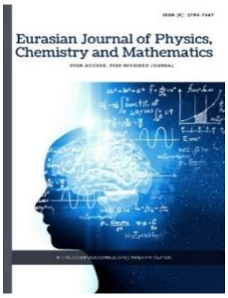

Features Of Geometry In ${}^{2}R_{5}$

Mamadaliev B. M

Fergana State University, Fergana, Uzbekistan.
e-mail: mamadaliyev_botirjon@mail.ru

ABSTRACT

The article is devoted to the study of the geometry of subspaces of a five-dimensional pseudo-Euclidean space. This space is attractive because all kinds of semi-Euclidean, semi-pseudo-Euclidean, hyperbolic three-dimensional spaces with projective metrics are realized in its subspaces. In the sphere of the imaginary radius of space, de Sitter space is realized. An interpretation of de Sitter space in a four-dimensional pseudo-Euclidean space is proved.

Keywords: Five-dimensional pseudo-Euclidean space of index two, semi-Euclidean space, sphere of imaginary radius, hyperbolic geometry, de Sitter space, isotropic cone.

Introduction

Difficulties in studying two-dimensional surfaces in five-dimensional space are associated with its normal space. This normal space is a three-dimensional subspace of the space under consideration. Therefore, the classification of the geometry of three-dimensional subspaces of a five-dimensional space of index two is of scientific interest.

The geometry of a three-dimensional subspace is generated by the metric of the enveloping space and essentially depends on the degeneracy of this metric.

Results and Discussion

Under five-dimensional pseudo-Euclidean space of index two ${}^{2}R_{5}$, we mean the Riemann space with the metric [1]

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2 - dx_5^2. \qquad (1)$$

Space ${}^{2}R_{5}$ is defined as an affine space A_5 with the scalar product of vectors $\vec{X}(x_1, x_2, x_3, x_4, x_5)$, $\vec{Y}(y_1, y_2, y_3, y_4, y_5)$

$$\left(\vec{X} \cdot \vec{Y}\right) = x_1 y_1 + x_2 y_2 + x_3 y_3 - x_4 y_4 - x_5 y_5 \qquad (2)$$

in some affine coordinate system with the origin point $O\{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4, \vec{e}_5\}$ and basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4, \vec{e}_5\}$.

This definition of space ${}^{2}R_{5}$ makes it possible to reveal the property of the elements of space by analogy with Euclidean space.

It is known from [2], that the norm of vector $\left|\vec{X}\right| = \sqrt{\left(\vec{X} \cdot \vec{X}\right)}$ is defined as the square root of the scalar square of the vector, and the distance between points is defined as the norm of the vector connecting these points.

The distance between points in ${}^{2}R_{5}$ is not positive definite. It can be real, imaginary, and zero when the points do not match. It is obvious, that the norm of vectors also takes real, imaginary values or zero when the vector is not equal to zero vector.

The set of vectors with zero norms form an isotropic cone of space ${}^{2}R_{5}$. It is given in affine coordinates by the following equation

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2 = 0. \qquad (3)$$

This is a cone with a two-dimensional hyperbolic-type surface with a flat apex and three-dimensional generatrices of a hyperplane in a five-dimensional space.

The study of geometry of ${}^2R_5$ is attractive, because it exists as a subspace of all kinds of three-dimensional spaces, except for elliptic ones with projective metrics determined by the Cayley-Klein theory set forth in the Erlangen Program [3].

According to Cayley-Klein, the number of $n-$dimensional spaces with projective metrics is 3^n. In the three-dimensional case, the number is 27 [4].

Here there is a space with projective metrics in the sense of Cayley-Klein. It is a three-dimensional space with a metric that preserves space on itself when mapped linearly. The corresponding linear transformation is called the motion of this space.

The study of the geometry of a pseudo-Euclidean space shows [5] that the geometry of some pseudo-Euclidean spaces differs little from another or does not introduce anything essentially new into the geometry of another pseudo-Euclidean space. Let us explain this property of space using the example of three-dimensional pseudo-Euclidean spaces.

