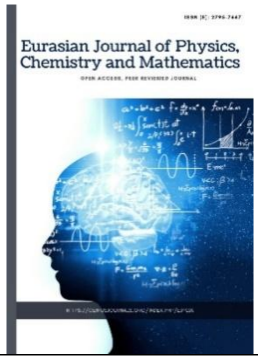

Use Of Differential Equations In Solving Filtering Problems, Solution By Euler, Heun And Runge-Kutt Methods.

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ABSTRACT

In this article, it is known from ordinary differential equation courses, if $y' = f(x,y)$ in a differential equation $f(x,y)$ and if the equation is complex, to find the appropriate solution, it is important to use approximate methods. In this article, the concepts and examples of the approximate calculation of differential equations solved with respect to the derivative are considered.

Keywords: Differential equations, approximate solution, analytical solution, Cauchy problem, Euler method, Heun method, Runge-Kutta method, approximation.

Introduction

In recent years, creating a mathematical model of many life processes and solving it with mathematical methods has become widespread among mathematicians. These processes are inextricably linked with the development of medicine and techniques. Fractional integrals and derivatives have many applications in the fields of physics, biology, medicine and technology, and it is important in the development of these fields.

For this reason, in recent years, mathematicians are increasingly interested in studying differential and partial differential equations with fractional derivatives. In our country, attention has been paid to differential equations and mathematical physics, which have scientific and practical application of fundamental sciences.

In the process of learning the course of differential equations, we studied the methods of solving differential equations with special forms. These methods cannot cover many other situations. That is why the search for universal methods that do not depend on the form of the equation [2-3].

Methods

The development of computing machines has made it possible to successfully apply almost any number of methods. Let's start with the Cauchy problem for first-order differential equations.

Let's say

$$y' = f(x,y), \quad x_0 \le x \le b \qquad (1)$$

Differential equation in such form

$$y(x_0) = y_0 \qquad (2)$$

Let the problem of finding a solution satisfying the initial condition, that is, the Cauchy problem, be given. In general, the Cauchy problem cannot always be found. There are ways to find the general solution of (1) only in certain representations of the function $f(x,y)$. In many cases, methods of approximate solution of differential equations are used in practical problems. It is assumed that the conditions of the theorem about the existence and uniqueness of the solution are fulfilled. Let the function $f(x,y)$ around the point $M_0(x_0; y_0)$ be continuous in x

and satisfy the Lipshis condition in y.

Euler's method: (1)-(2) We expand the solution of the Cauchy problem $y(x)$ into a Taylor series around the point x_0:

$$y(x) = y + (x - x_0)y'(x_0) + \frac{(x - x_0)^2}{2!}y''(x_0) + \frac{(x - x_0)^3}{3!}y'''(x_0) + \ldots (3)$$

We take the first two terms of the Taylor series around the point $x_0$and discard the remaining terms, as a result we arrive at the following approximate formula

$$y(x) \approx y_0 + (x - x_0)y' \quad (4)$$

if we use the form of y'in formula (1), then formula (4) can be written in the following form:

$$y(x) \approx y_0 + (x - x_0)f(x_0; y_0) \quad (5)$$

To generalize the formula (5) to the interval $x_0 \le x \le b$, we divide this interval into n parts. Splitting step:

$$h = \frac{b - x_0}{n}; \quad x_i = x_0 + ih, \; i = 0,1,2,3,\ldots,n$$

We aim to find the solution of the problem in the form of a table at points x_i. We find the approximate values of $y(x_i)$ according to formula (5):

$$y_{i+1} \approx y_i + h \cdot f(x_i, y_i) \qquad i = 0,1,2,3,\ldots,n-1 \quad (6)$$

where $y_{i+1} = y(x_{i+1}),\ y_i = y(x_i)$. This formula is called Euler's method. Euler's method is a universal method and does not depend on the form of $f(x, y)$, but the error is relatively large. The error at each step is of the order of $0(h^2)$, and this error increases step-by-step until the error can increase to $0(h)$ until we reach point b. In the coordinate plane (x_0, y_0); (x_1, y_1); … , (x_n, y_n) the broken line formed by connecting the points with straight line segments is the graph of the integral curve. The differential equations branch of mathematics originated from solving practical problems. Therefore, creating universal methods similar to formula (6) has been a constant problem. Representatives of various fields have also tried to solve this problem. A clear example of this is the Runge-Kutta method, which was created by scientists, one physicist and one astronomer. (3) served as the basis for the creation of this method. Unlike Euler's method (6), they used five terms instead of two. In addition, in the Runge-Kutta method, formulas that do not require the calculation of derivatives included in the series (3) were proposed. Without dwelling on the theoretical origin of these formulas, we will dwell on working formulas.

