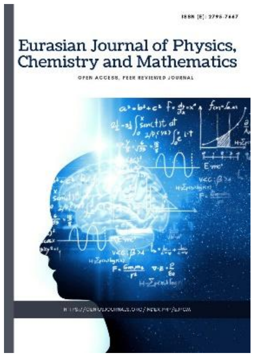

The characteristic polynomial and the Cayley-Hamilton theorem. The trace and the determinant maps

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ABSTRACT
In this paper, we provid a information the characteristic polynomial and the Cayley-Hamelton theorem, the trace and the determinant maps discuss the application of the theorem, and demonstrate its implementation in the OMOUS- 2024 and BIMO- 2026 olympiads.

Keywords: The characteristic polynomial and the Cayley-Hamelton theorem, the trace and the determinant maps and application to examples

Let $A \in M_2(C)$. The characteristic polynomial of A is by definition the polynomial denoted $\det(XI_2 - A)$ and defined by

$$\det(XI_2 - A) = X^2 - Tr(A)X + \det A$$

We note straight away that AB and BA have the same characteristic polynomial for all matrices $A, B \in M_2(C)$, since AB and BA have the same trace and the same determinant, by results established in the previous section. In particular, if P is invertible, then A and PAP^{-1} have the same characteristic polynomial.

The notation $\det(XI_2 - A)$ is rather suggestive, and it is indeed coherent, in the sense that for any complex number z, if we evaluate the characteristic polynomial of A at z, we obtain precisely the determinant of the matrix $zI_2 - A$. More generally, we have the following very useful:

Problem 1. For any two matrices $A, B \in M_2(C)$ there is a complex number u such that

$$\det(A + zB) = \det A + uz + \det B \cdot z^2$$

for all complex number z. If A, B have integer entries, then u is integer.

Solution. Write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} \alpha & \beta \\ \lambda & \delta \end{bmatrix}$.

Then

$$\det(A + zB) = \begin{bmatrix} a + z\alpha & b + z\beta \\ c + z\lambda & d + z\delta \end{bmatrix} =$$

$$= z^2(\alpha\delta - \beta\lambda) + z(a\delta + d\alpha - \beta c - \lambda b) + ad - bc.$$

Since $\alpha\delta - \beta\lambda = \det B$ and $ad - bc = \det A$, the result follows.

In other words, for any two matrices $A, B \in M_2(C)$ we can define a quadratic polynomial $\det(A + XB)$ which evaluated at any complex number z gives $\det(A + zB)$. Moreover, $\det(A + XB)$ has constant term $\det A$ and leading term B, and if λ have rational/integer/real entries, then this polynomial has rational/integer/real cofficients. Before moving on, let us practice some problems to better digest these ideas.

We introduce now two fundamental invariants of a 2×2 matrix, which will be generalized and extensively studied in subsequent chapters for $n\times n$ matrices:

Definition . Consider a matrix $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}\in M_2\left(C\right)$. We define the trace of A as

$$Tr(A)=a_{11}+a_{22}$$

the determinant of A as

$$\det A = a_{11}a_{22}-a_{12}a_{21}.$$

We also write

$$\det A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}$$

for the determinant of A .

We obtain in this way two maps

$$Tr,\det : \mathrm{M}_2\left(C\right)\to C$$

which essentially govern the theory of a 2×2 matrices. The following proposition summarizes the main properties of the trace map. The second property is absolutely fundamental. Recall that ${}^tA$ is the transpose of the matrix A .

Proposition. For all matrices $A,B\in M_2\left(C\right)$ and all complex numbers $z\in C$ we have

a) $Tr\left(A+zB\right)=Tr\left(A\right)+zTr\left(B\right)$

b) $Tr\left(AB\right)=Tr\left(BA\right)$

c) $Tr\left({}^tA\right)=Tr\left(A\right)$.

Proof. Properties a) and c) are readily checked, so let us focus on property $b)$. Write

$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix} \text{ and } B=\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{bmatrix}$$

Then

$$AB=\begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22}\end{bmatrix}$$

and

$$BA=\begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22}\end{bmatrix}.$$

Thus

$$Tr\left(AB\right)=Tr\left(BA\right).$$

Problem 2 (OMOUS-2024). Let $A,B\in M_2(R)$ matrices such that

$a)$ B is invertible

$b)$ $A^2+B^2=2AB$

$c)$ $\det(A^d-B^d)=\det(A^d+B^d)$ for $d=\{2020;2021\}$.

Prove that $\det\left(A^{2022}-I_2\right)=\det\left(A^{2022}+I_2\right)$.

Solution.

Lemma: If $A,B\in M_2(R)$ and $A^2+B^2=2AB$, then $AB=BA$.

Proof: From condition of lemma

$$\left(A-B\right)^2=AB-BA.$$

Then

$$Tr\left(A-B\right)^2=Tr(AB-BA)=0$$

and

$$Tr\left(A-B\right)^3=Tr((A-B)(\mathrm{AB}-\mathrm{BA}))=Tr(A^2B-ABA-BA$$

$$=Tr\left(A^2B\right)-Tr\left(ABA\right)-Tr\left(BAB\right)+Tr\left(B^2A\right)=0$$

Let α_1 and α_2 are eigenvalues of $(A-B)$ matrix, then

$$\begin{matrix}\alpha_1{}^2+\alpha_2{}^2=0 \\ \alpha_1{}^3+\alpha_2{}^3=0\end{matrix} \text{ or } \alpha_1=\alpha_2=0.$$

So we got $\left(A-B\right)$ matrix is nilpotent and $\left(A-B\right)^2=O_2$. From here $AB=BA$.

By using lemma 1 and condition $a)$ we have $AB^{-1}=B^{-1}A$. Multiplying both sides of condition $c)$ to the B^{-d}

$$\det(A^d B^{-d} - I_2) = \det\left(A^d B^{-d} + I_2\right)$$

or

$$\det((AB^{-1})^d - I_2) = \det\left((AB^{-1})^d + I_2\right)$$

Let $C = AB^{-1}$ and has eigenvalues α and β, then we get

$$\left(\alpha^d - 1\right)(\beta^d - 1) = \left(\alpha^d + 1\right)(\beta^d + 1)$$

$$\alpha^d + \beta^d = 0 \text{ or } \alpha = \beta.$$

Taking consideration A and B^{-1} matrices are commute, we know matrices A and B has common eigenvector. Let x is common eigenvector of matrix A and B^{-1} λ and μ are eigenvalues of A and B^{-1}.

$$AB^{-1}x = \gamma\delta x = 0 \Rightarrow \gamma = 0$$

so A matrix is nilpotent and $A^2 = O_2$.

Then

$$\det(A^{2022} - I_2) = \det\left(A^{2022} + I_2\right) = \det\left(\mp I_2\right) = 1.$$

Problem 3.(BIMO-2026). Let A be a $n \times n$ -matrix with real entries. Assume that we have $A^3 + A = 0$, then $trA\det\left(2I + A\right)$.

Solution. From $A^3 + A = 0$ we obtain

$$A\left(A^2 + I\right) = 0.$$

Hence every eigenvalue λ of A satisfies $\lambda\left(\lambda^2 + 1\right) = 0$, so $\lambda \in \{0, i, -i\}$. Since A has real entries, nonreal eigenvalus occur in conjugate pairs. Therefore the sum of all eigenvalues is zero, and $tr\left(A\right) = 0$. Consequently, $trA\det\left(2I + A\right) = 0$.

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