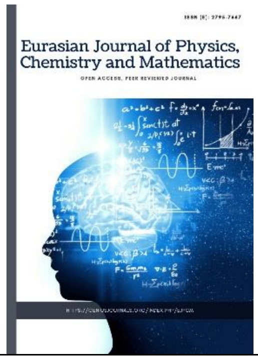

Construction Of The Wienberg Equation For The Eigenvectors Of The Family Of Operator Matrices

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ABSTRACT

In this paper we consider the family of operator matrices $H(K)$ of order three which depends on the parameter K. This family of operator matrices are act in the three-particle cut subspace of the bosonic Fock space. We constract the Weinberg equation for the eigenvectors of $H(K)$.

Keywords: Weinberg equation, eigenvalue, eigenvector, operator matrix, cut subspace, bosonic Fock space.

Introduction. In the theory of solid-state physics [1], quantum field theory [2], statistical physics [3,4], fluid mechanics [5], magnetohydrodynamics [6] and quantum mechanics [7] some important problems arise where the number of quasi-particles is finite, but not fixed. Notice that the corresponding Hamiltonian to such systems has operator matrix representation. Operator matrices are matrices the entries of which are linear operators between Banach or Hilbert spaces [8]. Every bounded linear operator can be written as a block operator matrix if the space in which it acts is decomposed in two or more more components. Operator matrices arise in various areas of mathematics and its applications.

In the present paper we consider a family of 3×3 operator matrices $H(K)$, $K\in \mathrm{T}:=(-\pi;\pi]$. These operator matrices are associated with the lattice systems describing two identical bosons and one particle, another nature in interactions, without conservation of the number of particles. They act in the direct sum of zero-, one- and two-particle subspaces of the bosonic Fock space. We discuss the case where the dispersion function $\varepsilon(\cdot)$ has the form $\varepsilon(x)=1-\cos(nx)$ with $n>1$. We denote by Λ the set of points T where the function $\varepsilon(\cdot)$ takes its (global) minimum. Our main aim is to constract the Weinberg equation for the eigenvectors of the family of operator matrices H(K).

Family of 3×3 operator matrices and main results

Let T be the one-dimensional torus, $H_0 := C$ be the field of complex numbers, $\mathrm{H}_1 := L_2(\mathrm{T})$ be the Hilbert space of square integrable (complex) functions defined on T and $\mathrm{H}_2 := L_2^s(\mathrm{T}^2)$ be the Hilbert space of square integrable (complex) symmetric functions defined on T^2. The Hilbert space $\mathrm{H} := \mathrm{H}_0 \oplus \mathrm{H}_1 \oplus \mathrm{H}_2$ is called three-particle cut subspace of a bosonic Fock space $\mathrm{F}_s(L_2(\mathrm{T}))$ over $L_2(\mathrm{T})$, respectively.

In the present paper we consider a family of 3×3 operator matrices $H(K)$, $K\in\mathrm{T}$ acting in the Hilbert space H as

$$H(K):=\begin{pmatrix} H_{00}(K) & H_{01} & 0 \\ H_{01}^{*} & H_{11}(K) & H_{12} \\ 0 & H_{12}^{*} & H_{22}(K) \end{pmatrix}$$

with the entries

$$H_{00}(K)f_0 = w_0(K)f_0, \quad H_{01}f_1 = \int_{\mathrm{T}} v(t)f_1(t)dt,$$

$$(H_{11}(K)f_1)(x) = w_1(K;x)f_1(x), \quad (H_{12}f_2)(x) = \int_{\mathrm{T}} v(t)f_2(x,t)dt,$$

$$(H_{22}(K)f_2)(x,y) = w_2(K;x,y)f_2(x,y), \quad f_i \in \mathrm{H}_i, \quad i=0,1,2$$

where H_{ij}^{*} $(i<j)$ denotes the adjoint operator to H_{ij}.

Here $w_0(\cdot)$ is a real-valued bounded function on T, the function $v(\cdot)$ is a real-valued analytic on T, the functions $w_1(\cdot;\cdot)$ and $w_2(\cdot;\cdot,\cdot)$ are defined by the equalities

$$w_1(K;x):=l_1\varepsilon(x)+l_2\varepsilon(K-x)+1, \quad w_2(K;x,y):=l_1\varepsilon(x)+l_1\varepsilon(y)+l_2\varepsilon(K-x-y),$$

respectively, with $l_1,l_2>0$ and

$$\varepsilon(x):=1-\cos(nx), \quad n\in N.$$

Under these assumptions the operator $H(K)$ is bounded and self-adjoint.

