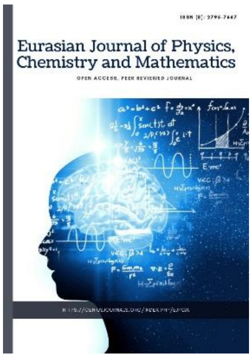


Numerical Range Of The Finite Rank Fredholm Integral Operator

Jasmina T. Husenova
Bukhara State University, Bukhara Uzbekistan
j.t.husenovna@buxdu.uz
ORCID 0009-0003-3365-7922

ABSTRACT

In the present paper we consider the Fredholm integral operator T with, rank $n, n \in N$ in the Hilbert space $L_2[-\pi;\pi]$. Firstly, the numerical range of the Fredholm integral operator $T_k,\ k = 1,2,\dots,n$ of rank one is investigated. Then the operator T is represented as $T = T_1 + T_2 + \dots + T_n$ and its *numerical* range is described by the numerical range of $T_k,\ k = 1,2,\dots,n$ under natural condition. It's given an example of a class of parameter functions where this natural condition were fulfilled.

Keywords: Numerical range, integral operator, kernel, Fredholm operator, parameter function, rank of operator.

1. Introduction.

In mathematics, Fredholm operators are certain operators that arise in the Fredholm theory of integral equations. They are named in honor of Erik Ivar Fredholm.

The operator T acting in the Hilbert space $L_2[-\pi;\pi]$ as

$$(Tf)(x) = \int_{-\pi}^{\pi} K(x,t)f(t)dt \qquad (1)$$

is called a Fredholm integral operator, where the function $K(\cdot,\cdot)$ is defined on $[-\pi;\pi]^2$ and called the kernel function of the Fredholm integral operator.

The simplest kernels are separable (degenerate), which have the form:

$$K(x,y) = \sum_{i=1}^{n} v_i(x)v_i(y). \qquad (2)$$

It is easy to see that the function $K(x,y)$ is a finite sum of separated products.

More complicated kernels are non-separable. Here are spme examples of such kernel functions:

1) $K(x,y) = \frac{1}{x-y}$ – Hilbert transform:
2) $K(x,y) = e^{-ixy}$- Fourier transform;
3) $K(x,y) = e^{-xy}$- Laplace transform.

In models of solid state physics [1,2] and also in lattice quantum field theory [3], one considers two-particle Schroedinger operators, which are lattice analogs of the two-particle Schroedinger operators in the continuous space. In most cases, Friedrichs models corresponding to the two-particle discrete Schroedinger operators are studied. In this case, the non-perturbed operator of the Friedrichs model is the multiplication operator. And the perturbation operator is a Fredholm integral operator with a kernel function of the form (2). In the paper [4] two Friedrichs models with rank two perturbation are considered and using the spectrum of these models the conditions for the existence of the eigenvalues of the lattice three-particle model Hamiltonian are found. In the paper [5] the Friedrichs model with rank two perturbation, related with the two quantum particle system on 3D integer lattice is considered and the number and location of the eigenvalues of this model is investigated. In the papers [6,7] using the spectral properties of the Friedrichs model with finite rank perturbation the essential and discrete spectrum of the lattice model operator associated with the system of three particles are studied.

In the present paper we consider the Fredholm integral operator with a kernel function of the form (2). We describe its numerical range.

2. Fredholm integral operator and its spectrum.

In the Hilbert space $L_2[-\pi;\pi]$ we consider the operator of the form

$$(T_i f)(x) = v_i(x) \int_{-\pi}^{\pi} v_i(t) f(t) dt \qquad (3)$$

for $i = 1,2,\dots,n$. Here the functions $v_i(\cdot)$,$i = 1,2,\dots,n$ are real-valued continuous and linearly independent functions defined on $[-\pi;\pi]$.

