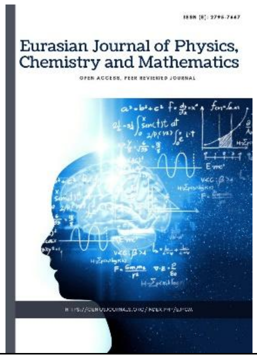


# Four-Dimensional Algebras $A_4$ With Radical $R(A_4)$ Of Dimension 0 And 1"

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ABSTRACT

The structure constants of the algebra $A_4$ define, in four-dimensional affine space, a system of curvilinear coordinates with constant connection coefficients. The study of curvilinear coordinates with constant connection coefficients is related to the study and classification of commutative and associative algebras $A_4$. Therefore, this work considers the classification of commutative and associative algebras $A_4$ with radical $R(A_4)$ of dimension 0 and 1.

**Keywords:** Algebra, commutativity, associativity, radical, real numbers, ideal, simple, semisimple, basis.

The theory of commutative and associative algebras is a well-developed branch of algebra. At the same time, in many problems of geometry, it becomes necessary to use certain concepts from linear algebra [1]. Therefore, the problem of classifying the entire set of associative and commutative algebras is of particular interest. In the present work, this problem is solved for four-dimensional algebras with a radical $R(A_4)$ of dimension 0 and 1.

In the classification of four-dimensional algebras over the field of real numbers, well-known theorems of linear algebra are used [2], [3].

**Theorem 1.** If a commutative and associative algebra A is semisimple, then it decomposes into a direct sum of a certain number of algebras of complex numbers **C** and real numbers **R**.

**Theorem 2.** If $R(A)$ is the radical of an associative algebra $A$, then the factor algebra $A/R(A)$ is semisimple.

**Theorem 3.** In an arbitrary associative algebra A there exists a semisimple subalgebra U complementary to the maximal nilpotent ideal $R(A)$.

Let $C^k_{ij}$ – be the structure constants of the algebra $A$. The multiplication law of the basis vectors $\varepsilon_k$ in the algebra $A$ is given by the formula:

$$\varepsilon_i\varepsilon_j = C^k_{ij}\varepsilon_k \,. \tag{1}$$

The conditions of commutativity and associativity of algebra $A$ are respectively equivalent to the identities

$$C^k_{ij} = C^k_{ji}\,, \quad C^m_{ij}C^s_{mk} = C^m_{jk}C^s_{mi}$$

which these structural constants satisfy. These relations are equivalent to the relations

$$\varepsilon_i\varepsilon_j = \varepsilon_j\varepsilon_i\,, \quad (\varepsilon_i\varepsilon_j)\varepsilon_k = \varepsilon_i(\varepsilon_j\varepsilon_k)\,. \tag{2}$$

Let's consider a four-dimensional commutative and associative algebra $A_4$ с with basis $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$. If the algebra $A_4$ полупростой, is semisimple, then this algebra does not have a radical of dimension different from zero. Therefore, for a semisimple algebra, $dimR(A_4)$=0 and there are three possible cases:

**I.** $A_4 = C \oplus C$ ; **II.** $A_4 = C \oplus R \oplus R$ ; **III.** $A_4 = R \oplus R \oplus R \oplus R$.

The multiplication tables of the basis vectors of these algebras are respectively written in the form:

I)
$$\varepsilon_1\varepsilon_1 = \varepsilon_1,\ \varepsilon_1\varepsilon_2 = \varepsilon_2\varepsilon_1 = \varepsilon_2,\ \varepsilon_1\varepsilon_3 = \varepsilon_3\varepsilon_1 = 0,\ \varepsilon_1\varepsilon_4 = \varepsilon_4\varepsilon_1 = 0,$$
$$\varepsilon_2\varepsilon_2 = -\varepsilon_1,\ \varepsilon_2\varepsilon_3 = \varepsilon_3\varepsilon_2 = 0,\ \varepsilon_2\varepsilon_4 = \varepsilon_4\varepsilon_2 = 0,$$
$$\varepsilon_3\varepsilon_3 = \varepsilon_3,\ \varepsilon_3\varepsilon_4 = \varepsilon_4\varepsilon_3 = \varepsilon_4,\ \varepsilon_4\varepsilon_4 = -\varepsilon_3.$$