We remark that the operators H_{01} and H_{12}, resp. H_{01}^{*} and H_{12}^{*} are called annihilation resp. creation operators, respectively. It is clear that

$$H_{01}^{*}:H_0\to H_1, \quad (H_{01}^{*}f_0)(x)=v(x)f_0, \quad f_0\in H_0;$$

$$H_{12}^{*}:H_1\to H_2, \ (H_{12}^{*}f_1)(x,y)=\frac{1}{2}(v(x)f_1(y)+v(y)f_1(x)), \quad f_1\in H_1.$$

The essential and discrete spectrum of $H(K)$ is studied in [9-11]. Similar operator matrix with fixed K is considered in many works, see for example [12-15].

Constraction of the Weinberg equation. In the domain $C\setminus[m_K(x),M_K(x)]$ we consider the following regular function

$$\Delta(K;x;\lambda):=\omega_1(K;x)-\lambda-\frac{1}{2}\int_T\frac{v^2(t)dt}{\omega_2(K;x;t)-\lambda},$$

where the numbers $m_K(x)$ and $M_K(x)$ are defined as follows:

$$m_K(x):=\min_{y\in T}\omega_2(K;x;y); \qquad M_K(x):=\max_{y\in T}\omega_2(K;x;y).$$

We introduce the following definitions: we define the set σ_K as the set of numbers $\lambda\in C$ such that $\Delta(K;x;\lambda)=0$ for some $x\in T$ and

$$m_K:=\min_{x,y\in T}\omega_2(K;x;y); \qquad M_K:=\max_{x,y\in T}\omega_2(K;x;y)$$

The following theorem describes the position of the essential spectrum of the operator matrix $H(K)$.

Theorem 1. The equality $\sigma_{ess}(H(K))=[m_K;M_K]\cup\sigma_K$ holds for the essential spectrum $\sigma_{ess}(H(K))$ of the operator matrix $H(K)$.

Definition 1. The sets σ_K and $[m_K;M_K]$ are respectively called the “two-particle” and “three-particle” branches of the essential spectrum of $H(K)$.

For each fixed $\lambda\in C\setminus\sigma_{ess}(H(K))$ we define

$$W(K;\lambda)=\begin{pmatrix} W_{00}(K;\lambda) & W_{01}(K;\lambda) & 0 \\ W_{10}(K;\lambda) & W_{11}(K;\lambda) & 0 \\ W_{20}(K;\lambda) & W_{21}(K;\lambda) & 0 \end{pmatrix},$$

where

$$(W_{00}(K;\lambda)f_0)_0=(\omega_0(K)-\lambda+1)f_0,\ (W_{01}(K;\lambda)f_1)_0=\int_T v(t)f_1(t)dt,\ (W_{10}(K;\lambda)f_0)_1=-\frac{v(x)f_0}{\Delta(K;x;\lambda)},$$

$$\Delta(K;x;\lambda)=\omega_1(K;x)-\lambda-\frac{1}{2}\int_T\frac{v^2(t)dt}{\omega_2(K;x;t)-\lambda},\ (W_{11}(\lambda)f_1)_1=\frac{1}{2}\frac{v(x)}{\Delta(K;x;\lambda)}\cdot\int_T\frac{v(t)f_1(t)}{\omega_2(K;x;t)-\lambda}dt$$

$$(W_{20}(K;\lambda)f_0)_1(x;y)=\left(\frac{v(x)}{2(\omega_2(K;x;y)-\lambda)}\frac{v(y)}{\Delta(K;y;\lambda)}-\frac{v(x)}{2(\omega_2(K;x;y)-\lambda)}\frac{v(x)}{\Delta(K;x;\lambda)}\right)f_0$$

$$(W_{21}(K;\lambda)f_0)_2(x;y)=-\frac{v(x)}{4(\omega_2(K;x;y)-\lambda)}\cdot\frac{v(y)}{\Delta(K;y;\lambda)}\cdot\int_T\frac{v(t)f_1(t)}{\omega_2(K;y;t)-\lambda}dt-$$

$$-\frac{v(x)}{4(\omega_2(K;x;y)-\lambda)}\cdot\frac{v(x)}{\Delta(K;x;\lambda)}\cdot\int_T\frac{v(t)f_1(t)}{\omega_2(K;x;t)-\lambda}dt$$

Definition 2. The equation $f=W(K;\lambda)f$ is called the Weinberg equation.

Theorem 2. If the number λ is an eigenvalue of an operator $H(K)$, and f eigenvector corresponding to this eigenvalue, then the vector f satisfies the Weinberg equation $W(K;\lambda)f=f$.

