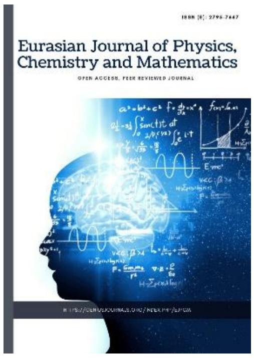

Mathematical Foundations Of Inverse And Ill-Posed Problems

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ABSTRACT

This article examines the fundamental concepts of inverse and ill-posed problems, their classification, and theoretical foundations in the context of mathematical physics. Issues related to the formulation of inverse problems based on differential equations and boundary conditions are discussed, including the correctness and stability of their solutions. The differences between weakly and strongly ill-posed problems are also analyzed, along with their practical applications and methods for stabilizing solutions.

Keywords: inverse problem, ill-posed problem, differential equation, integral equation, mathematical physics, regularization.

The mathematical model of many physical processes is expressed in the form of differential, integral, or integro-differential equations. Such equations play an important role in describing such complex processes as heat exchange, wave propagation, the theory of elasticity, electromagnetic fields, and the dynamics of liquids and gases in the fields of nature and technology. Particularly in mathematical physics, partial differential equations (PDEs) are the main object of research, since they allow modeling processes depending on space and time through functions of many variables. In mathematical physics, the following types of problems are usually posed: some differential equation and additional conditions (initial or boundary conditions) that its solution must satisfy. These additional conditions distinguish a unique or specific solution of the differential equation from the set of general solutions, which has a physical meaning. In other words, additional conditions clarify the physical content of the problem and ensure the completeness of the mathematical model.

Differential equations in partial derivatives are classified as elliptic, parabolic, and hyperbolic based on their characteristic properties. For example, the Laplace equation belongs to the elliptic type and represents stationary processes; The heat equation describes the process of heat propagation as a parabolic type equation; The wave equation belongs to the hyperbolic type and represents vibrational and wave processes. Different problems are posed for each type.

For hyperbolic and parabolic equations, the Cauchy problem (with initial conditions) and mixed problems are usually posed. For equations of elliptic type, boundary value problems, in particular, Dirichlet and Neumann problems, occupy an important place. In the Dirichlet problem, the value of the function is given on the boundary, and in the Neumann problem, the value of the normal derivative on the boundary is determined. Such problems are called correctly posed if they satisfy the conditions of existence, uniqueness, and continuous dependence of the solution on the given data. According to academic tradition, such problems are considered correct (or well-posed) problems in mathematical physics.

In a direct problem, it is usually assumed that a series of functions is given. In particular, the coefficients, free terms, and functions participating in the initial or boundary conditions of the differential equation are assumed to be known. It is required to find the unknown function based on these given functions. If the problem is correctly formulated in Hadamard's sense, then the solution exists, is unique, and is stable (i.e., not sensitive to small changes) with respect to the initial data.

Let us now assume that some of the functions given above are unknown. For example, if one of the coefficients of the equation or the function under the boundary condition is undefined, it is required to find it based on additional experimental data. In such cases, the problem of determining unknown functions arises. In mathematical physics, such problems are called inverse problems. In inverse problems, additional information is usually provided about the solution of the direct problem or some of its properties, and unknown parameters or functions are restored based on this information. Inverse problems are extremely important from a practical point of view and are used in many areas, such as geophysics, tomography, medical diagnostics, acoustics, and remote sensing. However, they often fall under the category of ill-posed problems, since the conditions of existence, uniqueness, or stability of the solution can be violated.

Incorrect problems are usually divided into two types: weakly incorrect and strongly incorrect problems. If a problem is ill-posed in one functional space but possesses the property of correctness in another, narrower or specially chosen space, then such a problem is called a weakly ill-posed problem. In this case, by choosing the space appropriately, the problem remains solvable. Conversely, if the problem does not satisfy the conditions of existence, uniqueness, or stability in any natural space, it is called a strongly ill-posed problem.

Thus, in mathematical physics, the study of forward and inverse problems, the determination of their regularity properties, and the development of solution stabilization methods are considered important directions of modern mathematical research. This area requires deep theoretical analysis and, on the practical side, the development of efficient numerical methods. An example of this is the following problem [1].

Example 1. Determination of the thermal conductivity coefficient.

Thermal equation

$$u_t = (k(x)u_x)_x \quad (1)$$

Here $k(x)$ is the unknown thermal conductivity coefficient. If the initial condition is given, and the temperature or heat flow at the boundary is measured for a certain time, then it is required to find. This is the inverse problem, since it is required to determine the unknown parameter based on additional information about the solution. Such problems are usually incorrect (small errors lead to large deviations).