II)
$$\varepsilon_1\varepsilon_1 = \varepsilon_1,\ \varepsilon_1\varepsilon_2 = \varepsilon_2\varepsilon_1 = \varepsilon_2,\ \varepsilon_1\varepsilon_3 = \varepsilon_3\varepsilon_1 = 0,\ \varepsilon_1\varepsilon_4 = \varepsilon_4\varepsilon_1 = 0,$$
$$\varepsilon_2\varepsilon_2 = -\varepsilon_1,\ \varepsilon_2\varepsilon_3 = \varepsilon_3\varepsilon_2 = 0,\ \varepsilon_2\varepsilon_4 = \varepsilon_4\varepsilon_2 = 0,$$
$$\varepsilon_3\varepsilon_3 = \varepsilon_3,\ \varepsilon_3\varepsilon_4 = \varepsilon_4\varepsilon_3 = 0,\ \varepsilon_4\varepsilon_4 = \varepsilon_4.$$

III)
$$\varepsilon_1\varepsilon_1 = \varepsilon_1,\ \varepsilon_1\varepsilon_2 = \varepsilon_2\varepsilon_1 = \varepsilon_1\varepsilon_3 = \varepsilon_3\varepsilon_1 = \varepsilon_1\varepsilon_4 = \varepsilon_4\varepsilon_1 = 0,\ \varepsilon_2\varepsilon_2 = \varepsilon_2,$$
$$\varepsilon_2\varepsilon_3 = \varepsilon_3\varepsilon_2 = \varepsilon_2\varepsilon_4 = \varepsilon_4\varepsilon_2 = 0,\ \varepsilon_3\varepsilon_3 = \varepsilon_3,\ \varepsilon_3\varepsilon_4 = \varepsilon_4\varepsilon_3 = 0,\ \varepsilon_4\varepsilon_4 = \varepsilon_4.$$

Using formula (1), the structure constants of the above-mentioned algebras I-III are determined.
The structure constants for an algebra of type I are defined as:

$$\varepsilon_1 = \varepsilon_1\varepsilon_1 = C_{11}^1\varepsilon_1 + C_{11}^2\varepsilon_2 + C_{11}^3\varepsilon_3 + C_{11}^4\varepsilon_4 \Rightarrow C_{11}^1 = 1,\ C_{11}^2 = C_{11}^3 = C_{11}^4 = 0,$$
$$\varepsilon_1\varepsilon_2 = \varepsilon_2\varepsilon_1 = \varepsilon_2 \Rightarrow C_{12}^2 = C_{21}^2 = 1,\ C_{12}^1 = C_{21}^1 = C_{12}^3 = C_{21}^3 = C_{12}^4 = C_{21}^4 = 0,$$
$$\varepsilon_1\varepsilon_3 = \varepsilon_3\varepsilon_1 = 0 \Rightarrow C_{13}^1 = C_{31}^1 = C_{13}^2 = C_{31}^2 = C_{13}^3 = C_{31}^3 = C_{13}^4 = C_{31}^4 = 0,$$
$$\varepsilon_1\varepsilon_4 = \varepsilon_4\varepsilon_1 = 0 \Rightarrow C_{14}^1 = C_{41}^1 = C_{14}^2 = C_{41}^2 = C_{14}^3 = C_{41}^3 = C_{14}^4 = C_{41}^4 = 0,$$
$$\varepsilon_2\varepsilon_2 = -\varepsilon_1 \Rightarrow C_{22}^1 = -1,\ C_{22}^2 = C_{22}^3 = C_{22}^4 = 0,$$
$$\varepsilon_2\varepsilon_3 = \varepsilon_3\varepsilon_2 = 0 \Rightarrow C_{23}^1 = C_{32}^1 = C_{23}^2 = C_{32}^2 = C_{23}^3 = C_{32}^3 = C_{23}^4 = C_{32}^4 = 0,$$
$$\varepsilon_3\varepsilon_3 = \varepsilon_3 \Rightarrow C_{33}^3 = 1,\ C_{33}^1 = C_{33}^2 = C_{33}^4 = 0,$$
$$\varepsilon_3\varepsilon_4 = \varepsilon_4\varepsilon_3 = \varepsilon_4 \Rightarrow C_{34}^4 = C_{43}^4 = 1,\ C_{34}^1 = C_{43}^1 = C_{34}^2 = C_{43}^2 = C_{34}^3 = C_{43}^3 = 0,$$
$$\varepsilon_4\varepsilon_4 = -\varepsilon_3 \Rightarrow C_{44}^3 = -1,\ C_{44}^1 = C_{44}^2 = C_{44}^4 = 0.$$

