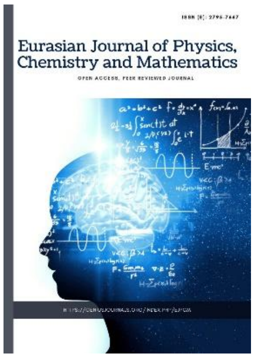

Exictence Of Generalized Solution Of Equations Of A Hyperbolic Type

Daniyarova Gulayim Kuwatbayevna

Mathematic teacher of number of 27th secondary school

ABSTRACT

Initial-boundary value problems for equations of a hyperbolic type are widely used in mathematical physics, technical science and other fields. For instance, describing a large class of natural phenomena: solving of wave equations, concepts of the classical and generalized suolution, as well as their difference and existence is given the target thesis For the different types of equations constructing of boundary conditions and their types, Cauchy problems with initial-boundary value problems are covered for the mixed issues. Classical mathematical physics problems are presented brought solving boundary value problems for second-order particular derivative differential equations. In the circumstance, finding and learning the initial-boundary value problems for the higher-order particular derivative mixed equations have become an urgent problem. Existence of the solutions of particular derivative mixed problems constructing with initial-boundary value conditions and researched finding of them. When none of the classical solution in the given problem, might be find generalized solution with the generalized functions corresponding to that. funksiyalar bilan umumlashgan yechimni topish mumkin.

Keywords: equations of mathematical physics, hyperbolic equation, generalized solution, classic solution, existence of generalized solution, uniqueness of solution, smoothness of generalized solution

1. Uniqueness of solution. Let some D be a bounded field in $n-$dimensional space R_n ($x=(x_1,...,x_n)-$point of the space). Look a bounded cylinder $Q_T=\{x\in D, 0<t<T\}$ with height $T>0$ in $(n+1)-$dimensional space $R_{n+1}=R_n\times\{-\infty<t<+\infty\}$. Let us denote by Γ_T lateral surface $\{x\in\partial D, 0<t<T\}$ of cylinder Q_T, and through D_τ $-$the section $\{x\in D, t=\tau\}$ of this cylinder plane $t=\tau$; in particular, the upper rolling of cylinder Q_T is $D_T=\{x\in D, t=T\}$, and its lower base - $D_0=\{x\in D, t=0\}$.

In cylinder Q_T at some $T>0$ we consider the hyperbolic equation

$$\mathcal{L}\, u \equiv u_{tt}-div(k(x)\nabla u)+a(x)=f(x,t), \tag{1}$$

which $k(x) \in C^1(\bar{D})$, $a(x) \in C(\bar{D})$, $k(x) \ge k_0 = const > 0$.

The function $u(x,t)$ belonging to the space $C^2(Q_T) \cap \cap C^1(Q_T \cup \Gamma_T \cup \bar{D}_0)$ satisfying in Q_T the equation (1) on D_0 the initial conditions

$$u\big|_{t=0} = \varphi, \tag{2}$$

$$u_t\big|_{t=0} = \psi, \tag{3}$$

and Γ_T on one of the boundary conditions

$$u\big|_{\Gamma_T} = \chi$$

or

$$\left(\frac{\partial u}{\partial n} + \sigma u\right)\bigg|_{\Gamma_T} = \chi,$$

where $\sigma -$ is some continuous function on Γ_T is called the (classical) solution of the first or, correspondingly, the third mixed problem for the equation (1).

Since the case of non-homogeneous boundary conditions is easily reduced to the case of homogeneous boundary conditions, then in the future we will consider homogeneous boundary conditions

$$u\big|_{\Gamma_T} = 0, \tag{4}$$

and

$$\left(\frac{\partial u}{\partial n} + \sigma u\right)\bigg|_{\Gamma_T} = 0. \tag{5}$$

Let us assume that the coefficient $a(x)$ in equation (1) is non-negative in Q_T, and the function σ in the boundary condition (5) depends only on x, $\sigma = \sigma(x)$, and is non-negative in Γ_T.

