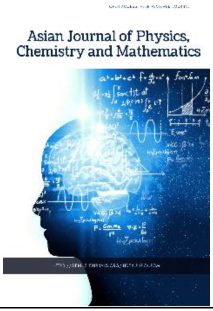

A New Genera Integral Operator Defines Harmonic Multivalent Functions

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ABSTRACT

The main goal of this paper is to introduce the general integral operator's definition of a class of multivalent harmonic functions. Coefficients calculation, extreme point and distortion theorem, convolution property, radii of starlikeness, and convexity are some of the geometric properties we obtain.

Keywords: Convolution, Distortion theorem, General integral operator, Multivalent harmonic functions, Radii of starlikeness and convexity

1. Introduction

The complex domain $T \subset C$ harmonic, the function $\omega = r + iu$ is said to be continuous. We can write $\omega = p + v$, In any simply connected domain T, where p and v are analytic in T [1,2,3], if real harmonic is r and u in T. Clune and Shel-Smal[4] are two examples. $P(\iota)$ denotes the family of all multivalent harmonic functions $\omega = p + \bar{v}$ [5,6], that are sense-preserving in the open unit disc $\Delta = \{t: |t| < 1\}$, where

$$p(t) = t^{\iota} + \sum_{k=2}^{\infty} d_{k+\iota-1} z^{k+\iota-1} \ , \quad v(t) = \sum_{k=1}^{\infty} e_{k+\iota-1}\, z^{k+\iota-1} \, . \qquad (1)$$

Recently Muhammed and Darius [7] defined by $M(e_i; l_j; a)\omega(t): A \to A$:

$$M(e_i; l_j; a)\omega(t) = t^{\iota} + \sum_{k=2}^{\infty} \frac{(e_1, a)_{k-1} \ldots (e_s, a)_{k-1}}{(a, a)_{k-1} \ \ldots (l_1, a)_{k-1} \ \ldots (l_x, a)_{k-1}} \, d_{k+\iota-1} t^{k+\iota-1} \, . \qquad (2)$$

The Srivastva-Atiya[8] operator $N_{c,e}: A \to A$ is defined in:

$$N_{c,e}\omega(t) = t^{\iota} + \sum_{k=2}^{\infty} \left(\frac{1+e}{k+e}\right)^{c} d_{k+\iota-1} t^{k+\iota-1} \, , \qquad (3)$$

where $t \in \Delta, e \in \mathbb{C}/\{0, -1, -2, \ldots\}, c \in \mathbb{C}$ and $\omega \in A$. Linear operator $N_{c,e}$ as well as

$N_{c,e}\omega(t) = R_{c,e} * \omega(t) = (1+e)^c(\emptyset(t,c,e) - e^{-c}) * \omega(t)$, $\Phi(t,c,e) = \sum_{k=0}^{\infty} \frac{t^k}{(k+e)^c}$, the renowned Horwitz –Lerich zeta function (see[8-10]), as well as

$$N_{ic}(t) = \sum_{k=0}^{\infty} \frac{t^k}{(k)^c} = t\Phi(t,c,1).$$

The linear operator$N_a^{c,e}(e_i, l_j)(\omega): A \to A$ and by [11,12] as

$$N_a^{c,e}(e_i, l_j)\omega(t) = t^{\iota} + \sum_{k=2}^{\infty} \frac{(e_1,a)_{k-1} \ \dots (e_s,a)_{k-1}}{(a,a)_{k-1} \ \dots (l_1,a)_{k-1} \ \dots (l_x,a)_{k-1}} (\frac{1+e}{k+e})^c \, d_{k+\iota-1} t^{k+\iota-1} \qquad ,(4)$$

$(t \in \Delta, e \in \mathbb{C}/\{0,-1,-2,\dots\}, c \in \mathbb{C}, l_j \in \mathbb{C}/\{0,-1,-2,-3,\dots\}, |a| < 1$ and $s = x+1, x \in \mathbb{N}_0$.

This class of multivalent harmonic functions was created by$N_k^*(\iota, \alpha_1, \mu)$, a favorable condition

$$\Re\mathrm{e}\left\{\frac{\left(N_a^{c,e}(d_i,e_j)p(t)\right)^{\eta} - \left(N_a^{c,e}(e_i,l_j)v(t)\right)^{\eta}}{t\left(N_a^{c,e}(d_i,e_j)p(t)\right)^{\eta-1} + z\left(N_a^{c,e}(e_i,l_j)v(t)\right)^{\eta-1}}\right\} \geq \iota\mu, \qquad (5)$$

for $\iota \geq 1,\ 0 \leq \mu < 1,\ |t| < 1$.