Let a three-dimensional pseudo-Euclidean space ${}^1R_3$, called the Minkowski space, in Cartesian coordinates, have a metric of the form

$$ds^2 = dx^2 + dy^2 - dz^2. \quad (4)$$

Consider pseudo-Euclidean space ${}^2R_3$ with the following metric in the same coordinate system

$$ds^2 = -dx^2 - dy^2 + dz^2. \quad (5)$$

Fixing the same coordinate system in two spaces is called the superimposed space method. In this method, a one-to-one correspondence is determined between the points of these spaces, comparing points with the same coordinates. Naturally, with such a correspondence, the geometric images of the structures under consideration are represented by one figure. Moreover, between the metric of space ${}^1R_3$ and the metric of space ${}^2R_3$ the difference is an imaginary factor. Therefore, $ds^2_{{}^1R_3} = -ds^2_{{}^1R_3}$.

It can be said that the image of point $(x_0,\ y_0,\ z_0) \in {}^1R_3$ is point $(ix_0,\ iy_0,\ iz_0) \in {}^2R_3$.

Therefore, they differ only by a factor, which is an imaginary unit. The geometry of space ${}^2R_3$ does not differ significantly from the geometry of space ${}^1R_3$. This means that the study of the geometry of space ${}^2R_3$ is of no interest if the geometry of space ${}^1R_3$ is studied.

Among the 27-three-dimensional spaces, there are 9 spaces that differ from each other only by an imaginary factor of coordinates. We do not consider these spaces.

When we talk about five-dimensional pseudo-Euclidean space, there is a three-dimensional subspace. Let us show one of the spaces that differ from each other only by an imaginary factor of coordinates.

Theorem 1. In a pseudo-Euclidean space ${}^2R_5$, there are all three-dimensional spaces with projective spaces as their subspaces, except for elliptic spaces.

Proof. Let us start with the classical three-dimensional Euclidean space R_3. The subspace, which is the 3D hypersurface defined by $x_4 = x_5 = 0$, is a 3D Euclidean space. This is an almost obvious fact. There is a three-dimensional space ${}^1R_3$ in ${}^2R_5$. It is hypersurface $x_3 = x_5 = 0$ or hyperplane $x_1 = x_2 = 0$.

Since there is no four-dimensional Euclidean space in ${}^2R_5$, then there is no three-dimensional elliptic space S_3, which is isometric to the geometry of the sphere of four-dimensional Euclidean space.

The three-dimensional hyperbolic spaces ${}^1S_3$ and ${}^2S_3$ are isometric to the geometries of the spheres of pseudo-Euclidean spaces ${}^1R_4$ and ${}^2R_4$, respectively. In ${}^2R_5$, the four-dimensional subspaces $x_5 = 0$ are pseudo-Euclidean space ${}^1R_4$ and subspace $x_3 = 0$ is pseudo-Euclidean space ${}^2R_4$.

In addition to the above, there are a number of semi-Euclidean $R_3^1,\ R_3^2,\ R_3^{12}$ and semi-pseudo-Euclidean ${}^{01}R_3^1,\ {}^{01}R_3^2$ spaces. These spaces are spaces with a degenerate metric [2]. Among three-dimensional spaces, there are elliptic and hyperbolic spaces with a degenerate metric. These spaces are isometric to spheres of four-dimensional semi-Euclidean and semi-pseudo-Euclidean spaces. Therefore, it is necessary to prove the existence of four-dimensional subspaces of subspace ${}^2R_5$ which is a four-dimensional semi-Euclidean and a semi-pseudo-Euclidean space. The theorem is proven.

Space $R_3^1 = \Gamma_3$ is called Galilean space. Galilean space R_3^1 is a subspace of $M\left(x,\ y,\ y,\ z,\ z\right) \subset {}^2R_5$. This assertion was proven in [6].

Three-dimensional space R_3^2 is called an isotropic space and has a degenerate metric if

$$ds^2 = \begin{cases} dx^2 + dy^2 \\ dz^2 \quad когда \quad dx^2 + dy^2 = 0. \end{cases}$$

(*)

Consider subspace $U\left(X,\ Y,\ Z\right)$ of space ${}^2R_5$, where $X = x_1,\ Y = x_2$ and $Z = x_3$, and condition $x_3^2 = x_4^2 + x_5^2$ is satisfied. It is easy to prove that under these conditions metric (1) turns into degenerate metric (*).