Let the function $u(x,t)$ be a solution to one of the problems (1) - (4) or (1), (2), (3), (5), and the right side $f(x,t)$ of the equation (1) belongs to $L_2(Q_T)$. Let's take an arbitrary δ, $0 < \delta < T$. We multiply (1) by the function $\upsilon(x,t)$ belonging to $C^1(\bar{Q}_{T-\delta})$ and satisfying the condition

$$\upsilon\big|_{D_{T-\delta}} = 0, \tag{6}$$

and integrate the obtained equality over cylinder $Q_{T-\delta}$. Since $u_{tt}\upsilon = \; = (u_t\upsilon)_t - u_t\upsilon_t$, a $\upsilon div(k\nabla u) = div(k\upsilon\nabla u) - k\nabla u \nabla \upsilon$, then, considering the initial condition (3) and condition (6) using the Ostrogradskian formula, we obtain

$$\int\limits_{Q_{T-\delta}} f\upsilon dxdt = \int\limits_{Q_{T-\delta}} ((u_t\upsilon)_t - div(k\upsilon\nabla u))dxdt +$$

$$+\int\limits_{Q_{T-\delta}}(k\nabla u\nabla\upsilon+au\upsilon-u_t\upsilon_t)dxdt=$$

$$=\int\limits_{D_{T-\delta}}u_t\upsilon dx-\int\limits_{D_0}u_t\upsilon dx-\int\limits_{\Gamma_{T-\delta}}k\frac{\partial u}{\partial n}\upsilon dSdt+$$

$$+\int\limits_{Q_{T-\delta}}(k\nabla u\nabla\upsilon+au\upsilon-u_t\upsilon_t)dxdt=-\int\limits_{D_0}\psi\upsilon dx-\int\limits_{\Gamma_{T-\delta}}k\upsilon\frac{\partial u}{\partial n}dSdt+$$

$$+\int\limits_{Q_{T-\delta}}(k\nabla u\nabla\upsilon+au\upsilon-u_t\upsilon_t)dxdt. \tag{7}$$

If $u(x,t)$ is the solution of the third (or second) mixed problem, then, due to (5), it follows from the last equality that $u(x,t)$ satisfies the integral identity

$$\int\limits_{Q_{T-\delta}}(k\nabla u\nabla\upsilon+au\upsilon-u_t\upsilon_t)dxdt+\int\limits_{\Gamma_{T-\delta}}k\sigma u\upsilon dSdt=\int\limits_{Q_{T-\delta}}f\upsilon dxdt+\int\limits_{D_0}\psi\upsilon dx,$$

for all $\upsilon(x,t)$ from $C^1(\bar{Q}_{T-\delta})$ for which condition (6) is satisfied, and consequently, for all $\upsilon(x,t)$ from $H^1(Q_{T-\delta})$ for which condition (6) is satisfied.

A function belonging to space $H^1(Q_T)$ is called a generalized solution in Q_T of the first mixed problem (1) - (4), if it satisfies the initial condition (2), boundary condition (4), and identity

$$\int\limits_{Q_T}(k\nabla u\nabla\upsilon+au\upsilon-u_t\upsilon_t)dxdt=\int\limits_{D_0}\psi\upsilon dx+\int\limits_{Q_T}f\upsilon dxdt, \tag{9}$$

for all $\upsilon\in H^1(Q_T)$ for which conditions (4) and condition

$$\upsilon\big|_{D_T}=0, \tag{10}$$

are satisfied.

The function i belonging to the space $H^1(Q_T)$ is called the generalized solution in Q_T of the third (second for $\sigma=0$) mixed problem (1), (2), (3), (5) if it satisfies the initial condition (2) and the identity

$$\int\limits_{Q_T}(k\nabla u\nabla\upsilon+au\upsilon-u_t\upsilon_t)dxdt+\int\limits_{\Gamma_T}k\sigma u\upsilon dSdt=\int\limits_{D_0}\psi\upsilon dx+\int\limits_{Q_T}f\upsilon dxdt \tag{11}$$

for all $\upsilon\in H^1(Q_T)$ for which condition (10) is satisfied.

Lemma 1. If the generalized solution of the problem (1) - (4) or the problem (1), (2), (3), (5) belongs to the space $H^2(Q_T)$, then it is the solution of the corresponding problem. If the generalized solution of the problem (1) - (4) or the problem (1), (2), (3), (5) belongs to $C^2(Q_T)\cap C^1(Q_T\cup\Gamma_T\cup\bar{D}_0)$, then it is a classical solution of the corresponding problem.