Similarly, for algebra II, the structure constants have the form:

$$C_{ij}^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad C_{ij}^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$C_{ij}^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad C_{ij}^4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The structure constants of an algebra of type III are written in the form:

$$C_{ij}^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad C_{ij}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C_{ij}^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad C_{ij}^4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

If the algebra $A_4$ is not semisimple, then there exists a non-zero radical $R(A_4)$.
Let $dimR(A_4)=1$ и $\varepsilon_4$ – be a basis element in the radical $R(A_4)$. Then, according to Theorem 2, the factor algebra $A_4/R(A_4)$ will be a three-dimensional semisimple algebra.
Hence, the following two cases are possible:
1. In the case $A_4 / R(A_4) = C \oplus R$, according to Theorem 3, the multiplication law of the basis elements $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$, $\varepsilon_4$ in the algebra $A_4$ is given by:

$$\begin{aligned} &\varepsilon_1\varepsilon_1 = \varepsilon_1,\ \varepsilon_1\varepsilon_2 = \varepsilon_2\varepsilon_1 = \varepsilon_2,\ \varepsilon_1\varepsilon_3 = \varepsilon_3\varepsilon_1 = 0, \varepsilon_1\varepsilon_4 = \varepsilon_4\varepsilon_1 = \lambda\varepsilon_4, \\ &\varepsilon_2\varepsilon_2 = -\varepsilon_1,\ \varepsilon_2\varepsilon_3 = \varepsilon_3\varepsilon_2 = 0,\ \varepsilon_2\varepsilon_4 = \varepsilon_4\varepsilon_2 = \mu\varepsilon_4, \\ &\varepsilon_3\varepsilon_3 = \varepsilon_3,\ \varepsilon_3\varepsilon_4 = \varepsilon_4\varepsilon_3 = \nu\varepsilon_4,\ \varepsilon_4\varepsilon_4 = 0, \end{aligned} \tag{3}$$

where is $\lambda, \mu, \nu \in R$.
From the associativity condition (2) of the algebra $A_4$, we obtain the following systems of equations:

$$\lambda = \lambda^2,\ \lambda = -\mu^2,\ \nu = \nu^2,\ \mu = \lambda\mu,\ \lambda\nu = 0,\ \mu\nu = 0.$$

The solution to this system of equations is:

$$\begin{cases} \lambda = 0, \\ \mu = 0, \\ \nu = 0 \end{cases} \text{или} \quad \begin{cases} \lambda = 0, \\ \mu = 0, \\ \nu = 1. \end{cases}$$

At the same time, for the first case expression (3) is written in the form,

$$\begin{aligned} &\varepsilon_1\varepsilon_1 = \varepsilon_1,\ \varepsilon_1\varepsilon_2 = \varepsilon_2\varepsilon_1 = \varepsilon_2,\ \varepsilon_1\varepsilon_3 = \varepsilon_3\varepsilon_1 = \varepsilon_1\varepsilon_4 = \varepsilon_4\varepsilon_1 = 0, \\ &\varepsilon_2\varepsilon_2 = -\varepsilon_1,\ \varepsilon_2\varepsilon_3 = \varepsilon_3\varepsilon_2 = \varepsilon_2\varepsilon_4 = \varepsilon_4\varepsilon_2 = 0, \\ &\varepsilon_3\varepsilon_3 = \varepsilon_3,\ \varepsilon_3\varepsilon_4 = \varepsilon_4\varepsilon_3 = \varepsilon_4\varepsilon_4 = 0, \end{aligned} \tag{4}$$

and for the second case,

$$\begin{aligned} &\varepsilon_1\varepsilon_1 = \varepsilon_1,\ \varepsilon_1\varepsilon_2 = \varepsilon_2\varepsilon_1 = \varepsilon_2,\ \varepsilon_1\varepsilon_3 = \varepsilon_3\varepsilon_1 = \varepsilon_1\varepsilon_4 = \varepsilon_4\varepsilon_1 = 0, \\ &\varepsilon_2\varepsilon_2 = -\varepsilon_1,\ \varepsilon_2\varepsilon_3 = \varepsilon_3\varepsilon_2 = \varepsilon_2\varepsilon_4 = \varepsilon_4\varepsilon_2 = 0, \\ &\varepsilon_3\varepsilon_3 = \varepsilon_3,\ \varepsilon_3\varepsilon_4 = \varepsilon_4\varepsilon_3 = \varepsilon_4,\ \varepsilon_4\varepsilon_4 = 0. \end{aligned} \tag{5}$$