Solution. Given equation:

$$u_t = (k(x)u_x)_x, \quad 0 < x < L, t > 0$$

(2)

Assume the length of the rod $L = 1$, boundary conditions

$$u(0,t) = 0, u(1,t) = 0$$

(3)

initial conditions

$$u(x,0) = sin(\pi x)$$

(4)

Let us assume that as a result of the experiment, the solution is determined as follows. $u(x,t) = e^{-\pi^2 t} sin(\pi x)$

Now we will find the unknown. First, let's calculate the derivatives.

Time derivative:

$$u_t = -\pi^2 e^{-\pi^2 t} sin(\pi x)$$

Spatial derivative:

$$u_{xx} = -\pi^2 e^{-\pi^2 t} sin(\pi x).$$

Substituting the derivatives into equation (2), we obtain the following equality for *k.*

$$-\pi^2 e^{-\pi^2 t} sin(\pi x) = k(-\pi^2 e^{-\pi^2 t} sin(\pi x))$$

By performing the reduction, we divide both sides by a common factor, and the result is $1 = k$.

Result. Consequently, if the temperature distribution in the experiment is observed in the form $u(x,t) = e^{-\pi^2 t} sin(\pi x)$, then the thermal conductivity coefficient $k = 1$ is determined. This is an example of a simplified inverse problem for the **heat equation**.

Determining the thermal conductivity coefficient $k(x)$ based on additional measurement data is a widely used inverse problem in practice. This problem is usually modeled based on the heat equation and serves to determine unknown physical parameters in real processes.

Example 2. The mathematical model of the process of chemical kinetics is expressed by the Cauchy problem for a system of linear differential equations [2].

$$\frac{dC_i}{dt} = a_{i1}c_1(t) + \ldots + a_{in}c_n(t) \quad (5)$$

$$C_i(t_0) = \overline{C}_i \ , \ i = 1,2,\ldots,n \quad (6)$$

The function $C_i(t)$ represents the concentration of the *i-th* substance participating in the process at time *t*. The constant a_{ij} characterizes the rate of change of the *i-th* concentration of the *i-th* substance participating in the process. Let us formulate the inverse problem for the system of differential equations (5). In some time interval $t \hat{I} [t_1, t_2]$, the concentration of substance $C_i(t)$ $i = 1,2,\ldots,N$ changes, and it is required to determine the parameters a_{ij}. That is, according to the solution of the system of differential equations (5), its coefficients are determined. This inverse problem can also be considered in the following two scenarios:

1. Initial conditions (4) are known, i.e., $\overline{C}_i(t)$ is given and solution $C_i(t)$ varies with corresponding $\overline{C}_i$,

2. $\overline{C}_i$ is unknown, it is necessary to determine them together with a_{ij}.

Example 3. Solving the heat equation "backward" in time

$$u_t = a^2 u_{xx}$$

Let $u(x,T)$ be known, i.e., the temperature at the last moment of time is given. Now ask to find the initial $u(x,0)$. This is a matter of reverse time. The problem is incorrect, because a small measurement error leads to a very large resultant error. Therefore, this problem is considered strongly incorrect.

Inverse and ill-posed problems are one of the important and relevant areas of mathematical physics. In such problems, the result or final state is known, and the initial causes or internal parameters of the process are determined. That is, in simple (correct) problems, one moves from cause to effect, while in inverse problems, one moves from effect to cause. This is precisely what makes them complex and often unstable.

The main feature of ill-posed problems is that even small measurement errors can lead to large deviations in the solution. In real life, measurements are always taken with a certain degree of error. Therefore, when solving inverse problems in practice, it is necessary to use special stabilization methods. Such problems are encountered in many practical areas. For example, determining the internal temperature of a material during thermal processes, restoring the properties of Earth's layers in geophysical research, diagnosing internal pathologies in medicine, and identifying latent defects in industry are inverse problems. Based on the data observed in these processes, unknown parameters are determined.

At the same time, problems such as solving the heat equation "backward" in time, although they have a theoretical solution, require special mathematical methods due to their instability. This shows the interrelationship of mathematical physics, numerical calculation methods, and practical modeling.

In general, inverse and incorrect problems are an integral part of modern science and technology. Their study has not only theoretical significance, but also provides important practical results in industry, energy, medicine, and natural sciences. Therefore, this area is considered one of the priority and promising areas of scientific research.

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