Heun‘s method: Heun's method is one of the simplest forms of Runge-Kutta methods, and it improves on Euler's method. It is sometimes called the 'improved Euler's method'. This method is used to find approximate solutions to differential equations:

$$y' = f(x, y), \; y(x_0) = y_0$$

Heun usuli quyidagi bosqichlarda bajariladi:

1. First guess (Euler stage):

$$\tilde{y} = y_n + h \cdot f(x_n, y_n)$$

2. Next value:

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_n, \tilde{y})]$$

Given: $y' = f(x, y)$, $y(x_0) = y_0$, step h.

For each stage n = 0,1,...:

1) $k_1 = f(x_n, y_n)$
2) Predictor: $\tilde{y}_{n+1} = y_n + h \cdot k_1$
3) $k_2 = f(x_n + h, \tilde{y}_{n+1})$
4) Corrector: $y_{n+1} = y_n + (h/2) \cdot (k_1 + k_2)$
5) $x_{n+1} = x_n + h$

Heun's method gives a more accurate approximation than Euler's method.

Runge-Kutte method:

Approximate solution of the Cauchy problem in the initial conditions of the differential equation $y' = f(x, y),\ x_0 \le x \le b,\ y = (x_0) = y_0$ by the Runge-Kutta method was carried out with the following formulas.

$$\begin{cases} K_1 = h \times f(x_i, y_i) \\ K_2 = h \times f\left(x_i + \frac{h}{2}, y_i + \frac{K_1}{2}\right) \\ K_3 = h \times f\left(x_i + \frac{h}{2}, y_i + \frac{K_2}{2}\right) \\ K_4 = h \times f(x_i + h,, y_i + K_3) \\ y_{i+1} = y_i + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4); \\ \quad i = 0,1,2,...,n-1. \end{cases}$$

Example 1. $y' = x^2 + y = f(x,y);\ y(0) = 0{,}3$ Find the approximate value of the solution satisfying the initial condition $x = 0{,}3$ using the Euler, Heun and Runge-Kutta methods and compare it with the real solution. Here $h = 0{,}1$.

Solving. Euler's method:

$x_0 = 1; \quad x_i + i \cdot h = 1 + i \cdot 0.2; \ y_0 = (1) = 2$

$y_1 = y_0 + h * f(x_0, y_0) = 2 + 0.2 * (1^2 + 2) = 2.6$

$y_2 = y_1 + h * f(x_1, y_1) = 2.6 + 0.2 * (1.2^2 + 2.6) = 3.408$

$y_3 = y_2 + h * f(x_2, y_2) = 3.408 + 0.2 * (1.4^2 + 3.408) = 4.4816$

$y_4 = y_3 + h * f(x_3, y_3) = 4.4816 + 0.2 * (1.6^2 + 4.4816) = 5.890$

$y_5 = y_4 + h * f(x_4, y_4) = 5.890 + 0.2 * (1.8^2 + 5.890) = 7.716$

Thus, the approximate solution of the differential equation by Euler's method is shown in the following table:

x_i	1	1.2	1.4	1.6	1.8	2.0
y_i	2	2.6	3.408	4.4816	5.890	7.716

Heun‘s method:

Values calculated using the Heun method (from 0 to 1):

$$\begin{cases} k_1 = 0^2 + 0.3 = 0.3 \\ \tilde{y}_1 = 0.3 + 0.1 \cdot 0.3 = 0.33 \\ k_2 = 0.1^2 + 0.33 = 0.01 + 0.33 = 0.34 \\ y_1 = 0.3 + 0.05 \cdot 0.64 = 0.332 \\ x_1 = 0 + 0.1 = 0.1 \end{cases}$$

$$\begin{cases} k_1 = 0.1^2 + 0.332 = 0.342 \\ \tilde{y}_2 = 0.332 + 0.1 \cdot 0.342 = 0.3662 \\ k_2 = 0.2^2 + 0.3662 = 0.4062 \\ y_2 = 0.332 + 0.05 \cdot 0.7482 = 0.36941 \\ x_2 = 0.1 + 0.1 = 0.2 \end{cases}$$

$$\begin{cases} k_1 = 0.2^2 + 0.36941 = 0.40941 \\ \tilde{y}_3 = 0.36941 + 0.1 \cdot 0.40941 = 0.410351 \\ k_2 = 0.3^2 + 0.410351 = 0.500351 \\ y_3 = 0.36941 + 0.05 \cdot 0.909761 = 0.414898 \\ x_3 = 0.2 + 0.1 = 0.3 \end{cases}$$

$$\begin{cases} k_1 = 0.3^2 + 0.414898 = 0.504898 \\ \tilde{y}_4 = 0.414898 + 0.1 \cdot 0.504898 = 0.4653878 \\ k_2 = 0.4^2 + 0.4653878 = 0.6253878 \\ y_4 = 0.414898 + 0.05 \cdot 1.1302858 = 0.471412 \\ x_4 = 0.3 + 0.1 = 0.4 \end{cases}$$

$$\begin{cases} k_1 = 0.4^2 + 0.471412 = 0.631412 \\ \tilde{y}_5 = 0.471412 + 0.1 \cdot 0.631412 = 0.534553 \\ k_2 = 0.5^2 + 0.534553 = 0.784553 \\ y_5 = 0.471412 + 0.05 \cdot 1.415965 = 0.542211 \\ x_5 = 0.4 + 0.1 = 0.5 \end{cases}$$

$$\begin{cases} k_1 = 0.5^2 + 0.542211 = 0.792211 \\ \tilde{y}_6 = 0.542211 + 0.1 \cdot 0.792211 = 0.621432 \\ k_2 = 0.6^2 + 0.621432 = 0.981432 \\ y_6 = 0.542211 + 0.05 \cdot 1.773643 = 0.630893 \\ x_6 = 0.5 + 0.1 = 0.6 \end{cases}$$

$$\begin{cases} k_1 = 0.6^2 + 0.630893 = 0.990893 \\ \tilde{y}_7 = 0.630893 + 0.1 \cdot 0.990893 = 0.7299823 \\ k_2 = 0.7^2 + 0.7299823 = 1.2199823 \\ y_7 = 0.630893 + 0.05 \cdot 2.2108753 = 0.741436 \\ x_7 = 0.6 + 0.1 = 0.7 \end{cases}$$

$$\begin{cases} k_1 = 0.7^2 + 0.741436 = 1.231436 \\ \tilde{y}_8 = 0.741436 + 0.1 \cdot 1.231436 = 0.8645796 \\ k_2 = 0.8^2 + 0.8645796 = 1.5045796 \\ y_8 = 0.741436 + 0.05 \cdot 2.7360156 = 0.878237 \\ x_8 = 0.7 + 0.1 = 0.8 \end{cases}$$

$$\begin{cases} k_1 = 0.8^2 + 0.878237 = 1.518237 \\ \tilde{y}_9 = 0.878237 + 0.1 \cdot 1.518237 = 1.0300607 \\ k_2 = 0.9^2 + 1.0300607 = 1.8400607 \\ y_9 = 0.878237 + 0.05 \cdot 3.3583477 = 1.046152 \\ x_8 = 0.8 + 0.1 = 0.9 \end{cases}$$

$$\begin{cases} k_1 = 0.9^2 + 1.046152 = 1.856152 \\ \tilde{y}_{10} = 1.046152 + 0.1 \cdot 1.856152 = 1.231767 \\ k_2 = 1^2 + 1.231767 = 2.231767 \\ y_{10} = 1.046152 + 0.05 \cdot 4.087919 = 1.250548 \\ x_8 = 0.9 + 0.1 = 1 \end{cases}$$

n	x_n	$y_{n(Heun)}$	y(x) clear	mistake
0	0.000000	0.300000	0.300000	0.000000
1	0.100000	0.332000	0.331893	0.000107
2	0.200000	0.369410	0.369226	0.000184
3	0.300000	0.414898	0.414675	0.000223
4	0.400000	0.471412	0.471197	0.000216
5	0.500000	0.542211	0.542059	0.000152
6	0.600000	0.630893	0.630873	0.000020
7	0.700000	0.741436	0.741631	0.000195
8	0.800000	0.878237	0.878744	0.000507
9	0.900000	1.046152	1.047087	0.000935
10	1.000000	1.250548	1.252048	1.001500