An algebra whose multiplication law of the basis elements is given in the form (4) is called an algebra of type IV. Similarly, an algebra of type V is defined by relations (5).
The structure constants of the algebra of type IV have the form:

$$C_{ij}^{1}=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad C_{ij}^{2}=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

$$C_{ij}^{3}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix},\quad C_{ij}^{4}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

and for the type V,

$$C_{ij}^{1}=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad C_{ij}^{2}=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

$$C_{ij}^{3}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix},\quad C_{ij}^{4}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}.$$

2. For this case $A_4/R(A_4)=R\oplus R\oplus R$ we write the multiplication law of the basis elements of the algebra as $A_4$:

$$\begin{aligned}&\varepsilon_1\varepsilon_1=\varepsilon_1,\ \varepsilon_1\varepsilon_2=\varepsilon_2\varepsilon_1=\varepsilon_1\varepsilon_3=\varepsilon_3\varepsilon_1=0,\ \varepsilon_1\varepsilon_4=\varepsilon_4\varepsilon_1=\lambda\varepsilon_4,\\&\varepsilon_2\varepsilon_2=\varepsilon_2,\ \varepsilon_2\varepsilon_3=\varepsilon_3\varepsilon_2=0,\ \varepsilon_2\varepsilon_4=\varepsilon_4\varepsilon_2=\mu\varepsilon_4,\\&\varepsilon_3\varepsilon_3=\varepsilon_3,\ \varepsilon_3\varepsilon_4=\varepsilon_4\varepsilon_3=\nu\varepsilon_4,\ \varepsilon_4\varepsilon_4=0.\end{aligned} \tag{6}$$

From the associativity conditions of the algebra $A_4$, we obtain the following systems of equations:

$$\begin{cases}\lambda=\lambda^2,\ \mu=\mu^2,\ \nu=\nu^2,\\ \lambda\mu=0,\ \lambda\nu=0,\ \nu\mu=0.\end{cases}$$

This system of equations has the following solutions:

2.1) $\lambda=1\ \Rightarrow\ \mu=\nu=0;$

2.2) $\mu=1\ \Rightarrow\ \lambda=\nu=0;$

2.3) $\nu=1\ \Rightarrow\ \lambda=\mu=0;$

2.4) $\lambda=0,\ \mu=0,\ \nu=0.$

The multiplication table (6) of the basis elements of the algebra in case (2.1) is written in the form:

| | $\varepsilon_1$ | $\varepsilon_2$ | $\varepsilon_3$ | $\varepsilon_4$ |
|---|---|---|---|---|
| $\varepsilon_1$ | $\varepsilon_1$ | 0 | 0 | $\varepsilon_4$ |
| $\varepsilon_2$ | 0 | $\varepsilon_2$ | 0 | 0 |
| $\varepsilon_3$ | 0 | 0 | $\varepsilon_3$ | 0 |
| $\varepsilon_4$ | $\varepsilon_4$ | 0 | 0 | 0 |

(7)

In case (2.2), after substituting the basis elements according to the formulas:

$$\overline{\varepsilon}_1 = \varepsilon_2, \quad \overline{\varepsilon}_2 = \varepsilon_1, \quad \overline{\varepsilon}_3 = \varepsilon_3, \quad \overline{\varepsilon}_4 = \varepsilon_4$$

...once again transforms the multiplication law (6) into table (7). Similarly, in case (2.3), after substituting the basis elements:

$$\overline{\varepsilon}_1 = \varepsilon_3, \quad \overline{\varepsilon}_2 = \varepsilon_2, \quad \overline{\varepsilon}_3 = \varepsilon_1, \quad \overline{\varepsilon}_4 = \varepsilon_4$$

...the multiplication law (6) is again reduced to table (7). In case (2.4), the multiplication law (6) is rewritten as follows:

| | $\varepsilon_1$ | $\varepsilon_2$ | $\varepsilon_3$ | $\varepsilon_4$ |
|---|---|---|---|---|
| $\varepsilon_1$ | $\varepsilon_1$ | 0 | 0 | 0 |
| $\varepsilon_2$ | 0 | $\varepsilon_2$ | 0 | 0 |
| $\varepsilon_3$ | 0 | 0 | $\varepsilon_3$ | 0 |
| $\varepsilon_4$ | 0 | 0 | 0 | 0 |

(8)

Thus, the following theorem holds.

**Theorem**. There exist seven pairwise non-isomorphic four-dimensional commutative and associative algebras with a radical of dimension 0 and 1.

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