The multvalent harmonic functionsω in $N_k^*(\iota, \alpha_1, \mu)$ such that $\omega$and v are function of

$$p(t) = t^{\iota} - \sum_{k=2}^{\infty} |d_{k+\iota-1}| t^{k+\iota-1} \ , \ \ v(t) = \sum_{k=1}^{\infty} |e_{k+\iota-1}| \, t^{k+\iota-1} \, . \qquad (6)$$

2. MAIN RESULTS

We demonstrate that the class $N_k^*(\iota, \alpha_1, \mu)$ satisfies the necessary coefficient requirements

Theorem 2.1: Assume $\omega \in N_k^*(\iota, \alpha_1, \mu)$ if and only if

$$\sum_{k=2}^{\infty} (k + \iota(1-\mu) - \eta) \left[\frac{(k+\iota-1)!}{(k+\iota-\eta)!}\right] \frac{(e_1,a)_{k-1} \ \dots (e_s,a)_{k-1}}{(a,a)_{k-1} \ \dots (l_1,a)_{k-1} \ \dots (l_x,a)_{k-1}} (\frac{1+e}{k+e})^c |d_{k+\iota-1}| +$$

$$\sum_{k=1}^{\infty} (k + \iota(1+\mu) - \eta) \left[\frac{(k+\iota-1)!}{(k+\iota-\eta)!}\right] \frac{(e_1,a)_{k-1} \ \dots (e_s,a)_{k-1}}{(a,a)_{k-1} \ \dots (l_1,a)_{k-1} \ \dots (l_x,a)_{k-1}} (\frac{1+e}{k+e})^c \, |e_{k+\iota-1}|$$

$$\leq (\eta - \mu)[(\iota)^{\eta}(\eta-1)!], \qquad (7)$$

$(t \in \Delta, e \in \mathbb{C}/\{0,-1,-2,\dots\}, c \in \mathbb{C}, l_j \in \mathbb{C}/\{0,-1,-2,-3,\dots\}, |a| < 1$and $s = x+1, x \in \mathbb{N}_0$.

Proof: Assuming that the condition is both necessary and sufficient for ω via (5), we obtain

$$\Re e\left\{\frac{(\iota)^{\eta}(\eta)!\,t^{\eta-1}-\sum_{k=2}^{\infty}\left[\frac{(k+\iota-1)!}{(k+\iota-1-\eta)!}\right]\frac{(e_1,a)_{k-1}\ \dots(e_s,a)_{k-1}}{(a,a)_{k-1}\ \dots(l_1,a)_{k-1}\ \dots(l_x,a)_{k-1}}(\frac{1+e}{k+e})^{c}d_{k+\iota-1}t^{k+\iota-}}{t\left[(\iota)^{\eta-1}(\eta-1)!\,t^{\eta-2}-\sum_{k=2}^{\infty}\left[\frac{(k+\iota-1)!}{\left(k+\iota-1-(\eta-1)\right)!}\right]\frac{(e_1,a)_{k-1}\ \dots(e_s,a)_{k-1}}{(a,a)_{k-1}\ \dots(l_1,a)_{k-1}\ \dots(l_x,a)_{k-1}}(\frac{1+e}{k+e})^{c}d_{k+\iota-1}t \quad \sum_{k=1}^{\infty}\left[\frac{(k+\iota-1)!}{\left(k+\iota-1-(\eta-1)\right)!}\right]\frac{(e_1,a)_{k-1}\ \dots(e_s,a)_{k-1}}{(a,a)_{k-1}\ \dots(l_1,a)_{k-1}\ \dots(l_x,a)_{k-1}}(\frac{1+e}{k+e})^{c}\,e_{k+\iota-1}\,t^{k+\iota-1-(\eta-}\right.}\right.$$

$$-$$

$$\Re e\left\{\frac{\sum_{k=1}^{\infty}\left[\frac{(k+\iota-1)!}{(k+\iota-1-\eta)!}\right]\frac{(e_1,a)_{k-1}\ \dots(e_s,a)_{k-1}}{(a,a)_{k-1}\ \dots(l_1,a)_{k-1}\ \dots(l_x,a)_{k-1}}(\frac{1+e}{k+e})^{c}\,e_{k+\iota-1}\,t^{\iota+k-1-\eta}}{t\left[(\iota)^{\eta-1}(\eta-1)!\,t^{\eta-1-1}-\sum_{k=2}^{\infty}\left[\frac{(k+\iota-1)!}{\left(k+\iota-1-(\eta-1)\right)!}\right]\frac{(e_1,a)_{k-1}\ \dots(e_s,a)_{k-1}}{(a,a)_{k-1}\ \dots(l_1,a)_{k-1}\ \dots(l_x,a)_{k-1}}(\frac{1+e}{k+e})^{c}d_{k+\iota-} \quad \sum_{k=1}^{\infty}\left[\frac{(k+\iota-1)!}{\left(k+\iota-1-(\eta-1)\right)!}\right]\frac{(e_1,a)_{k-1}\ \dots(e_s,a)_{k-1}}{(a,a)_{k-1}\ \dots(l_1,a)_{k-1}\ \dots(l_x,a)_{k-1}}(\frac{1+e}{k+e})^{c}\,e_{k+\iota-1}\,t^{k+\iota-1-(\eta}\right.}\right.$$

$$\geq \iota\mu \quad (8)$$

If all t values satisfy the criteria,
When selecting values for t on the positive true axis, where $0 \leq t < 1$, we get