Likewise, we can prove that subspace $N\left(x,\ y,\ z,\ y,\ z\right)$ is a pseudo-Galilean space in space ${}^{01}R_3^1$, and subspace $H\left(x_1,\ x_2,\ x_3,\ x_4,\ x_5\right)$ with conditions $x = x_1,\ y = x_5$ and $z = x_4^2 = x_2^2 + x_3^2$ is space ${}^{01}R_3^1$.

To prove the existence of semi-hyperbolic spaces ${}^{01}S_3^1,\ {}^{01}S_3^2$, it suffices to prove the existence of four-dimensional semi-Euclidean and semi-pseudo-Euclidean spaces in spheres of which these spaces are realized.

Subspace $M\left(x_1,\ x_2,\ x_3,\ x_4,\ x_5\right)$ of space ${}^2R_5$ is semi-pseudo-Euclidean space ${}^{11}R_4^3$. In the sphere of space ${}^{10}R_4^3$, semi-hyperbolic space ${}^{10}S_3^2$ is realized.

Space ${}^{01}S_3^1$ is the sphere of subspace $W\left(x_1,\ x_2,\ x_3,\ x_4,\ x_5\right) \in {}^2R_5$ where $x_3^2 = x_4^2 + x_5^2$. Subspace W is semi-pseudo-Euclidean space R_4.

The sphere of space ${}^2R_5$, defined as the locus of points equidistant from a given point depending on the radius, is divided into three types. A sphere of real radius, a sphere of imaginary radius, and a sphere of zero radius coinciding with an isotropic cone.

When the center of the sphere is at the origin, the equation for a sphere of real radius is

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2 = r^2.$$

and for a sphere of imaginary radius, it is

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2 = -r^2. \qquad (6)$$

A sphere of space ${}^2R_5$ is a surface of constant curvature; curvature $R > 0$ for a sphere of real radius, $R < 0$ for a sphere of imaginary radius.

The space of constant curvature $R < 0$ is called de Sitter space of second kind. It has topology $S_1 \times R_3$ and can be represented as a sphere of imaginary radius [7], [8].

This definition of de Sitter space of second kind is given by Hawking in [9].

In the monograph by B. A. Rosenfeld [2], a set of points of a sphere of imaginary radius of space ${}^2R_5$ with identified diametrically opposite points is called hyperbolic space ${}^2S_4$. Obviously, these definitions are equivalent. Hyperbolic space ${}^2S_4$ is de Sitter space of second kind.

When the equation of a sphere of imaginary radius is given in spherical coordinates

$$\begin{cases} x_1 = r\cos t\, sh\chi \sin\theta \sin\phi \\ x_2 = r\cos t\, sh\chi \sin\theta \cos\phi \\ x_3 = r\cos t\, sh\chi \cos\theta \\ x_4 = r\cos t\, ch\chi \\ x_5 = r\sin t \end{cases}$$

then the metric on the sphere is

$$ds^2 = -dt^2 + \cos^2 t\left\{d\chi^2 + sh^2\chi\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right\}$$

.

When the equation of a sphere is given in cylindrical coordinates

$$\begin{cases} x_1 = shr\sin\theta\sin\phi \\ x_2 = shr\sin\theta\cos\phi \\ x_3 = shr\cos\theta \\ x_4 = chr \\ x_5 = t' \end{cases}$$

then the metric on this sphere has the following form

$$ds^2 = -dt'^2 + dr^2 + sh^2 r\left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

In [10], the Pogorelovsky analog in the mapping of hyperbolic spaces in a pseudo-Euclidean space is given. Let us construct this mapping for space ${}^2S_3$.

For convenience, the coordinate system in the space under study ${}^2R_5$ is considered a Cartesian system. Then for the unit sphere of imaginary radius of space ${}^2R_5$, plane $x_5 = 1$ is a tangent plane. On this tangent plane, vectors $\{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\}$ of the enveloping space can be taken as basis vectors. Therefore, the geometry on this plane ${}^1R_4$ is the Minkowski four-dimensional space [11]-[12], [16]-[18].