Runge-Kutta method:

In $y' = x^2 + y = f(x,y); \quad y = (1) = 2, \quad [1;2]$, solve approximately with steps $h = 0.2$:

$$\begin{cases} K_1 = h \cdot f(x_0, y_0) = 0.2*(1^2+2) = 0.6 \\ K_2 = h \cdot f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = 0.2*\left((1+\frac{0.2}{2})^2 + 2 + \frac{0.6}{2}\right) = 0.702 \\ K_3 = h \cdot f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = 0.2*\left((1+\frac{0.2}{2})^2 + 2 + \frac{0.702}{2}\right) = 0.7122 \\ K_4 = h \cdot f(x_0 + h, y_0 + K_3) = 0.2*\left((1+0.2)^2 + 2 + 0.7122\right) = 0.83044 \\ y_1 = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = \\ 2 + \frac{1}{6}(0.6 + 2*0.702 + 2*0.7122 + 0.83044) = 2.71; \end{cases}$$

$$\begin{cases} K_1 = h \cdot f(x_1, y_1) = 0.2*(1^2+2.71) = 0.83 \\ K_2 = h \cdot f\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right) = 0.2*\left((1.2+\frac{0.2}{2})^2 + 2.71 + \frac{0.83}{2}\right) = 0.963 \\ K_3 = h \cdot f\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right) = 0.2*\left((1.2+\frac{0.2}{2})^2 + 2.71 + \frac{0.963}{2}\right) = 0.9763 \\ K_4 = h \cdot f(x_1 + h, y_1 + K_3) = 0.2*\left((1.2+0.2)^2 + 2.71 + 0.9763\right) = 0.987 \\ y_2 = y_1 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = \\ 2 + \frac{1}{6}(0.83 + 2*0.963 + 2*0.9763 + 0.987) = 2.95; \end{cases}$$

$$\begin{cases} K_1=h\cdot f(x_2,y_2)=0.2*\left(1.4^2+2.95\right)=0.982 \\ K_2=h\cdot f\left(x_2+\frac{h}{2},y_2+\frac{K_1}{2}\right)=0.2*\left((1.4+\frac{0.2}{2})^2+2.95+\frac{0.982}{2}\right)=1.1382 \\ K_3=h\cdot f\left(x_2+\frac{h}{2},y_2+\frac{K_2}{2}\right)=0.2*\left((1.4+\frac{0.2}{2})^2+2.95+\frac{1.1382}{2}\right)=1.154 \\ K_4=h\cdot f(x_2+h,y_2+K_3)=0.2*\left((1.4+0.2)^2+2.95+1.154\right)=1.3328 \\ y_3=y_2+\frac{1}{6}(K_1+2K_2+2K_3+K_4)= \\ 2.95+\frac{1}{6}(0.982+2*1.1382+2*1.154+1.3328)=4.1; \end{cases}$$

$$\begin{cases} K_1=h\cdot f(x_3,y_3)=0.2*\left(1.6^2+4.1\right)=1.332 \\ K_2=h\cdot f\left(x_3+\frac{h}{2},y_3+\frac{K_1}{2}\right)=0.2*\left((1.6+\frac{0.2}{2})^2+4.1+\frac{1.332}{2}\right)=1.531 \\ K_3=h\cdot f\left(x_3+\frac{h}{2},y_3+\frac{K_2}{2}\right)=0.2*\left((1.6+\frac{0.2}{2})^2+4.1+\frac{1.531}{2}\right)=1.551 \\ K_4=h\cdot f(x_3+h,y_3+K_3)=0.2*\left((1.6+0.2)^2+4.1+1.551\right)=1.778 \\ y_4=y_3+\frac{1}{6}(K_1+2K_2+2K_3+K_4)= \\ 4.1+\frac{1}{6}(1.332+2*1.531+2*1.551+1.778)=5.65; \end{cases}$$