$$\frac{(\iota)^{\eta}(\eta)!-\sum_{k=2}^{\infty}\left[\frac{(k+\iota-1)!}{(k+\iota-1-\eta)!}\right]\frac{(e_1,a)_{k-1}\ \dots(e_s,a)_{k-1}}{(a,a)_{k-1}\ \dots(l_1,a)_{k-1}\ \dots(l_x,a)_{k-1}}(\frac{1+e}{k+e})^{c}d_{k+\iota-1}t^{k+\iota-2(\eta-1)}}{(\iota)^{\eta-1}(\eta-1)!\,t^{\eta-1-1}-\sum_{k=2}^{\infty}\left[\frac{(k+\iota-1)!}{\left(k+\iota-1-(\eta-1)\right)!}\right]\frac{(e_1,a)_{k-1}\ \dots(e_s,a)_{k-1}}{(a,a)_{k-1}\ \dots(l_1,a)_{k-1}\ \dots(l_x,a)_{k-1}}(\frac{1+e}{k+e})^{c}d_{k+\iota-1}t^{k+\iota-} \quad \sum_{k=1}^{\infty}\left[\frac{(k+\iota-1)!}{\left(k+\iota-1-(\eta-1)\right)!}\right]\frac{(e_1,a)_{k-1}\ \dots(e_s,a)_{k-1}}{(a,a)_{k-1}\ \dots(l_1,a)_{k-1}\ \dots(l_x,a)_{k-1}}(\frac{1+e}{k+e})^{c}\,b_{k+\iota-1}\,t^{k+\iota-2(\eta-1)}}$$

$$-\frac{\sum_{k=1}^{\infty}\left[\frac{(k+\iota-1)!}{(k+\iota-1-\eta)!}\right]\frac{(e_1,a)_{k-1}\ \dots(e_s,a)_{k-1}}{(a,a)_{k-1}\ \dots(l_1,a)_{k-1}\ \dots(l_x,a)_{k-1}}(\frac{1+e}{k+e})^{c}\,e_{k+\iota-1}\,t^{k+\iota-2(\eta-1)}}{(\iota)^{\eta-1}(\eta-1)!\,t^{\eta-1-1}-\sum_{k=2}^{\infty}\left[\frac{(k+\iota-1)!}{\left(k+\iota-1-(\eta-1)\right)!}\right]\frac{(e_1,a)_{k-1}\ \dots(e_s,a)_{k-1}}{(a,a)_{k-1}\ \dots(l_1,a)_{k-1}\ \dots(l_x,a)_{k-1}}(\frac{1+e}{k+e})^{c}d_{k+\iota-1}t^{k+\iota} \quad \sum_{k=1}^{\infty}\left[\frac{(k+\iota-1)!}{\left(k+\iota-1-(\eta-1)\right)!}\right]\frac{(e_1,a)_{k-1}\ \dots(e_s,a)_{k-1}}{(a,a)_{k-1}\ \dots(l_1,a)_{k-1}\ \dots(l_x,a)_{k-1}}(\frac{1+e}{k+e})^{c}\,e_{k+\iota-1}\,t^{k+\iota-2(\eta-1)}}$$

$$\geq \iota\mu \quad \mathbf{(9)}$$

When condition (7) fails, the numerator in equation (8) becomes negative as it approaches one. This issue with the situation for $\omega(t) \in N_k^*(\iota,\alpha_1,\mu)$, and as a result, the verification is completed.

Corollary 2.2[13,14]: If $\eta = 1$, then$\omega \in N_k^*(\iota,\alpha_1,\mu)$ if and only if

$$\sum_{k=2}^{\infty}(k+\iota(1-\mu)-1)\ \frac{(e_1,a)_{k-1}\ \dots(e_s,a)_{k-1}}{(a,a)_{k-1}\ \dots(l_1,a)_{k-1}\ \dots(l_x,a)_{k-1}}(\frac{1+e}{k+e})^{c}|d_{k+\iota-1}| +$$

$$\sum_{k=1}^{\infty}(k+\iota(1+\mu)-1)\frac{(e_1,a)_{k-1}\ \dots(e_s,a)_{k-1}}{(a,a)_{k-1}\ \dots(l_1,a)_{k-1}\ \dots(l_x,a)_{k-1}}(\frac{1+e}{k+e})^{c}\,|e_{k+\iota-1}| \leq (1-\mu)\iota,$$

$(t \in \Delta, e \in \mathbb{C}/\{0,-1,-2,\dots\}, c \in \mathbb{C}, l_j \in \mathbb{C}/\{0,-1,-2,-3,\dots\}, |a| < 1$and $s = x+1, x \in \mathbb{N}_0$.