Proof. In the proof of this theorem, we consider the equation with respect to the eigenvalue for the operator $H(K)$:

$$H(K)f=\lambda f.$$

Let us consider

$$H(K)f=\begin{pmatrix} H_{00}(K) & H_{01} & 0 \\ H_{01}^* & H_{11}(K) & H_{12} \\ 0 & H_{12}^* & H_{22}(K) \end{pmatrix}\begin{pmatrix} f_0 \\ f_1 \\ f_2 \end{pmatrix}=\begin{pmatrix} H_{00}(K)f_0+H_{01}f_1 \\ H_{01}^*f_0+H_{11}(K)f_1+H_{12}f_2 \\ H_{12}^*f_1+H_{22}(K)f_2 \end{pmatrix},$$

The equation $H(K)f=\lambda f$ can be written in the form

$$\begin{pmatrix} H_{00}(K)f_0+H_{01}f_1 \\ H_{01}^*f_0+H_{11}(K)f_1+H_{12}f_2 \\ H_{12}^*f_1+H_{22}(K)f_2 \end{pmatrix}=\begin{pmatrix} \lambda f_0 \\ \lambda f_1 \\ \lambda f_2 \end{pmatrix}.$$

The above equation can be written as the following system of linear equations:

$$\begin{cases} H_{00}(K)f_0+H_{01}f_1=\lambda f_0 \\ H_{01}^*f_0+H_{11}(K)f_1+H_{12}f_2=\lambda f_1 \\ H_{12}^*f_1+H_{22}(K)f_2=\lambda f_2 \end{cases} \qquad (1)$$

We can write the system of linear equations (1) as follows

$$\begin{cases} \omega_0(K)f_0+\int_T v(t)f_1(t)ds=\lambda f_0 \\ v(x)f_0+\omega_1(K;x)f_1(x)+\int_T v(t)f_2(x;t)dt=\lambda f_1(x) \\ \frac{1}{2}(v(x)f_1(y)+v(y)f_1(x))+\omega_2(K;x;y)f_2(x;y)=\lambda f_2(x;y). \end{cases} \qquad (2)$$

We simplify the system of linear equations (2):

$$\begin{cases} (\omega_0(K)-\lambda_0)\, f_0 + \int\limits_T v(t)\, f_1(t)dt = 0 \\ v(x)\, f_0 + (\omega_1(K,x)-\lambda_0)\, f_1(x) + \int\limits_T v(t)\, f_2(x;t)dt = 0 \\ \frac{1}{2}(v(x)\, f_1(y) + v(y)\, f_1(x)) + (\omega_2(K;x;y)-\lambda_0)\, f_2(x;y) = 0 \end{cases} \qquad (3)$$

$\omega_2(K;x;y)-\lambda \neq 0$ is true for $x, y \in T$. Therefore, from the third equation of the system of equations (3) the function $f_2(x;y)$ can be found in the following form:

$$f_2(p;q) = -\frac{1}{2}\,\frac{(v(x)\, f_1(y) + v(y)\, f_1(x))}{\omega_2(K;x;y)-\lambda} \qquad (4)$$

Now we substitute the expression (4) for $f_2(x;y)$ into the second equation of the system of equations (3) and write it as follows:

$$\begin{cases} (\omega_0(K)-\lambda)\, f_0 + \int\limits_T v(t)\, f_1(t)dt = 0 \\ v(x)\, f_0 + (\omega_1(K;x)-\lambda)\, f_1(x) + \int\limits_T v_1(s)\left(-\frac{1}{2}\,\frac{(v(x)\, f_1(t) + v(t)\, f_1(x))}{\omega_2(K;x;t)-\lambda}\right)dt = 0 \end{cases}$$

or, simplifying our last system, we write as follows:

$$\begin{cases} (\omega_0(K)-\lambda)\, f_0 + \int\limits_T v(t)\, f_1(t)dt = 0 \\ v(x)\, f_0 + (\omega_1(K;x)-\lambda)\, f_1(x) - \frac{1}{2}\int\limits_T \frac{v(t)v(x)\, f_1(t)}{\omega_2(K;x;t)-\lambda}\, dt - \frac{1}{2}\int\limits_T \frac{v^2(t)\, f_1(x)}{\omega_2(K;x;t)-\lambda} dt = 0 \end{cases} \qquad (5)$$

or

$$\begin{cases} (\omega_0(K)-\lambda)\, f_0 + \int\limits_T v(t)\, f_1(t)ds = 0 \\ v(x)\, f_0 + (\omega_1(K;x)-\lambda)\, f_1(x) - \frac{1}{2}v(x)\int\limits_T \frac{v(t)\, f_1(t)}{\omega_2(K;x;t)-\lambda}\, ds - \frac{1}{2}\, f_1(x)\int\limits_T \frac{v^2(t)}{\omega_2(K;x;t)-\lambda} dt = 0 \end{cases} \qquad (6)$$