Theorem 1. Each of the problems (1) - (4) and (1), (2), (3), (5) can have more than one generalized solution.

Consequence 1. Each of the problems (1) - (4) and (1), (2), (3), (5) cannot have more than one classical solution.

Consequence 2. Each of the problems (1) - (4) and (1), (2), (3), (5) can have no more than one solution to the problem.

2. Existence of a generalized solution. Let us now proceed to the proof of the existence of solutions to the problems (1) - (4) and (1), (2), (3), (5). For this, we use the Fourier method, which means that the solution of the mixed problem is sought in the form of a series of eigenfunctions of the corresponding elliptic boundary value problem..

Let $\upsilon(x)-$ be the generalized eigenfunction of the first boundary value problem

$$div(k\nabla\upsilon)-a\upsilon=\lambda\upsilon,\ \ x\in D, \tag{12}$$

$$\upsilon\big|_{\partial D}=0$$

or the third (when the second) boundary value problem

$$div(k\nabla\upsilon)-a\upsilon=\lambda\upsilon, \qquad x\in D,$$

$$\left.\left(\frac{\partial\upsilon}{\partial n}+\sigma\upsilon\right)\right|_{\partial D}=0 \tag{13}$$

($\lambda-$corresponding eigenvalue). This means that in the case of the first boundary value problem $\upsilon\in\overset{\circ}{H}{}^1(D)$ and for all $\eta\in\overset{\circ}{H}{}^1(D)$

$$\int_D(k\nabla\upsilon\nabla\eta+a\upsilon\eta)dx+\lambda\int_D\upsilon\eta dx=0, \tag{14}$$

and in the case of the third (second) boundary value problem $\upsilon\in H^1(D)$ and for all $\eta\in H^1(D)$

$$\int_D(k\nabla\upsilon\nabla\eta+a\upsilon\eta)dx+\int_{\partial D}k\sigma\upsilon\eta dS+\lambda\int_D\upsilon\eta dx=0, \tag{15}$$

Consider an orthonormal system υ_1, υ_2,... in $L_2(D)$ consisting of all generalized eigenfunctions of the problem (12) or, respectively, the problem (13); Let us assume that the initial functions $\varphi(x)$ and $\psi(x)$ in (2) and (3) belong to $L_2(D)$, and the function $f(x,t)\in L_2(Q_T)$. According to Fubini's theorem, $f(x,t)\in L_2(D)$ for almost everywhere $t\in(0,T)$. Let us decompose the functions $\varphi(x)$ and $\psi(x)$ and the function $f(x,t)$ for almost all values of $t\in(0,T)$ into Fourier series according to the system $\upsilon_1(x)$, $\upsilon_2(x)$,... of generalized eigenfunctions of the problem (12), if the problem (1) - (4) is considered, or the problem (13), if the problem (1), (2), (3), (5) is considered

$$\varphi(x)=\sum_{k=1}^{\infty}\varphi_k\upsilon_k(x),\ \ \psi(x)=\sum_{k=1}^{\infty}\psi_k\upsilon_k(x),\ \ f(x,t)=\sum_{k=1}^{\infty}f_k\upsilon_k(x), \tag{16}$$

where $\varphi_k = (\varphi, \upsilon_k)_{L_2(D)}$, $\psi_k = (\psi, \upsilon_k)_{L_2(D)}$, and $f_k(t) = \int\limits_D f(x,t)\upsilon_k dx$, $k = 1,2,\ldots$. Let us first take as the initial functions in (2) and (3) the functions $\varphi_k \upsilon_k(x)$ and $\psi_k \upsilon_k(x) - k - e$ - "harmonics" from the series (16), and as the function in the right-hand side of the equation (1) - the function $f_k(t)\upsilon_k(x)$, $k \geq 1$. Consider the function

$$u_k(x,t) = U_k(t)\upsilon_k(x), \tag{17}$$

where

$$U_k(t) = \varphi_k \cos\sqrt{-\lambda_k}\,t + \frac{\psi_k}{\sqrt{-\lambda_k}}\sin\sqrt{-\lambda_k}\,t + \frac{1}{\sqrt{-\lambda_k}}\int\limits_0^t f_k(\tau)\sin\sqrt{-\lambda_k}(t-\tau)d\tau; \tag{18}$$