3. EXTREME POINT

Theorem 3.1: Suppose $\omega(t)$ given and (6). Then $\omega \in N_k^*(\iota, \alpha_1, \mu)$ if and only if

$\omega(t) = \sum_{k=1}^{\infty}(X_{k+\iota-1}\, p_{k+\iota-1}(t) + Y_{k+\iota-1} v_{k+\iota-1}(t))$,
where

$$p_\iota(t) = t^\iota,$$

$$p_{k+\iota-1}(t) = t^\iota - \sum_{k=2}^{\infty} \frac{(r-\mu)[(\iota)^r(r-1)!]}{(k+\iota(1-\mu)-r)\left[\frac{(k+\iota-1)!}{(k+\iota-r)!}\right] \frac{(e_1,a)_{k-1} \dots (e_s,a)_{k-1}}{(a,a)_{k-1} \dots (l_1,a)_{k-1} \dots (l_x,a)_{k-1}} (\frac{1+e}{k+e})^c} t^{k+\iota-1-r},$$

and

$$v_{k+\iota-1}(t) = t^\iota - \sum_{k=1}^{\infty} \frac{(u-\mu)[(\iota)^r(r-1)!]}{(k+\iota(1+\mu)-r)\left[\frac{(k+\iota-1)!}{(k+\iota-r)!}\right] \frac{(e_1,a)_{k-1} \dots (e_s,a)_{k-1}}{(a,a)_{k-1} \dots (l_1,a)_{k-1} \dots (l_x,a)_{k-1}} (\frac{1+e}{k+e})^c} t^{k+\iota-1-r},$$

$$p_{k+\iota-1}(t) \geq 0 \ , v_{k+\iota-1}(t) \geq 0 \, , x_\iota = 1 - \sum_{k=2}^{\infty} X_{k+\iota-1} + \sum_{k=1}^{\infty} Y_{k+\iota-1}.$$

The extrme points of $\omega \in N_k^*(\iota, \alpha_1, \mu)$ are $\{ p_{k+\iota-1}\}$ and$\{v_{k+\iota-1}\}$.

Proof: Suppose

$$\omega(t) = \sum_{k=1}^{\infty}(X_{k+\iota-1}\, p_{k+\iota-1}(t) + Y_{k+\iota-1} v_{k+\iota-1}(t)) = \sum_{k=1}^{\infty}(X_{k+\iota-1} + Y_{k+\iota-1}) t^\iota$$

$$-\sum_{k=2}^{\infty} \frac{(r-\mu)[(\iota)^r(r-1)!]}{(k+\iota(1-\mu)-r)\left[\frac{(k+\iota-1)!}{(k+\iota-r)!}\right] \frac{(e_1,a)_{k-1} \dots (e_s,a)_{k-1}}{(a,a)_{k-1} \dots (l_1,a)_{k-1} \dots (l_x,a)_{k-1}} (\frac{1+e}{k+e})^c} X_{k+\iota-1} t^{k+\iota-1-r}$$

$$+\sum_{k=1}^{\infty} \frac{(r-\mu)[(\iota)^r(r-1)!]}{(k+\iota(1+\mu)-r)\left[\frac{(k+\iota-1)!}{(k+\iota-r)!}\right] \frac{(e_1,a)_{k-1} \dots (e_s,a)_{k-1}}{(a,a)_{k-1} \dots (l_1,a)_{k-1} \dots (l_x,a)_{k-1}} (\frac{1+e}{k+e})^c} Y_{k+\iota-1} t^{k+\iota-1-r}$$

$$= t^\iota - \sum_{k=2}^{\infty} \frac{(r-\mu)[(\iota)^r(r-1)!]}{(k+\iota(1-\mu)-r)\left[\frac{(k+\iota-1)!}{(k+\iota-r)!}\right] \frac{(e_1,a)_{k-1} \dots (e_s,a)_{k-1}}{(a,a)_{k-1} \dots (l_1,a)_{k-1} \dots (l_x,a)_{k-1}} (\frac{1+e}{k+e})^c} X_{k+\iota-1} t^{k+\iota-1}$$

$$+\sum_{k=1}^{\infty} \frac{(r-\mu)[(\iota)^r(r-1)!]}{(k+\iota(1-\mu)-r)\left[\frac{(k+\iota-1)!}{(k+\iota-r)!}\right] \frac{(e_1,a)_{k-1} \dots (e_s,a)_{k-1}}{(a,a)_{k-1} \dots (l_1,a)_{k-1} \dots (l_x,a)_{k-1}} (\frac{1+e}{k+e})^c} Y_{k+\iota-1} t^{k+\iota-1-r}.$$

Additionally,

$$\sum_{k=2}^{\infty} \frac{(k+\iota(1-\mu)-r)\left[\frac{(k+\iota-1)!}{(k+\iota-r)!}\right] \frac{(e_1,a)_{k-1} \dots (e_s,a)_{k-1}}{(a,a)_{k-1} \dots (l_1,a)_{k-1} \dots (l_x,a)_{k-1}} (\frac{1+e}{k+e})^c}{(r-\mu)[(\iota)^r(r-1)!]} |d_{k+\iota-1}|$$

$$+\sum_{k=1}^{\infty} \frac{(k+\iota(1-\mu)-r)\left[\frac{(k+\iota-1)!}{(k+\iota-r)!}\right] \frac{(e_1,a)_{k-1} \dots (e_s,a)_{k-1}}{(a,a)_{k-1} \dots (l_1,a)_{k-1} \dots (l_x,a)_{k-1}} (\frac{1+e}{k+e})^c}{(r-\mu)[(\iota)^r(r-1)!]} |e_{k+\iota-1}|$$

