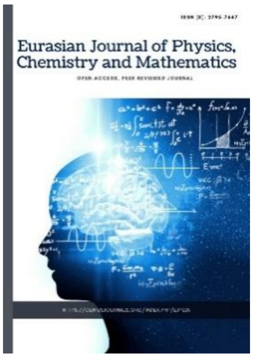

About Two-Dimensional QNSO

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ABSTRACT

In the world, many scientific and practical problems in dynamical systems are modeled by evolutionary operators of stochastic and non-stochastic cubic matrices. In many biological and physical systems their behaviour can be given by dynamics of quadratic (non-)stochastic operators defined by cubic matrices. A quadratic stochastic process is defined by a family of cubic matrices which satisfy Kolmogorov-Chapman equation with respect to a fixed multiplication and stochastisity of such matrices. Therefore, the study of the dynamics of nonlinear operators constructed using stochastic and non-stochastic cubic matrices that preserve simplex remains one of the important and urgent tasks.

In the book [12] entitled "Population dynamics: algebraic and probabilistic approach" which is written by U. A.Rozikov in 2020, the probabilistic approach part presents Markov processes of cubic stochastic (in a fixed sense) matrices (MPCSM), which are continuous-time dynamical systems, whose states are stochastic cubic matrices satisfying an analogue of the Kolmogorov-Chapman equation. MPCSM considered for two specially chosen notions of stochastic cubic matrices and two Maksimov's multiplications of such matrices.Time-dependent behavior of such processes is given with applications to a population with the possibility of twin births.

Nowadays in the world, the theory of nonlinear dynamical systems of mathematics is used as the main tool for solving many practical problems. The study of the dynamics of quadratic stochastic and non-stochastic operators generated by cubic matrices leads to many problems due to the non-linearity of the operator. In particular, targeted scientific research includes finding fixed points and determining their types, describing the set of periodic points and determining their type, and describing the set of limit points of trajectories.

The study of the dynamics of modern quadratic operators is reduced to the multiplication of cubic matrices, the main problem here is that the general rule for the multiplication of cubic matrices has not yet been determined. Nevertheless, many significant results have been achieved today. As the main tasks and areas of activity of mathematics, the conduct of scientific research at the level of international standards in the priority areas of functional analysis, mathematical physics and statistical physics are determined. It is important to develop the theory of piecewise smooth dynamical systems in order to use scientific results in related fields of science to ensure the implementation of the solution.

Keywords:	dynamical systems, evolutionary operators, stochastic cubic matrices, non-stochastic cubic matrices

Let $I=\{1,2,\dots,m\}$. A *distribution* (or *state*) of the set I is a probability measure $x=(x_1,\dots,x_m)$, i.e. an element of the simplex:

$$S^{m-1}=\left\{x\in R^m : x_i\geq 0, \sum_{i=1}^m x_i=1\right\}. \quad (1)$$

The quadratic stochastic operator (QSO) is a mapping of the simplex S^{m-1} (see for [1] more details) into itself, of the form

$$V: x'_k=\sum_{i,j=1}^m P_{ij,k}x_ix_j,\ k=1,\dots,m \quad (2)$$

where $P_{ij,k}$ are coefficients of heredity and

$$P_{ij,k}\geq 0,\ P_{ij,k}=P_{ji,k}, \sum_{k=1}^m P_{ij,k}=1,\ i,j,k=1,\dots,m, \quad (3)$$

Thus, each quadratic stochastic operator V can be uniquely defined by a cubic matrix $\mathbf{P}=\left(P_{ij,k}\right)_{i,j,k=1}^n$ with conditions (3).

For a given $x^{(0)}\in S^{m-1}$ the trajectory (orbit)

$$\{x^{(n)}\},\ n=0,1,2,\dots \text{ of } x^{(0)}$$

under the action of QSO (1.1.3) is defined by

$$x^{(n+1)}=V(x^{(n)}),\ \ n=0,1,2,\dots.$$

One of the main problems in mathematical biology consists in the study of the asymptotical behavior of the trajectories.

Let us give main definitions (see [2]).

Let f be a function defined on some topological space X.

- Denote $f^n(x)$, meaning f is applied to $x\in X$ iteratively n times.
- Let A be a subset of X. Then $f(A)=\{f(x): x\in A\}$.
- If $f(A)\subset A$, then A is an invariant set under function f.

Topologically transitive: A continuous map $f: X\to X$ is said to be topologically transitive if, for every pair of non-empty open sets $A,B\subset X$, there exists an integer n such that $f^n(A)\cap B\neq\varnothing.$

Sensitivity to initial condition. Let X be a metric space and $f: X \to X$ be a map. f has sensitive dependence on initial conditions if there exists $\delta > 0$ such that, for any $x \in X$ and for any $r > 0$ there exists $y \in B_r(x)$ and $n \in N$ such that $f^n(y) \notin B_\delta(f^n(x))$.