Obviously, the function $U_k(t)$ belongs to $H^2(0,T)$, satisfies the initial conditions $U_k(0) = \varphi_k$, $U_k'(0) = \psi_k$ at $t = 0$, and for $t \in (0,T)$ is a solution to the equation

$$U_k'' - \lambda_k U_k = f_k, \qquad k = 1,2,\ldots \tag{19}$$

We will show that if $\upsilon_k(x)$ and λ_k are generalized eigenfunctions and corresponding eigenvalues of the problem (12) (or problem (13)), then the function $u_k(x,t)$ is a generalized solution of the first (respectively third or second) mixed problem for the equation

$$u_{tt} - div(k(x)\nabla u) + au = f_k(t)\upsilon_k(x)$$

with initial conditions

$$u\big|_{t=0} = \varphi_k \upsilon_k(x), \qquad u_t\big|_{t=0} = \psi_k \upsilon_k(x).$$

Indeed, the function $u_k(x,t) \in H^1(Q_T)$, on D_0 satisfies the initial condition (2) and in the case of the first mixed problem - the boundary condition (4). We will show that the function $u_k(x,t)$ in the case of the first mixed problem satisfies the integral identity

$$\int\limits_{Q_T}(k\nabla u_k \nabla \upsilon + au_k\upsilon - u_{kt}\upsilon_t)dxdt = \psi_k \int\limits_{D_0}\upsilon_k(x)\upsilon dx + \int\limits_{Q_T} f_k(t)\upsilon_k(x)\upsilon dxdt \tag{9_k}$$

for all υ functions belonging to the space $H^1(Q_T)$ satisfying the conditions (4) and (10), and in the case of the second and third mixed problems - the identity

$$\int\limits_{Q_T}(k\nabla u_k \nabla \upsilon + au_k\upsilon - u_{kt}\upsilon_t)dxdt + \int\limits_{\Gamma_T} k\sigma u_k \upsilon dSdt =$$

$$= \psi_k \int\limits_{D_0}\upsilon_k(x)\upsilon dx + \int\limits_{Q_T} f_k(t)\upsilon_k(x)\upsilon dxdt \tag{11_k}$$

for all υ satisfying the condition (10) from $H^1(Q_T)$. Obviously, it is sufficient to establish the validity of the identities (9_k) and (11_k) only for all υ functions that are continuously differentiable in $\bar{Q}_T$ and satisfy the conditions (4) and (10), respectively, the condition (10).

If the partial sums of the series $\sum_{k=1}^{N}\varphi_k\upsilon_k(x)$ and $\sum_{k=1}^{N}\psi_k\upsilon_k(x)$ from (16) at some N are taken as the initial functions in (2) and (3), and the partial sum of its Fourier series $\sum_{k=1}^{N}f_k(t)\upsilon_k(x)$ is taken as the function f in (1), then the generalized solution of the problem (1) - (4) ((1), (2), (3), (5)) will be the function

$$S_N(x,t)=\sum_{k=1}^{N}u_k(x,t)=\sum_{k=1}^{N}U_k(t)\upsilon_k(x).$$

Therefore, it is natural to expect that under certain assumptions relative to φ, ψ and f, the solution of the problem (1) - (4) ((1), (2), (3), (5)) can be represented as a series

$$u(x,t)=\sum_{k=1}^{\infty}U_k(t)\upsilon_k(x), \tag{20}$$

where υ_1, $\upsilon_2,\ldots -$ generalized eigenfunctions of the problem (12) (respectively (13)).

Literature

1. Михайлов В.П. Дифференциальные уравнения в частных производных. М. Наука, 1983 год.

2. Тихонов А.Н., Самарский А.А., Уравнения математической физики, М-Л., изд. 2-е, Гостехиздат, 1953.

3. Ладыженская О.А. Краевые задачи математической физики. Наука, 1973 год.

4. Петровский И.Г. Лекции об уравнениях с частными производными. Физматгиз, 1961 год.

5. Владимиров В.С. Уравнения математической физики. Наука, 1971.

6. Ш.Г.Касимов, Т.Н.Аликулов, Ш.К.Отаев, Ғ.С.Хаитбоев, М.М.Бабаев “Математик физиканинг замонавий усуллари” 1-2 том.