}{3}a\right)^2}{a\sqrt{a^2 - \left(\frac{4}{3}a\right)^2}} = \frac{b}{a}\cdot\frac{8a^2 - 9a^2}{8\cdot\frac{\sqrt{7}}{4}a} = -\frac{4b}{2\sqrt{7}} < 0.$$

When passing through the point $x = \frac{a}{\sqrt{2}}$, the derivative changes its sign from "+" to "−". Therefore, this point corresponds to the maximum area of the rectangle. When $x = \frac{a}{\sqrt{2}}$, its other dimension is

$$y = \frac{b}{a}\cdot\sqrt{a^2 - \frac{a^2}{2}} = \frac{b}{a}\cdot\frac{a}{\sqrt{2}} = \frac{b}{\sqrt{2}}.$$

3. It is required to make a cylindrical bucket of volume V from sheet metal. What should the height of the cylinder and the radius of its base be so that the minimum amount of material is used?

Solution: Let the radius of the base of the bucket be x. We find the volume of the bucket (Figure 3):

$$V = \pi x^2 h \Rightarrow h = \frac{V}{\pi x^2}.$$

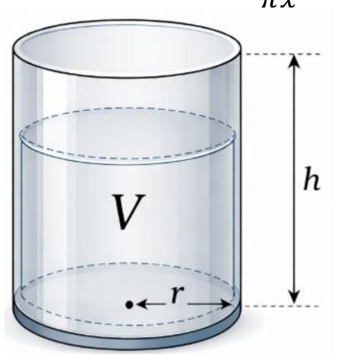

Figure 3.

We find the total surface area of the bucket:

$$S = \pi x^2 + 2\pi x h.$$

We form the function whose extremum is to be investigated:

$$S = \pi x^2 + \frac{2V}{x}.$$

We determine the condition under which this function has a minimum value:

$$S'(x) = 2\pi x - \frac{2V}{x^2}.$$

Equating the derivative to 0, we solve the obtained equation:

$$2\pi x - \frac{2V}{x^2} = 0 \Rightarrow \pi x^3 - v = 0 \Rightarrow x = \sqrt[3]{\frac{V}{\pi}}.$$

We find the second-order derivative of the function $S(x)$:

$$S''(x) = 2\pi + \frac{4V}{x^3}.$$

This derivative is positive when $x = \sqrt[3]{\frac{V}{\pi}}$:

$$S''\left(\sqrt[3]{\frac{V}{\pi}}\right) = 2\pi + \frac{4V}{V/\pi} = 2\pi + 4\pi = 6\pi > 0.$$

Therefore, the function $S(x)$ attains its minimum value at the point $x = \sqrt[3]{\frac{V}{\pi}}$, which gives the solution of the problem:

$$h = \frac{V}{\pi x^2} = V : \pi\sqrt[3]{\frac{V^2}{\pi^2}} = \frac{V}{\pi}\cdot\sqrt[3]{\frac{\pi^2}{V^2}} = \sqrt[3]{\frac{V}{\pi}} = x.$$

Thus, in order to use the minimum amount of material (sheet metal) in making the bucket, its dimensions must satisfy $x = r = h$.

4. In designing an alternating current transformer, an iron core whose cross-section consists of a square with four small squares cut from its corners is placed inside the coil. From a technical point of view, the area of this cross-section should be as large as possible. If the radius of the coil is R, what should the angle φ be in order for this area to be maximal?

Solution: Denoting the area of the iron core by S, from Figure 4 we obtain:

$$S = (MN)^2 - 4(PQ)^2$$

$$ON = R\cos\varphi, \qquad NQ = R\sin\varphi, \qquad MN = 2\cdot ON = 2R\cos\varphi$$

$$PQ = ON - NQ = R(\cos\varphi - \sin\varphi).$$

As a result, we obtain the following expression for the area S:

$$S = 4R^2(\sin^2\varphi - \sin 2\varphi).$$

We find the first-order derivative of S with respect to φ:

$$S' = 4R^2(2\cos 2\varphi - 2\sin\varphi\cdot\cos\varphi).$$

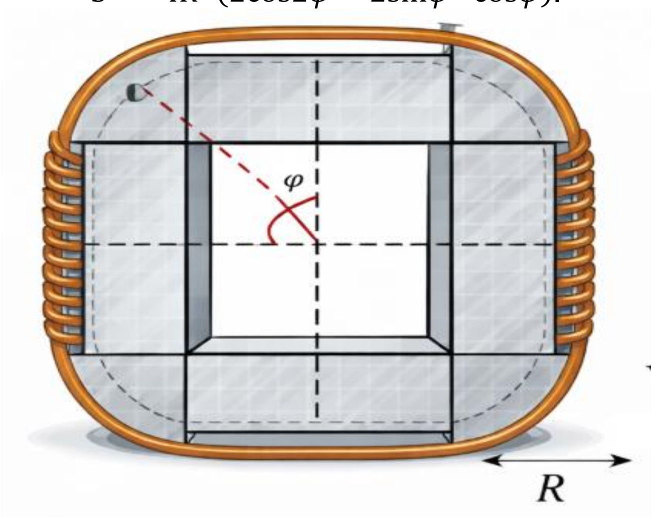

Figure 4.