Simplifying the system of equations (6) we can write it down

$$\begin{cases} (\omega_0(K)-\lambda)\, f_0 + \int\limits_T v(t)\, f_1(t)dt = 0 \\ v(x)\, f_0 + (\omega_1(K;x)-\lambda - \frac{1}{2}\int\limits_T \frac{v^2(t)}{\omega_2(K;x;t)-\lambda} dt)\, f_1(x) - \frac{1}{2}v(x)\int\limits_T \frac{v(t)\, f_1(t)}{\omega_2(K;x;t)-\lambda}\, dt = 0 \end{cases} \qquad (7)$$

We define the function in front of $f_1(x)$ in the second equation of the system of equations (7) as follows:

$$\Delta(K;x;\lambda) = \omega_1(K;x) - \lambda - \frac{1}{2}\int\limits_T \frac{v^2(t)dt}{\omega_2(K;x;t)-\lambda}.$$

It is known that the relation $\Delta(K;x;\lambda) \neq 0$ is valid for all $\lambda \notin \sigma$ and $p \in T$. Therefore, we can find $f_1(x)$ from the second equation of the system of equations (7):

$$f_1(x) = \frac{1}{2}\,\frac{v(x)}{\Delta(K;x;\lambda)} \cdot \int\limits_T \frac{v(t)\, f_1(t)}{\omega_2(K;x;t)-\lambda}\, dt - \frac{v(x)\, f_0}{\Delta(K;x;\lambda)}\ . \qquad (8)$$

We substitute the expression (8) for $f_1(x)$ into the expression (4):

$$f_2(x;y) = -\frac{1}{2}\,\frac{(v(x)\, f_1(y) + v(y)\, f_1(x))}{\omega_2(K;x;y)-\lambda} = -\frac{1}{2}\,\frac{v(x)\, f_1(y)}{\omega_2(K;x;y)-\lambda} - \frac{1}{2}\,\frac{v(y)\, f_1(x)}{\omega_2(K;x;y)-\lambda} =$$

$$= -\frac{v(x)}{2(\omega_2(K;x;y)-\lambda)}\cdot\left(\frac{1}{2}\frac{v(y)}{\Delta(K;y;\lambda)}\cdot\int_T \frac{v(t)f_1(t)}{\omega_2(K;y;t)-\lambda}dt - \frac{v(y)f_0}{\Delta(K;y;\lambda)}\right) -$$

$$-\frac{v(x)}{2(\omega_2(K;x;y)-\lambda)}\cdot\left(\frac{1}{2}\frac{v(x)}{\Delta(K;x;\lambda)}\cdot\int_T \frac{v(t)f_1(t)}{\omega_2(K;x;t)-\lambda}dt - \frac{v(x)f_0}{\Delta(K;x;\lambda)}\right). \quad (9)$$

We open the brackets on the right side of the expression (9),

$$f_2(x;y) = -\frac{v(x)}{4(\omega_2(K;x;y)-\lambda)}\cdot\frac{v(y)}{\Delta(K;y;\lambda)}\cdot\int_T \frac{v(t)f_1(t)}{\omega_2(K;y;t)-\lambda}dt + \frac{v(x)}{2(\omega_2(K;x;y)-\lambda)}\frac{v(y)f_0}{\Delta(K;y;\lambda)} -$$

$$-\frac{v(x)}{4(\omega_2(K;x;y)-\lambda)}\cdot\frac{v(x)}{\Delta(K;x;\lambda)}\cdot\int_T \frac{v(t)f_1(t)}{\omega_2(K;x;t)-\lambda}dt + \frac{v(x)}{2(\omega_2(K;x;y)-\lambda)}\frac{v(x)f_0}{\Delta(K;x;\lambda)}. \quad (10)$$

In the system of equations (7), we add to the right and left sides of the f_0 first equation. As a result,

$$f_0 + (\omega_0(K)-\lambda)f_0 + \int_{T^\nu} v(t)f_1(t)dt = 0 + f_0$$

is formed. Since this is f_0 common multiplier from the left side of the equation, we take it out of the parentheses. Then, it will be

$$(\omega_0(K)-\lambda+1)f_0 + \int_{T^\nu} v(t)f_1(t)dt = f_0 \quad (11)$$

Using equations (8), (10) and (11), we obtain the following matrix equation $W(K;\lambda)f = f$. Theorem is proved.

Conclusion. In the present paper a family of operator matrices $H(K)$ of order three associated with the energy operator of a system describing three particles in interaction, without conservation of the number of particles, is considered. This operator matrix acting in the direct sum of zero-particle, one-particle and two-particle subspaces of the bosonic Fock space. The location of the essential spectrum of $H(K)$ is described. The Wienberg equation for the eigen vector functions of the $H(K)$ is constructed. Using this equation one can prove the finiteness of the number of eigenvalues of the operator matrix $H(K)$.

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