$$\begin{cases} K_1=h\cdot f(x_4,y_4)=0.2*\left(1.8^2+5.65\right)=1.778 \\ K_2=h\cdot f\left(x_4+\frac{h}{2},y_4+\frac{K_1}{2}\right)=0.2*\left((1.8+\frac{0.2}{2})^2+5.65+\frac{1.778}{2}\right)=2.03 \\ K_3=h\cdot f\left(x_4+\frac{h}{2},y_4+\frac{K_2}{2}\right)=0.2*\left((1.8+\frac{0.2}{2})^2+5.65+\frac{2.03}{2}\right)=2.055 \\ K_4=h\cdot f(x_4+h,y_4+K_3)=0.2*\left((1.8+0.2)^2+5.65+2.055\right)=2.341 \\ y_5=y_5+\frac{1}{6}(K_1+2K_2+2K_3+K_4)= \\ 5.65+\frac{1}{6}(1.778+2*2.03+2*2.055+2.341)=7.698; \end{cases}$$

Thus, the approximate solution of the differential equation using the Runge-Kutta method is as follows:

x_i	1	1.2	1.4	1.6	1.8	2.0
y_i	2	2.71	2.95	4.1	5.65	7.698

For clarity, we also present the solution of the differential equation found analytically:
Solution:

$$y' = x^2 + y, \quad y(1) = 2$$

$$y' = x^2, \quad e^{\int -dx} = e^{-x}$$

$$y'e^{-x} - e^{-x}y = x^2e^{-x} \Rightarrow \left(ye^{-x}\right)' = x^2e^{-x}dx$$

$$y'e^{-x} = \int x^2e^{-x}dx = \begin{vmatrix} u=x^2 & dv=e^{-x}dx \\ du=2xdx & v=-e^{-x} \end{vmatrix} = -x^2e^{-x} + 2\int xe^{-x}dx$$

$$= -x^2e^{-x} + \int xe^{-x}dx = \begin{vmatrix} u=x & dv=e^{-x}dx \\ du=dx & v=-e^{-x} \end{vmatrix}$$

$$= -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C \Rightarrow$$

Now according to the value $y(1) = 2$:

$$2 = Ce - 1 - 2 - 2 \Rightarrow C = \frac{7}{2}$$

$$y = -x^2 - 2x + 7e^{x-1} - 2$$

Solution of a differential equation when solved analytically

$$y = (x) = -x^2 - 2x + 7e^{x-1} - 2$$

we can see that the approximate values we found are very close fits compared to the original solution values.

Example 2. Find the approximate value satisfying the initial conditio $y' = y - x,\ \ y(0) = 1,\ \ [0;3],\ \ h = 0,2$ by Euler, Heun and Runge-Kutta method and compare with the real solution.

Euler's method:

$$y_{i+1} \approx y_i + h \cdot f(x_i, y_i)\ \ i = 0, 1, 2, 3, ..., n-1$$

$$x_0 = 0,\ y_0 = 1,\ x_i = x_0 + 0.2 \cdot i = 0 + 0.2 \cdot i$$

$$y_1 = 1 + 0.2(1-0) = 1 + 0.2 = 1.2; \quad x_1 = 0.2$$

$$y_2 = 1.2 + 0.2(1.2-0.2) = 1.2 + 0.2 = 1.4; \quad x_2 = 0.4$$

$$y_3 = 1.4 + 0.2(1.4-0.4) = 1.4 + 0.2 = 1.6; \quad x_3 = 0.6$$

$$y_4 = 1.6 + 0.2(1.6-0.6) = 1.6 + 0.2 = 1.8; \quad x_4 = 0.8$$

$$y_5 = 1.8 + 0.2(1.8-0.8) = 1.8 + 0.2 = 2; \quad x_5 = 1$$

$$y_6 = 2 + 0.2(2-1) = 2 + 0.2 = 2.2; \quad x_6 = 1.2$$

$$y_7 = 2.2 + 0.2(2.2-1.2) = 2.2 + 0.2 = 2.4; \quad x_7 = 1.4$$

$$y_8 = 2.4 + 0.2(2.4-1.4) = 2.4 + 0.2 = 2.6; \quad x_8 = 1.6$$

$$y_9 = 2.6 + 0.2(2.6-1.6) = 2.6 + 0.2 = 2.8; \quad x_9 = 1.8$$

$$y_{10} = 2.8 + 0.2(2.8-1.8) = 2.8 + 0.2 = 3; \quad x_{10} = 2$$

$$y_{11} = 3 + 0.2(3-2) = 3 + 0.2 = 3.2; \quad x_{11} = 2.2$$

$$y_{12} = 3.2 + 0.2(3.2-2.2) = 3.2 + 0.2 = 3.4; \quad x_{12} = 2.4$$

$$y_{13} = 3.4 + 0.2(3.4 - 2.4) = 3.4 + 0.2 = 3.6; \quad x_{13} = 2.6$$

$$y_{14} = 3.6 + 0.2(3.6 - 2.6) = 3.6 + 0.2 = 3.8; \quad x_{14} = 2.8$$

$$y_{15} = 3.8 + 0.2(3.8 - 2.8) = 3.8 + 0.2 = 4; \quad x_{15} = 3$$

$$y_{16} = 4 + 0.2(4 - 3) = 4 + 0.2 = 1.2; \quad x_{16} = 4.2$$

x_i	0	0.2	0.4	0.6	0.8	1	1.2	1.4
y_i	1.2	1.4	1.6	1.8	2	2.2	2.4	2.6
x_i	1.6	1.8	2	2.2	2.4	2.6	2.8	3
y_i	2.8	3	3.2	3.4	3.6	3.8	4	4.2

Heun's method:

Values calculated using the Heun method (from 0 to 1):

$$\begin{cases} k_1 = 1 - 0 = 1 \\ \tilde{y}_1 = 1 + 0.2 \cdot 1 = 1.2 \\ k_2 = 1.2 - 0.2 = 1 \\ y_1 = 1 + 0.1 \cdot 2 = 1.2 \\ x_1 = 0 + 0.2 = 0.2 \end{cases}$$

$$\begin{cases} k_1 = 1.4 - 0.4 = 1 \\ \tilde{y}_3 = 1.4 + 0.2 \cdot 1 = 1.6 \\ k_2 = 1.6 - 0.6 = 1 \\ y_3 = 1.4 + 0.1 \cdot 2 = 1.6 \\ x_3 = 0.4 + 0.2 = 0.6 \end{cases}$$

$$\begin{cases} k_1 = 1.6 - 0.6 = 1 \\ \tilde{y}_4 = 1.6 + 0.2 \cdot 1 = 1.8 \\ k_2 = 1.8 - 0.8 = 1 \\ y_4 = 1.6 + 0.1 \cdot 2 = 1.8 \\ x_4 = 0.6 + 0.2 = 0.8 \end{cases}$$

$$\begin{cases} k_1 = 1.8 - 0.8 = 1 \\ \tilde{y}_5 = 1.8 + 0.2 \cdot 1 = 2 \\ k_2 = 2 - 1 = 1 \\ y_5 = 1.8 + 0.1 \cdot 2 = 2 \\ x_5 = 0.8 + 0.2 = 1 \end{cases}$$

$$\begin{cases} k_1 = 2 - 1 = 1 \\ \tilde{y}_6 = 2 + 0.2 \cdot 1 = 2.2 \\ k_2 = 2.2 - 1.2 = 1 \\ y_6 = 2 + 0.1 \cdot 2 = 2.2 \\ x_6 = 1 + 0.2 = 1.2 \end{cases}$$

$$\begin{cases} k_1 = 2.2 - 1.2 = 1 \\ \tilde{y}_7 = 2.2 + 0.2 \cdot 1 = 2.4 \\ k_2 = 2.4 - 1.4 = 1 \\ y_7 = 2.2 + 0.1 \cdot 2 = 2.4 \\ x_7 = 1.2 + 0.2 = 1.4 \end{cases}$$