Equating the derivative S' to 0 and solving the obtained equation $tg2\varphi = 2$, we find the critical point $\varphi \approx 31°43'$. For this value of φ, we have

$$S'' = -4R^2(4\sin 2\varphi + 2\cos\ 2\varphi) < 0.$$

According to the second derivative test, at the maximum point $\varphi_{\max} \approx 31°43'$, the area S attains its greatest value:

$$S_{\max} \approx 2.4R^2.$$

5. Find the point M on the parabola $y = x^2 - 2x - 8$ such that the tangent drawn to the parabola at this point is parallel to the straight line $2x + y + 4 = 0$.

Solution: We find the slope of the tangent to the parabola $y = x^2 - 2x - 8$:

$$k = y' = (x^2 - 2x - 8)' = 2x - 2.$$

We find the slope of the straight line $2x + y + 4 = 0$:

$$y = -2x - 4; \quad k = -2.$$

The tangent to the parabola and the straight line are parallel; therefore, their slopes are equal: $2x-2=-2$. Hence, the abscissa of the point of tangency is $x=0$. We find the ordinate of the point of tangency M from the equation of the parabola:

$$y|_{x=0} = 0^2 - 2\cdot 0 - 8 = -8,$$

Therefore,

$$M(0;-8).$$

6. Light sources with luminous intensities F_1 and F_2 are installed at the points $x=4$ and $x=5$, respectively. The distance between points A and B is equal to a. Find the point M on the segment AB where the illumination is minimal. The illumination at a point located at a distance r from a light source is inversely proportional to the square of the distance: $E = kF/r^2$, where F is the luminous intensity and E is the illumination.

Solution: The illumination at point M *is* $E = E_1 + E_2$. We determine the illumination at point M:

$$E = \frac{kF_1}{x^2} + \frac{kF_2}{(a-x)^2}.$$

We investigate the extremum of this function. We find its derivative:

$$E' = -\frac{2kF_1}{x^3} + \frac{2kF_2}{(a-x)^3}.$$

Equating this derivative to 0 and solving the obtained equation, we find the critical point:

$$-\frac{F_1}{x^3} + \frac{F_2}{(a-x)^3} = 0$$

$$\frac{F_2}{F_1} = \left(\frac{a-x}{x}\right)^3$$

$$\sqrt[3]{\frac{F_2}{F_1}} = \frac{a-x}{x}.$$

If we examine the sign of the derivative around the obtained point, we see that as it passes through this point, its sign changes from "–" to "+". Therefore, the function attains its minimum value at the obtained point. Thus, the point M with the minimum illumination is located at the distance

$$x = \frac{a\sqrt[3]{F_1}}{\sqrt[3]{F_1} + \sqrt[3]{F_2}}$$

from the light source with luminous intensity F_1 (Figure 5).

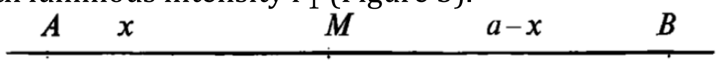

Figure 5.

7. The motion of a projectile in a vertical plane is described by the equations

$$x = v_0 t\cos\alpha, \qquad y = v_0 t\sin\alpha - \frac{gt^2}{2}.$$

Find the velocity and acceleration of the projectile at the initial moment of time ($g = 9.8$ m/s^2, v_0 and α are given constants). If $v_0 = 20$ m/s, $\alpha = 45°$, find the velocity of the projectile 1 second after the start of motion.

Solution: We find the velocity of the projectile. For this purpose, we determine the projections of the velocity from the equations of motion of the projectile:

$$x' = v_x = v_0\cos\alpha, \qquad y' = v_y = v_0\sin\alpha - gt.$$

We find the magnitude and direction of the velocity:

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(x')^2 + (y')^2} = \sqrt{v_0^2\cos^2\alpha + (v_0\sin\alpha - gt)^2} \qquad (1)$$

$$\cos\,(x,\wedge\, v) = \cos\ \alpha = \frac{v_x}{v} = \frac{v_0\cos\alpha}{v},$$

$$\cos(y,\wedge\, v) = \frac{v_0\sin\alpha - gt}{v}.$$

When $t = 0$, we have

$$v = v_0, \qquad \cos(x,\wedge v) = \cos\alpha\,, \cos(y,\wedge v) = \sin\alpha.$$