We note that the scalar product of the two elements f and g from $L_2[-\pi;\pi]$ is defined by

$$(f,g) = \int_{-\pi}^{\pi} f(t)\overline{g(t)}\, dt.$$

Analogously, the norm of the element $f \in L_2[-\pi;\pi]$ is defined by

$$\|f\| = \left(\int_{-\pi}^{\pi} |f(x)|^2 dx \right)^{\frac{1}{2}}$$

Using these formulas and corresponding definitions one can show that the operator T acting in the Hilbert space $L_2[-\pi;\pi]$ as

$$T = T_1 + T_2 + \dots + T_n$$

is linear, bounded and self-adjoint.

By the construction the equality

$$(Tf)(x) = \sum_{i=1}^{n} v_i(x) \int_{-\pi}^{\pi} v_i(t)\, f(t) dt$$

holds.

Recall that the number $\lambda = 0$ is an eigenvalue of the operator T with infinite multiplicity; an infinite set of (orthogonal) eigenfunctions f_m $(m = 1,2,\dots)$ characterized by

$$0 = (f_m, v_i) = \int_{-\pi}^{\pi} v_i(x) f_m(x) dx\,, \qquad i = 1,2,\dots,n.$$

There are n non-zero eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ which are zeros of the function

$$\Delta(\lambda) \coloneqq \det\left(\lambda\delta_{ij} - (v_i, v_j)\right)_{i,j=1}^{n},$$

where

$$\delta_{i,j} \coloneqq \begin{cases} 1, & if \quad i = j, \\ 0, & if \quad i \neq j. \end{cases}$$

Usually the function $\Delta(\cdot)$ is called a Fredholm determinant associated with the operator T.

For the discrete spectrum of the operator T we have the following equality

$$\sigma_{\text{disc}}(T) = \{\lambda \neq 0\colon\ \Delta(\lambda) = 0\}.$$

Also for the essential spectrum of T we have

$$\sigma_{\text{ess}}(T) = \{0\}.$$

For the sake of convenience in our further research, we require the condition

$$(v_i, v_j) = 0, \ i \neq j, \ \ i, j = 1,2, \dots, n \tag{4}$$

to be fulfilled. Then the function $\Delta(\cdot)$ can be rewritten as a product of the form

$$\Delta(\lambda) = (\lambda - \|v_1\|^2)(\lambda - \|v_2\|^2) \dots \dots \dots (\lambda - \|v_n\|^2).$$

Therefore

$$\sigma_{\text{disc}}(T) = \{ \|v_1\|^2, \|v_2\|^2, \dots, \|v_n\|^2\},$$
$$\sigma(T) = \sigma_{\text{pp}}(T) = \{0, \|v_1\|^2, \|v_2\|^2, \dots, \|v_n\|^2\}.$$

Simple calculations show that

$$\sigma(T_i) = \sigma_{\text{pp}}(T_i) = \{0, \|v_i\|^2\}, \quad i = 1,2, \dots, n.$$

Therefore, if the condition (4) is fulfilled, then we obtain the equality

$$\sigma(T) = \sigma(T_1) \cup \sigma(T_2) \cup \dots \cup \sigma(T_n)$$

for the spectrum of T.

3. The numerical range of T.

In the spectral theory of linear operators, the concept of spectrum is important. In many cases, we are faced with the problem of determining not the spectrum of a linear operator, but the domain where it is located. It is well known from the course of functional analysis that the spectrum of a linear operator lies in the complex plane. If the linear operator A defined in the Hilbert space X is bounded, then the spectrum of the operator A lies in a closed circle with radius $|| A||$ centered at the origin. If A is a self-adjoint operator, then its spectrum lies in the section $[-\big||A|\big|; ||A||]$.

In order to further improve these results, the concept of the numerical representation of a linear operator is introduced. For the reader's convenience when reading the text of the article, we provide some information related to the introduction and study of the concept of digital image.

Let A be the linearly bounded operator defined in the Hilbert space X. The symbols $|| \cdot ||$ and $(\cdot,\cdot)$denote norm and scalar product in the Hilbert space X, respectively.