$$\begin{cases} k_1 = 2.4 - 1.4 = 1 \\ \tilde{y}_8 = 2.4 + 0.2 \cdot 1 = 2.6 \\ k_2 = 2.6 - 1.6 = 1 \\ y_8 = 2.4 + 0.1 \cdot 2 = 2.6 \\ x_8 = 1.4 + 0.2 = 1.6 \end{cases}$$

$$\begin{cases} k_1 = 2.6 - 1.6 = 1 \\ \tilde{y}_9 = 2.6 + 0.2 \cdot 1 = 2.8 \\ k_2 = 2.8 - 1.8 = 1 \\ y_9 = 2.6 + 0.1 \cdot 2 = 2.8 \\ x_9 = 1.6 + 0.2 = 1.8 \end{cases}$$

$$\begin{cases} k_1 = 2.8 - 1.8 = 1 \\ \tilde{y}_{10} = 2.8 + 0.2 \cdot 1 = 3 \\ k_2 = 3 - 2 = 1 \\ y_{10} = 2.8 + 0.1 \cdot 2 = 3 \\ x_{10} = 1.8 + 0.2 = 2 \end{cases}$$

$$\begin{cases} k_1 = 3 - 2 = 1 \\ \tilde{y}_{11} = 3 + 0.2 \cdot 1 = 3.2 \\ k_2 = 3.2 - 2.2 = 1 \\ y_{11} = 3 + 0.1 \cdot 2 = 3.2 \\ x_{11} = 2 + 0.2 = 2.2 \end{cases}$$

$$\begin{cases} k_1 = 3.2 - 2.2 = 1 \\ \tilde{y}_{12} = 3.2 + 0.2 \cdot 1 = 3.4 \\ k_2 = 3.4 - 2.4 = 1 \\ y_{12} = 3.2 + 0.1 \cdot 2 = 3.4 \\ x_{12} = 2.2 + 0.2 = 2.4 \end{cases}$$

$$\begin{cases} k_1 = 3.4 - 2.4 = 1 \\ \tilde{y}_{13} = 3.4 + 0.2 \cdot 1 = 3.6 \\ k_2 = 3.6 - 2.6 = 1 \\ y_{13} = 3.4 + 0.1 \cdot 2 = 3.6 \\ x_{13} = 2.4 + 0.2 = 2.6 \end{cases}$$

$$\begin{cases} k_1 = 3.6 - 2.6 = 1 \\ \tilde{y}_{14} = 3.6 + 0.2 \cdot 1 = 3.6 \\ k_2 = 3.8 - 2.8 = 1 \\ y_{14} = 3.6 + 0.1 \cdot 2 = 3.8 \\ x_{14} = 2.6 + 0.2 = 2.8 \end{cases}$$

$$\begin{cases} k_1 = 3.8 - 2.8 = 1 \\ \tilde{y}_{15} = 3.8 + 0.2 \cdot 1 = 3.8 \\ k_2 = 4 - 3 = 1 \\ y_{15} = 3.8 + 0.1 \cdot 2 = 4 \\ x_{15} = 2.8 + 0.2 = 3 \end{cases}$$

$$\begin{cases} k_1 = 4 - 3 = 1 \\ \tilde{y}_{16} = 4 + 0.2 \cdot 1 = 4.2 \\ k_2 = 4.2 - 3.2 = 1 \\ y_{16} = 4 + 0.1 \cdot 2 = 4.2 \end{cases}$$

x_i	0	0.2	0.4	0.6	0.8	1	1.2	1.4
y_i	1.2	1.4	1.6	1.8	2	2.2	2.4	2.6
x_i	1.6	1.8	2	2.2	2.4	2.6	2.8	3
y_i	2.8	3	3.2	3.4	3.6	3.8	4	4.2