At the initial moment, the velocity makes an angle α with the Ox axis (Figure 6). We determine the projections of acceleration on the coordinate axes:

$$a_x = 0; \quad a_y = -g.$$

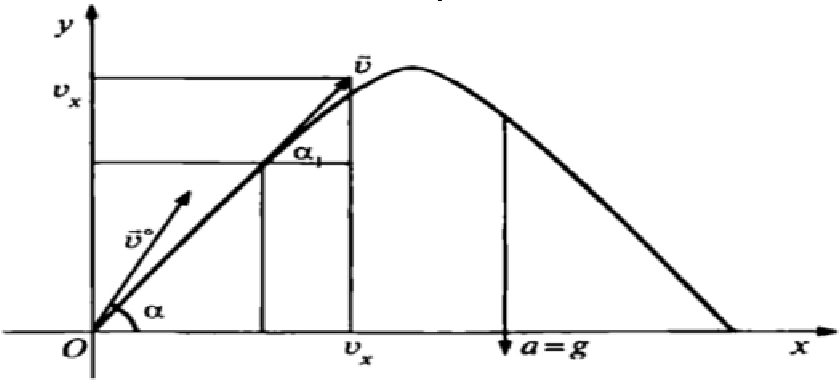

Figure 6.

The magnitude of acceleration is

$$a = \sqrt{a_x^2 + a_y^2} = g.$$

The acceleration of the projectile is directed vertically downward. At $t = 1$ s, the magnitude of the velocity vector is found using formula (1):

$$v = \sqrt{400\,\frac{\mathrm{m}^2}{\mathrm{s}^2}\cdot\frac{1}{2} + \left(20\,\frac{\mathrm{m}}{\mathrm{s}}\cdot\frac{\sqrt{2}}{2} - 10\,\frac{\mathrm{m}}{\mathrm{s}^2}\cdot 1\ \mathrm{s}\right)^2} \approx 14.7\,\frac{\mathrm{m}}{\mathrm{s}}.$$

We find the angle α_1 formed by the velocity vector $\vec{v}$ with the positive direction of the Ox axis:

$$tg\alpha_1 = \frac{v_y}{v_x} = \frac{v_0\sin\alpha - gt}{v_0\cos\alpha} = \frac{20\,\frac{\mathrm{m}}{\mathrm{s}}\cdot\frac{\sqrt{2}}{2} - 10\,\frac{\mathrm{m}}{\mathrm{s}^2}\cdot 1\ \mathrm{s}}{20\,\frac{\mathrm{m}}{\mathrm{s}}\cdot\frac{\sqrt{2}}{2}} = 0.3;$$

$$\alpha_1 = 17^\circ.$$

Results. In this article, the effectiveness of the theory of extrema in solving practical problems was demonstrated on the basis of a number of specific mathematical models. In particular, seven types of problems of mechanical, geometrical, physical, and technical content were considered, and in all of them the solution was obtained by determining the extrema of a function of one variable. In particular, in the problem of determining the maximum length of a wooden log in canals connected at a right angle, the function constructed on the basis of geometric constraints was minimized, and the optimal position was justified by means of the derivative. In the problem of a rectangle inscribed in an ellipse, the maximum value of the area function was determined, and the optimal dimensions were found using exact formulas. In the problem of making a cylindrical bucket, under the condition of minimizing material consumption, it was rigorously proved by means of the theory of extrema that the radius and height must be equal.

In addition, in the problem of maximizing the cross-sectional area of the iron core of a transformer, technical parameters were reduced to a mathematical model, and the optimal value of the angle was found. Through the tangent problem in analytic geometry, the condition of parallelism between a parabola and a straight line was expressed using the concept of derivative. In the problem related to light sources, the minimum value of illumination was determined, and the optimal point with physical meaning was found. In the problem of projectile motion, the dependence of velocity and acceleration on time was analyzed, and the significance of the theory of extrema in mechanical processes was shown.

The obtained results confirm that the theory of extrema is a universal method not only for purely mathematical problems, but also for optimizing real practical processes.

Conclusion. The results of the study show that the theory of extrema is one of the most important and practically effective branches of mathematical analysis. Through the examples presented in the article, it was substantiated that practical problems encountered in various fields of science can be successfully solved by means of a single mathematical approach — determining the extremum of a function.

The analysis of critical points using the derivative and the second-order derivative, the determination of maximum and minimum conditions, and the enrichment of the results with graphical and geometrical interpretations make the solution of problems more understandable and well-grounded. This makes it possible to use the theory of extrema as an important tool for ensuring the integration of theory and practice in teaching mathematical analysis at higher education institutions. At the same time, the problems considered in the article contribute to the development of students' analytical thinking, mathematical modeling, and optimal solution-finding skills. In the future, a deeper study of the applications of the theory of extrema to functions of several variables, constrained extrema, and variational problems will remain one of the relevant directions from both scientific and practical points of view.

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