$$=\sum_{k=2}^{\infty} \frac{(k+\iota(1-\mu)-r)\left[\frac{(k+\iota-1)!}{(k+\iota-r)!}\right] \frac{(e_1,a)_{k-1} \dots (e_s,a)_{k-1}}{(a,a)_{k-1} \dots (l_1,a)_{k-1} \dots (l_x,a)_{k-1}} (\frac{1+e}{k+e})^c}{(r-\mu)[(\iota)^r(r-1)!]} \times$$

$$\left(\sum_{k=2}^{\infty} \frac{(r-\mu)[(\iota)^r(r-1)!]}{(k+\iota(1-\mu)-r)\left[\frac{(k+\iota-1)!}{(k+\iota-r)!}\right] \frac{(e_1,a)_{k-1} \dots (e_s,a)_{k-1}}{(a,a)_{k-1} \dots (l_1,a)_{k-1} \dots (l_x,a)_{k-1}} (\frac{1+e}{k+e})^c} X_{k+\iota-1}\right)$$

$$+\sum_{k=1}^{\infty} \frac{(k+\iota(1-\mu)-r)\left[\frac{(k+\iota-1)!}{(k+\iota-r)!}\right] \frac{(e_1,a)_{k-1} \dots (e_s,a)_{k-1}}{(a,a)_{k-1} \dots (l_1,a)_{k-1} \dots (l_x,a)_{k-1}} (\frac{1+e}{k+e})^c}{(r-\mu)[(\iota)^r(r-1)!]} \times$$

$$\left(\sum_{k=1}^{\infty} \frac{(r-\mu)[(\iota)^r(r-1)!]}{(k+\iota(1-\mu)-r)\left[\frac{(k+\iota-1)!}{(k+\iota-r)!}\right] \frac{(e_1,a)_{k-1} \dots (e_s,a)_{k-1}}{(a,a)_{k-1} \dots (l_1,a)_{k-1} \dots (l_x,a)_{k-1}} (\frac{1+e}{k+e})^c} Y_{k+\iota-1}\right)$$

$$=\sum_{k=2}^{\infty} X_{k+\iota-1} + \sum_{k=1}^{\infty} Y_{k+\iota-1} = 1 - X_\iota \leq 1.$$

Hence, $\omega(t) \in N_k^*(\iota, \alpha_1, \mu)$.

Conversely, if $\omega(t) \in N_k^*(\iota, \alpha_1, \mu)$. Suppose

$$X_\iota = 1 - \sum_{k=2}^{\infty} X_{k+\iota-1} + \sum_{k=1}^{\infty} Y_{k+\iota-1}.$$

Set

$$X_{k+\iota-1} = (k+\iota(1-\mu)-r)\left[\frac{(k+\iota-1)!}{(k+\iota-r)!}\right] \frac{(e_1,a)_{k-1} \dots (e_s,a)_{k-1}}{(a,a)_{k-1} \dots (l_1,a)_{k-1} \dots (l_x,a)_{k-1}} (\frac{1+e}{k+e})^c |d_{k+\iota-1}|,$$

$$Y_{k+\iota-1} = (k+\iota(1+\mu)-r)\left[\frac{(k+\iota-1)!}{(k+\iota-r)!}\right] \frac{(e_1,a)_{k-1} \dots (e_s,a)_{k-1}}{(a,a)_{k-1} \dots (l_1,a)_{k-1} \dots (l_x,a)_{k-1}} (\frac{1+e}{k+e})^c |e_{k+\iota-1}|,$$

Now,

$$\omega(t) = t^\iota - \sum_{k=2}^{\infty} d_{k+\iota-1}\, t^{k+\iota-1-r} + \sum_{k=1}^{\infty} e_{k+\iota-1}\, \overline{t^{k+\iota-1-r}}$$

$$= t^\iota - \sum_{k=2}^{\infty} \frac{(r-\mu)[(\iota)^r(r-1)!]X_{k+\iota-1}}{(k+\iota(1+\mu)-r)\left[\frac{(k+\iota-1)!}{(k+\iota-r)!}\right] \frac{(e_1,a)_{k-1} \dots (e_s,a)_{k-1}}{(a,a)_{k-1} \dots (l_1,a)_{k-1} \dots (l_x,a)_{k-1}} (\frac{1+e}{k+e})^c} t^{k+\iota-1-r}$$

$$+\sum_{k=1}^{\infty}\frac{(r-\mu)[(\iota)^{r}(r-1)!]Y_{k+m-1}}{(k+\iota(1+\mu)-r)\left[\frac{(k+\iota-1)!}{(k+\iota-r)!}\right]\frac{(e_1,a)_{k-1}\ \dots(e_s,a)_{k-1}}{(a,a)_{k-1}\ \dots(l_1,a)_{k-1}\ \dots(l_x,a)_{k-1}}(\frac{1+e}{k+e})^{c}}\overline{t^{k+\iota-1-r}}$$