Periodic orbits. Let X be a topological space and $f: X \to X$ be a map. The point $x \in X$ is a fixed point for f if $f(x) = x$. The point $x \in X$ is a periodic point of period m for f if $f^m(x) = x$. The least positive m for which $f^m(x) = x$ is called the prime period of x [1]. We denote the set of all periodic points of (not necessarily prime) period m by $Per_m(f)$, and the set of all fixed points by $Fix(f)$. The set of all iterates of a periodic point form a periodic orbit.

To date, a rigorous mathematical definition of Chaos has not been agreed upon. However, many literatures accept the definition proposed by Devaney.

Chaos. Devaney's definition (see more details [3]) of chaos is stated as follows:

Definition 1. [2]. *A continuous map f is chaotic if f has an invariant set $A \subset X$ such that*

1) f satisfies sensitive dependence on its initial conditions on A,

2) The set of points initiating periodic orbits are dense in A, i.e., every point in the space is approached arbitrarily closely by periodic orbits.

3) f is topologically transitive on A.

In [4] it was observed that sensitive dependence on initial conditions follows as a mathematical consequence of the other two properties.

Definition 2. *A quadratic operator* (2), *preserving a simplex, is called non-stochastic (QnSO) if at least one of its coefficients $P_{ij,k}, i \neq j$ is negative.*

The following theorem gives conditions for coefficients of V to preserve the simplex.

Theorem 1. *For a quadratic operator V (given by* (1.1.2)), *to preserve a simplex S^{m-1} it is* ***sufficient*** *that*

i) $\sum_{k=1}^{m} P_{ij,k} = 1,\ i, j = 1, \dots, m;$

ii) $0 \le P_{ii,k} \le 1,\ i, k = 1, \dots, m;$

iii) $-\frac{1}{m-1}\sqrt{P_{ii,k}P_{jj,k}} \le P_{ij,k} \le 1 + \sqrt{(1-P_{ii,k})(1-P_{jj,k})}$

and ***necessary*** *that the conditions* (i), (ii) *and*

iii') $-\sqrt{P_{ii,k}P_{jj,k}} \le P_{ij,k} \le 1 + \sqrt{(1-P_{ii,k})(1-P_{jj,k})}$

are satisfied.

Now we study the following example of QnSO on the 2D-simplex S^2:

$$V:\begin{cases} x' = x^2 + y^2 + z^2 - axy - axz - ayz \\ y' = (2 + a)(x + z)y \\ z' = (2 + a)zx \end{cases} \quad (4)$$

where

$$a \in [-2;\ 2]$$

If this operator W takes a value in the interval a ∈ [0; 2] for the parameter a, then W is a quadratic non-stochastic operator (QnSO) according to Theorem 1.

$$V_0:\begin{cases} x' = x^2 + y^2 + z^2 - axy - axz - ayz \\ y' = (2 + a)(x + z)y \\ z' = (2 + a)zx \end{cases} \quad (2)$$

Fixed points. The fixed points of this operator are solutions to the system V_0 (x; y; z) =(x; y; z): The following lemma gives all fixed points of operator V_0:

Lemma 1. The set of all fixed points of operator (7) is

$$e_1 = (1;\ 0;\ 0);\ e_2 = (\frac{1}{2 + a}, \frac{1 + a}{2 + a}, 0)\ ;\ e_3 = (\frac{1}{2 + a}, 0, \frac{1 + a}{2 + a})$$

D**efinition 2**. A fixed point x^* of a mapping W is called hyperbolic point if its Jacobian JW at x^* has no eigenvalues on the unit circle;

attracting point if all the eigenvalues of the Jacobi matrix $JW\,(x^*)$ are less than 1 in absolute value;

repelling point if all the eigenvalues of the Jacobi matrix $JW\,(x^*)$ are greater than 1 in absolute value;

a **saddle** point otherwise.

The following theorem gives type (see [1] for definitions) of each fixed point of the operator W_2:

Theorem 2. For the operator W_2, the points e_1, e_2, e_3 are fixed points. Moreover,

1) the point e_2 is non-hyperbolic;

2) if $|a \in [0;\ 1)$, then the point e_1 is a repeller, point e_3 is a saddle;

3) if $a = 1$, then $e1$ and e_3 are non-hyperbolic;

4) if $a \in (1;\ 2]$, then $e_1$is a saddle and point e_3 is a repeller.

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