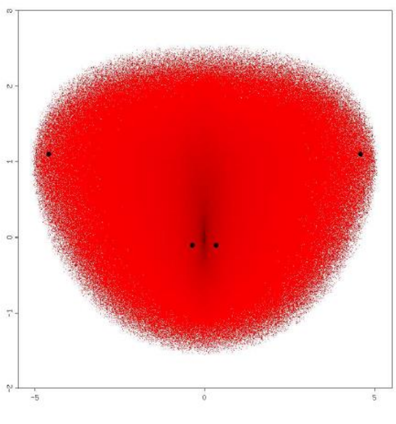
The set of the form

$$W(A) := \{(Ax, x) : x \in X : \big||x|\big| = 1\}$$

is called the numerical range of the operator A. This concept was introduced for the first time in 1918 by Tioplitz in the article [8] for matrices, and it was proved that all the eigenvalues of the matrix, that is, its spectrum, lie in the numerical range of the given matrix, and the boundary of the numerical range is a convex line. In his article [9], Hausdorff showed that a numerical range is convex as a set. To make it more understandable to the reader, we give an example of the numerical range of the matrix from a geometric point of view [10]. In the following picture the numerical range of the matrix

$$M := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -1 & i & 5i \\ -1 & -2 & -5i & i \end{pmatrix}$$

is given:



Then the properties listed above are valid not only for matrices, but also for an arbitrary linear bounded operator, and the spectrum of a linear bounded operator A defined in the Hilbert space X lies in the closure of its numerical range, i.e. the validity of the relation $\sigma(A) \subset \overline{W(H)}$ was excellently proved in the article [11].

We describe another important property of the numerical range of a linearly bounded operator:

$$W(A) \subset \{x \in X: ||x|| \leq ||A||\}.$$

It can be seen that the closure of the numerical range of a linearly bounded operator A is a set "smaller" than the circle $\{x \in X: ||x|| \leq ||A||\}$ containing its spectrum.

Two main problems can be distinguished from the above considerations. First, under what conditions is the numerical range of a linearly bounded operator a closed set? Second, in what cases do the numerical range and the spectrum of a linearly bounded operator overlap? In the article [12] for the Friedrichs model with two-dimensional perturbation, and in the article [13] for the generalized Friedrichs model, the conditions where the numerical range and the spectrum overlap were found. In addition, there are cases where the numerical range is a closed set. The methods of the theory of threshold phenomena were used in the conducted studies.

Let us formulate the first result of the paper.

Theorem 1. Let $k \in \{1,2,\dots,n\}$. For the numerical range $W(T_k)$ of the operator T_k we have

$$W(T_k) = [0; \|v_k\|^2].$$

Proof. As it is shown in Section 2, the number $\lambda_0 = 0$ is an eigenvalue of T_k with infinite multiplicity and the number $\lambda_k = \|v_k\|^2$ is a simple eigenvalue of T_k. We denote by $f_0 \in L_2[-\pi;\pi]$ and $f_k \in L_2[-\pi;\pi]$ the corresponding normed eigenfunction associated with the eigenvalue λ_0 and λ_k, respectively. For any $f \in L_2[-\pi;\pi]$ with $\|f\| = 1$ we have

$$(T_k f, f) \geq \inf_{||f||=1} (T_k f, f) = (T_k f_0, f_0) = \lambda_0 (f_0, f_0) = \lambda_0 ||f_0||^2 = \lambda_0 = 0$$

$$(T_k f, f) \leq \sup_{||f||=1} (T_k f, f) = (T_k f_k, f_k) = \lambda_k (f_k, f_k) = \lambda_k ||f_k||^2 = \lambda_k$$

that is,

$$\lambda_0 = 0 \leq (T_k f, f) \leq \lambda_k = \|v_k\|^2.$$

By the definition of the set $W(T_k)$ we obtain the equality $W(T_k) = [0; \|v_k\|^2]$. Theorem 1 is completely proved.

Now we formulate the next result of the paper.