Renge-Kutte method:

$$\begin{cases} K_1 = h \cdot f(x_i, y_i) \\ K_2 = h \cdot f\left(x_i + \frac{h}{2}, y_i + \frac{K_1}{2}\right) \\ K_3 = h \cdot f\left(x_i + \frac{h}{2}, y_i + \frac{K_2}{2}\right) \\ K_4 = h \cdot f(x_i + h, y_i + K_3) \\ y_{i+1} = y_i + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4); \end{cases}$$

$$i = 0, 1, 2, ..., \text{n}-1.$$

$$i = 0 \quad at \ x_0 = 0; \ y_0 = 1$$

$$K_1 = h \cdot f(x_0, y_0) = 0.2(1+0) = 0.2$$

$$K_2 = h \cdot f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = 0.2(1 + 0.1 - 0.1) = 0.2$$

$$K_3 = h \cdot f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = 0.2(0.1;1.1) = 0.2(1.2 - 0.2) = 0.2$$

$$K_4 = h \cdot f(x_i + h, y_i + K_3) = 0.2(0.2;1.2) = 0.2(1.2 - 0.2) = 0.2$$

$$\text{y}_1 = y_i + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 1 + \frac{1}{6}(0.2 + 0.4 + 0.4 + 0.2) =$$

$$= 1 + \frac{1}{6} \cdot 1.2 = 1.2; \quad x_1 = 0.2:$$

$$i = 1 \ \ at \ \ x_1 = 0.2; \quad y_1 = 1.2$$

$$K_1 = h \cdot f(x_1, y_1) = 0.2(1+0) = 0.2$$

$$K_2 = h \cdot f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = 0.2(1 + 0.1 - 0.1) = 0.2$$

$$K_3 = h \cdot f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = 0.2(0.1;1.1) = 0.2(1.2 - 0.2) = 0.2$$

$$K_4 = h \cdot f(x_i + h, y_i + K_3) = 0.2(0.2;1.2) = 0.2(1.2 - 0.2) = 0.2$$

$$\text{y}_1 = y_i + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 1 + \frac{1}{6}(0.2 + 0.4 + 0.4 + 0.2) =$$

$$= 1 + \frac{1}{6} \cdot 1.2 = 1.2; \quad x_1 = 0.2:$$

$$i = 2 \;\; at \;\; x_1 = 0.2; \;\; y_1 = 1.2$$

$$K_1 = h \cdot f\left(x_0, y_0\right) = 0.2\left(1+0\right) = 0.2$$

$$K_2 = h \cdot f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = 0.2\left(1+0.1-0.1\right) = 0.2$$

$$K_3 = h \cdot f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = 0.2\left(0.1;1.1\right) = 0.2\left(1.2-0.2\right) = 0.2$$

$$K_4 = h \cdot f\left(x_i + h, y_i + K_3\right) = 0.2\left(0.2;1.2\right) = 0.2\left(1.2-0.2\right) = 0.2$$

$$\mathrm{y}_{i+1} = y_i + \frac{1}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right) = 1 + \frac{1}{6}\left(0.2+0.4+0.4+0.2\right) =$$

$$= 1 + \frac{1}{6} \cdot 1.2 = 1.2; \quad x_1 = 0.2:$$

Now we find an analytical solution:

Solution:

$$y' = y - x; \qquad y' - y = -x$$

$y' + P\left(x\right)y = Q\left(x\right)$ - linear differential equation.

$y = u\left(x\right) \cdot v\left(x\right)$ -form we look for a solution

$$y' = u' \cdot v + u \cdot v'$$

$$u' \cdot v + u \cdot v' - u \cdot v = -x$$

$$u' \cdot v + u\left(v' - v\right) = -x$$

$$v' - v = 0$$

$$\frac{dv}{dx} - v = 0, \; \frac{dv}{v} - dx = 0, \; \frac{dv}{v} = dx \Rightarrow \int \frac{dv}{v} = \int dx$$

$$lnv = x; \quad v = e^x$$

$$u'v = -x \Rightarrow u'e^x = -x$$

$$u' = -xe^{-x}$$

$$u = -\int xe^{-x}dx + C = \begin{bmatrix} u=x & du=dx \\ dv=e^{-x} & v=-e^{-x} \end{bmatrix} =$$

$$= xe^{-x} - \int e^{-x}dx + C = xe^{-x} + e^{-x} + C.$$

$$y = u \cdot v = e^x\left(xe^{-x} + e^{-x} + C\right) = x + 1 + Ce^x$$

$$y = \left(0\right) = 1$$

$$1 = 0 + 1 + C$$

$$C = 0$$

Hence, the particular solution of the given differential equation

$$y = x + 1$$

Conclusion

From this, we can see that the corresponding values of the approximate solution found by Euler's method and the solution found by the analytical method are very close to each other.

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