$$=t^{\iota}-\sum_{k=2}^{\infty}[t^{\iota}-p_{k+\iota-1}(t)]X_{k+\iota-1}+\sum_{k=1}^{\infty}[t^{\iota}-v_{k+\iota-1}(t)]Y_{k+\iota-1}$$

$$=\left[1-\sum_{k=2}^{\infty}X_{k+\iota-1}-\sum_{n=1}^{\infty}Y_{k+\iota-1}\right]t^{\iota}+\sum_{k=2}^{\infty}X_{k+\iota-1}p_{k+\iota-1}(t)+\sum_{k=1}^{\infty}Y_{k+\iota-1}\,v_{k+\iota-1}(t)$$

$$=\sum_{k=2}^{\infty}X_{k+\iota-1}p_{k+\iota-1}(t)+\sum_{k=1}^{\infty}Y_{k+\iota-1}\,v_{k+\iota-1}(t).$$

4. The Distortion theorem

Theorem 4.1: Assume $\omega(t)\in N_k^{*}(\iota,\alpha_1,\mu)$ **.**Then for $|t|=j<1$, let

$$\psi_{n}=\frac{(e_1,a)_{k-1}\ \dots(e_s,a)_{k-1}}{(a,a)_{k-1}\ \dots(l_1,a)_{k-1}\ \dots(l_x,a)_{k-1}}(\frac{1+e}{k+e})^{c}|\omega(t)|$$

$$\le(1+|e_{\iota}|)j^{\iota}+j^{\iota+1}\left(\frac{\iota(1-\mu)}{(\iota(1-\mu)+r)|\psi_2|}-\frac{\iota(1+\mu)|e_{\iota}|}{(\iota(1-\mu)+r)|\psi_2|}\right)\qquad(10)$$

and

$$|\omega(t)|\ge(1+|e_{\iota}|)j^{\iota}-j^{\iota+1}\left(\frac{\iota(1-\mu)}{(\iota(1-\mu)+r)|\psi_2|}-\frac{\iota(1+\mu)|e_{\iota}|}{(\iota(1-\mu)+r)|\psi_2|}\right).$$

Proof: Since

$$\frac{(\iota(1-\mu)+r}{\iota(1-\mu)}|\psi_2|\sum_{k=2}^{\infty}(d_{k+\iota-1}+e_{k+\iota-1})\le\sum_{k=2}^{\infty}\frac{k+\iota(1-\mu)-r}{\iota(1-\mu)}(d_{k+\iota-1}+e_{k+\iota-1})|\psi_2|$$

$$\le\sum_{k=2}^{\infty}\left(\frac{k+\iota(1-\mu)-r}{\iota(1-\mu)}|d_{k+\iota-1}|+\frac{k+\iota(1-\mu)-r}{\iota(1-\mu)}|e_{k+\iota-1}|\right)|\psi_2|,$$

Theorem 2.2 is conclusion, we obtain

$$\sum_{k=2}^{\infty}(|d_{k+\iota-1}|+|e_{k+\iota-1}|)\le\frac{\iota(1-\mu)}{(\iota(1-\mu)+r)|\psi_2|}(1-|e_{\iota}|).\qquad(11)$$

Since$\omega(t)\in N_k^{*}(\iota,\alpha_1,\mu)$, and $|t|=j$

$$|\omega(t)|=|t^{\iota}|-\sum_{k=2}^{\infty}d_{k+\iota-1}\,t^{k+\iota-1}+\sum_{k=1}^{\infty}e_{k+\iota-1}\,\bar{t}^{k+\iota-1}$$

$$\le|t^{\iota}|+\sum_{k=2}^{\infty}d_{k+\iota-1}\,t^{k+\iota-1}+\sum_{k=1}^{\infty}e_{k+\iota-1}\,\bar{t}^{k+\iota-1}$$

$$= j^{\iota} + \sum_{k=2}^{\infty} |d_{k+\iota-1}| \, j^{k+\iota-1} + \sum_{k=1}^{\infty} |e_{k+\iota-1}| \, j^{k+\iota-1}$$

$$\leq (1 + |e_{\iota}|) j^{\iota} + \left(\sum_{k=2}^{\infty} d_{k+\iota-1} + e_{k+\iota-1} \right) j^{\iota+1}$$

$$\leq (1 + |e_{\iota}|) j^{\iota} + j^{\iota+1} \left(\frac{\iota(1-\mu)}{(\iota(1-\mu)+r)|\psi_2|} - \frac{\iota(1+\mu)|e_{\iota}|}{(\iota(1-\mu)+r)|\psi_2|} \right).$$

It yields the first outcome. Similarly, the lower bound is as follows.

$$|\omega(t)| \geq j^{\iota} - \sum_{k=2}^{\infty} |d_{k+\iota-1}| \, j^{k+\iota-1} + \sum_{k=1}^{\infty} |e_{k+\iota-1}| \, j^{k+\iota-1}$$

$$= (1 - |e_{\iota}|) j^{\iota} - \sum_{k=2}^{\infty} (|d_{k+\iota-1}| + \sum_{k=1}^{\infty} |e_{k+\iota-1}|) \, j^{k+\iota-1}$$

$$\geq (1 - |e_{\iota}|) j^{\iota} - j^{\iota+1} \left(\frac{\iota(1-\mu)}{(\iota(1-\mu)+r)|\psi_2|} - \frac{\iota(1+\mu)|e_{\iota}|}{(\iota(1-\mu)+r)|\psi_2|} \right).$$