Theorem 2. There exist indices $i_1, i_2, \dots, i_n \in \{1,2,\dots,n\}$ such that

$$W(T_{i_1}) \subset W(T_{i_2}) \subset \dots \subset W(T_{i_n}).$$

Proof. For the numbers $\|v_1\|^2, \|v_2\|^2, \dots, \|v_n\|^2$ there exist indices $i_1, i_2, \dots, i_n \in \{1,2,\dots,n\}$ such that

$$\|v_{i_1}\|^2 \leq \|v_{i_2}\|^2 \leq \dots \leq \|v_{i_n}\|^2.$$

Therefore,

$$\left[0; \|v_{i_1}\|^2\right] \subset \left[0; \|v_{i_2}\|^2\right] \subset \dots \subset \left[0; \|v_{i_n}\|^2\right]$$

that is,

$$W(T_{i_1}) \subset W(T_{i_2}) \subset \dots \subset W(T_{i_n}).$$

Theorem 2 is completely proved.

Now we formulate the third result paper.

Theorem 3. If for any $i \neq j$, $i,j = 1,2,\dots,n$ the condition (4) is fulfilled, then there exists index $k \in \{1,2,\dots,n\}$ such that

$$W(T_k) = W(T).$$

Proof. Let for any $i \neq j$, $i,j = 1,2,\dots,n$ the condition (4) be fulfilled. Then

$$\sigma_{\mathrm{disc}}(T) = \sigma_{\mathrm{disc}}(T_1) \cup \sigma_{\mathrm{disc}}(T_2) \cup \dots \cup \sigma_{\mathrm{disc}}(T_n)$$

and

$$\sigma(T)=\sigma_{\rm pp}(T)=\sigma_{\rm disc}(T_1)\cup\sigma_{\rm disc}(T_2)\cup\ldots\cup\sigma_{\rm disc}(T_n)\cup\{0\}.$$

Taking into account the last equality and the definition of the numerical range we have

$$W(T)=W(T_1)\cup W(T_2)\cup\ldots\cup W(T_n).$$

By theorem 2 there exist indices $i_1, i_2, \ldots, i_n \in \{1,2,\ldots,n\}$ such that

$$W(T_{i_1})\subset W(T_{i_2})\subset\cdots\subset W(T_{i_n}).$$

Taking into account the last two relations and setting $k=i_n$ we obtain the equality $W(T_k)=W(T)$. Theorem 3 is completely proved.

Note that, the class of parameter functions $v_1(\cdot), v_2(\cdot), \ldots, v_n(\cdot)$ satisfying the condition (4) for any $i\neq j,\ i,j=1,2,\ldots,n$ is non empty. For example, if $n=3$ and

$$v_1(x)=\begin{cases}\sin 2x, & if \quad x\in\left[\frac{\pi}{2};\pi\right]\\ 0, & if \quad x\in\left[-\pi;\frac{\pi}{2}\right]\end{cases};$$

$$v_2(x)=\begin{cases}\sin 2x, & if \quad x\in\left[0;\frac{\pi}{2}\right]\\ 0, & if \quad x\in[-\pi;0]\cup\left[\frac{\pi}{2};\pi\right]\end{cases};$$

$$v_1(x)=\begin{cases}\sin 2x, & if \quad x\in[-\pi;0]\\ 0, & if \quad x\in[0;\pi]\end{cases};$$

then these parameter functions are real-valued, continuous, linearly independent functions defined on $[-\pi;\pi]$, satisfying the condition (4) for any $i\neq j,\ i,j=1,2,3$.

We note that in the papers [14-17] the spectral properties of the Friedrichs model with finite rank perturbation using the Cramer method and methods of the Fredholm integral equations theory.

Conclusion. This paper is devoted to the investigation of the numerical range of the Fredholm integral operator T with, rank $n, n\in N$ in the Hilbert space $L_2[-\pi;\pi]$. As the first step, the numerical range of each Fredholm integral operator $T_k,\ k=1,2,\ldots,n$ of rank one is calculated. Then the numerical range of the Fredholm integral operator $T=T_1+T_2+\cdots+T_n$ is described by the numerical range of one of Fredholm integral operators $T_k,\ k=1,2,\ldots,n$ under natural condition. That is, it is shown that there exists the index $k\in\{1,2,\ldots,n\}$ such that $W(T_k)=W(T)$ if $(v_i,v_j)=0$ for all $i\neq j,\ i,j=1,2,\ldots,n$. At the end of the paper it is given an example of a class of parameter functions where this natural condition is fulfilled.

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