5. The Convolution property

We illustrate how to prove two theories, the first of which is the convolution theorem for the class $N_k^*(\iota, \alpha_1, \mu)$. Assume

$$\omega(t) = t^{\iota} - \sum_{k=2}^{\infty} d_{k+\iota-1} \, t^{k+\iota-1} + \sum_{k=1}^{\infty} e_{k+\iota-1} \bar{t}^{k+\iota-1}$$

$$v(t) = t^{\iota} - \sum_{k=2}^{\infty} f_{k+\iota-1} \, t^{k+\iota-1} + \sum_{k=1}^{\infty} h_{k+\iota-1} \bar{t}^{k+\iota-1}.$$

The complication of ω and v define by, [15,16]:

$$(\omega * v)(t) = \omega(t) * v(t) = t^{\iota} - \sum_{k=2}^{\infty} d_{k+\iota-1} f_{k+\iota-1} t^{k+\iota-1} + \sum_{k=1}^{\infty} e_{k+\iota-1} h_{k+\iota-1} \bar{t}^{k+\iota-1}.$$

Theorem 5.1: Suppose $\omega(t) \in N_k^*(\iota, \alpha_1, \mu)$ and $v(t) \in N_k^*(\iota, \alpha_1, \mu)$. Then $\omega * v \in N_k^*(\iota, \alpha_1, \mu) \subset N_k^*(\iota, \alpha_2, \mu)$.

Proof: Let

$$\omega(t) = t^{\iota} - \sum_{k=2}^{\infty} d_{k+\iota-1} \, t^{k+\iota-1} + \sum_{k=1}^{\infty} e_{k+\iota-1} \bar{t}^{k+\iota-1},$$

be in $N_k^*(\iota, \alpha_1, \mu)$ and

$$v(t) = t^{\iota} - \sum_{k=2}^{\infty} f_{k+\iota-1} \, t^{k+\iota-1} + \sum_{k=1}^{\infty} h_{k+\iota-1} \bar{t}^{k+\iota-1},$$

be in $N_k^*(\iota, \alpha_2, \mu)$.

Consider the concept of convolution functions$\omega * v$ the following:

$$\sum_{k=2}^{\infty}\frac{(k+\iota(1+\mu)-r)\left[\frac{(k+\iota-1)!}{(k+\iota-r)!}\right]\frac{(e_1,a)_{k-1}\ ...(e_s,a)_{k-1}}{(a,a)_{k-1}\ ...(l_1,a)_{k-1}\ ...(l_x,a)_{k-1}}(\frac{1+e}{k+e})^c}{(r-\mu)[(\iota)^r(r-1)!]}d_{k+\iota-1}\,f_{k+\iota-1}$$

$$+\sum_{k=2}^{\infty}\frac{(k+\iota(1+\mu)-r)\left[\frac{(k+\iota-1)!}{(k+\iota-r)!}\right]\frac{(e_1,a)_{k-1}\ ...(e_s,a)_{k-1}}{(a,a)_{k-1}\ ...(l_1,a)_{k-1}\ ...(l_x,a)_{k-1}}(\frac{1+e}{k+e})^c}{(r-\mu)[(\iota)^r(r-1)!]}e_{k+\iota-1}\,h_{k+\iota-1}$$

$$\leq\sum_{k=2}^{\infty}\frac{(k+\iota(1+\mu)-r)\left[\frac{(k+\iota-1)!}{(k+\iota-r)!}\right]\frac{(e_1,a)_{k-1}\ ...(e_s,a)_{k-1}}{(a,a)_{k-1}\ ...(l_1,a)_{k-1}\ ...(l_x,a)_{k-1}}(\frac{1+e}{k+e})^c}{(r-\mu)[(\iota)^r(r-1)!]}d_{k+\iota-1}$$

$$+\sum_{k=2}^{\infty}\frac{(k+\iota(1+\mu)-r)\left[\frac{(k+\iota-1)!}{(k+\iota-r)!}\right]\frac{(e_1,a)_{k-1}\ ...(e_s,a)_{k-1}}{(a,a)_{k-1}\ ...(l_1,a)_{k-1}\ ...(l_x,a)_{k-1}}(\frac{1+e}{k+e})^c}{(r-\mu)[(\iota)^r(r-1)!]}e_{k+\iota-1}\leq 1.$$

6. The Radii of starlikeness and convexity

Theorem 6.1: Let the function ω by (1) be in $N_k^*(\iota,\alpha_1,\mu)$. Then ω is multivalent starlike of order η in the disk $|t|<j_1N_k^*(\iota,\alpha_1,\mu)$,where

$J_1(\iota,\alpha_1,\mu,\eta\ )=$

$$\inf\left\{\sum_{k=2}^{\infty}\frac{(1-\eta)(k+\iota(1-\mu)-r)\left[\frac{(k+\iota-1)!}{(k+\iota-u)!}\right]\frac{(e_1,a)_{k-1}\ ...(e_s,a)_{k-1}}{(a,a)_{k-1}\ ...(l_1,a)_{k-1}\ ...(l_x,a)_{k-1}}(\frac{1+e}{k+e})^c}{(k+\iota+\eta\ )(r-\mu)[(\iota)^r(r-1)!]}\right\}^{\frac{1}{k-1}}$$

Proof: Prove it

$$\left|\frac{t\omega'(t)}{\omega(t)}+1\right|\leq 1-\eta,$$

$$\left|\frac{t\omega'(t)}{\omega(t)}+1\right|=\left|\frac{\sum_{k=2}^{\infty}(k+\iota-1)d_{k+\iota-1}\,t^{k+\iota-1}}{t^{\iota}+\sum_{k=2}^{\infty}d_{k+\iota-1}\,t^{k+\iota-1}}\right|\leq\frac{\sum_{k=2}^{\infty}(k+\iota-1)d_{k+\iota-1}\,|t|^{k-1}}{1-\sum_{k-=2}^{\infty}d_{k+\iota-1}\,|T|^{k-1}}.$$

Will be constrained by $1-\eta$,

$$\frac{\sum_{k=2}^{\infty}(k+\iota-1)d_{k+\iota-1}\,|t|^{k-1}}{1-\sum_{k=2}^{\infty}d_{k+\iota-1}\,|t|^{k-1}}\leq 1-\eta,\sum_{k=2}^{\infty}(k+\iota+\eta\ )d_{k+\iota-1}\,|t|^{k-1},$$

by Theorem 2.1, we have

$$\sum_{k=2}^{\infty}\frac{(k+\iota(1-\mu)-r)\left[\frac{(k+\iota-1)!}{(k+\iota-u)!}\right]\frac{(e_1,a)_{k-1}\ ...(e_s,a)_{k-1}}{(a,a)_{k-1}\ ...(l_1,a)_{k-1}\ ...(l_x,a)_{k-1}}(\frac{1+e}{k+e})^c}{(r-\mu)[(\iota)^r(r-1)!]}d_{k+\iota-1}\leq 1.$$

Hence,

$$|t|^{k-1} \le \sum_{k=2}^{\infty} \frac{(1-\eta)(k+\iota(1-\mu)-r)\left[\frac{(k+\iota-1)!}{(k+\iota-u)!}\right]\frac{(e_1,a)_{k-1}\ \dots (e_s,a)_{k-1}}{(a,a)_{k-1}\ \dots (l_1,a)_{k-1}\ \dots (l_x,a)_{k-1}}(\frac{1+e}{k+e})^c}{(k+\iota+\eta)(r-\mu)[(\iota)^r(r-1)!]},$$

$$|t| \le \left\{\sum_{k=2}^{\infty} \frac{(1-\eta)(k+\iota(1-\mu)-u)\left[\frac{(k+\iota-1)!}{(k+\iota-u)!}\right]\frac{(e_1,a)_{k-1}\ \dots (e_s,a)_{k-1}}{(a,a)_{k-1}\ \dots (l_1,a)_{k-1}\ \dots (l_x,a)_{k-1}}(\frac{1+e}{k+e})^c}{(k+\iota+\eta)(r-\mu)[(\iota)^r(r-1)!]}\right\}^{\frac{1}{k-1}}.$$

The proof of the theorem is now completed.

Theorem 6.2: The function $\omega(t)$ defined by (1) be in $N_k^*(\iota,\alpha_1,\mu)$.Then ω is multivalent convex of order η in the disk $|t| < j_2(\iota,\alpha_1,\mu,\eta)$, where

$r_2(\iota,\alpha_1,\mu,\delta) =$

$$\inf\left\{\sum_{k=2}^{\infty} \frac{(1-\delta)(k+\iota(1-\mu)-r)\left[\frac{(k+\iota-1)!}{(k+\iota-u)!}\right]\frac{(e_1,a)_{k-1}\ \dots (e_s,a)_{k-1}}{(a,a)_{k-1}\ \dots (l_1,a)_{k-1}\ \dots (l_x,a)_{k-1}}(\frac{1+e}{k+e})^c}{(k+\iota+\delta)(r-\mu)[(\iota)^r(r-1)!]}\right\}^{\frac{1}{k-1}}.$$

Proof: We can demonstrate this using the same method we used to prove Theorem 6.1.

$$\left|\frac{t\omega''(t)}{\omega'(t)}+2\right| \le 1-\delta, \qquad (0 \le \delta < 1).$$

In relation $|t| < j_2$ with the assistance of a Theorem 2.1, we now have proof of the Theorem 6.2.

7. Conclusions

We've shown that a class of harmonic multivalent functions can be described by the general integral operator, which leads to some interesting results. Finally, geometric properties such as coefficients conditions, extreme points, distortion theorem, convolution property, and starlikeness radii have been investigated.

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