Let X be the radius vector of a point in ${}^2S_3$. Then $X^2 = -1$. Therefore, for any point $(x_1, x_2, x_3, x_4, x_5) \in {}^1S_2$, condition (6) is satisfied.

Denote the central projection of point X on the tangent plane $x_5 = 1$ by T_X.

Then

$$T_X = \frac{X - (X\cdot\vec{e}_5)\vec{e}_5}{(X\cdot\vec{e}_5)}$$

or T_X has coordinates $\left(\frac{x_1}{x_5}, \frac{x_2}{x_5}, \frac{x_3}{x_5}, \frac{x_4}{x_5}, 1\right)$ on plaxne $x_5 = 1$.

Lemma. Under a mapping of T_X of $i-$dimensional planes of space ${}^2S_4$, non $i-$dimensional planes $(i = 0, 1, 2, 3)$ are mapped.

Proof. Since we are considering the projection of the sphere of space ${}^2R_5$ from the center onto the tangent plane, the points are mapped to points. Hence, the zero-dimensional plane is mapped to the zero-dimensional plane.

An $m-$dimensional plane of space ${}^2S_5$ refers to the intersection $(m+1)$ of the plane of space ${}^2R_5$ passing through the origin with the sphere (6).

However, any plane $(m+1)$ of space ${}^2R_5$ intersects plane $x_5 = 1$ along $m-$dimensional plane. Therefore, $m-$planes of space ${}^2S_5$ correspond to $m-$plane in plane $x_5 = 1$.

A two-dimensional plane passing through the origin of coordinates of space ${}^2R_5$ intersects a sphere of imaginary radius along a curve that is an arc of a circle of large radius. This curve is called the straight line of space ${}^2S_4$. The distance δ between points $A(\vec{X})$ and $B(\vec{Y})$ of space ${}^2S_4$ is determined using the radius vectors $\vec{X}$ and $\vec{Y}$ by the following formula

$$ch\delta = \frac{(\vec{X}\cdot\vec{Y})}{|\vec{X}||\vec{Y}|}.$$

If we denote the coordinates of vector T_X by u_i and T_Y by v_i, then the distance between the samples of points $A(\vec{X})$ and $B(\vec{Y})$ is calculated by the following formula

$$ch\delta = \frac{1-u_1v_1-u_2v_2-u_3v_3+u_4v_4}{\sqrt{1-u_1^2-u_2^2-u_3^2+u_4^2}\cdot\sqrt{1-v_1^2-v_2^2-v_3^2+v_4^2}}.$$

(7)

Since the coordinates of a point of a sphere of imaginary radius are subject to condition (7) and $x_5 \neq 0$, then for the points under consideration we obtain the following inequality

$$u_1^2+u_2^2+u_3^2-u_4^2 = 1-\frac{1}{x_5^2} < 1.$$

Hence, $1-u_1^2-u_2^2-u_3^2+u_4^2 > 0$. The lemma is proven.

We obtain the following corollary.

Corollary. Under a mapping of T_X, a point in space ${}^2S_4$ is mapped in the interior of a sphere of real radius of space ${}^1R_4$.

The sphere of unit real radius of space ${}^1R_4$ has the equation

$$u_1^2+u_2^2+u_3^2-u_4^2 = 1.$$

(8)

This is a hyperbolic surface, similar to a one-strip hyperboloid in Euclidean space. At $u_4 = 0$, that is, the section of the sphere by this hyperplane is a ball bounded by the unit sphere $u_1^2+u_2^2+u_3^2 = 1$. When $u_4 = a$, the radius of this ball increases and will be equal to $r^2 = 1+a^2$. In four-dimensional space ${}^1R_4$ the boundaries of these balls will be in the form of a one-block hyperboloid.

Moreover, from Lemma, it follows that the points of space ${}^2S_4$ are the points of the interior of this sphere. The straight, two-dimensional, and three-dimensional planes are expressed by the chords of this sphere of the corresponding dimension.

For $x_4 = 0$, a subspace of space ${}^2S_4$ is the Lobachevskii space ${}^1S_3$. The image of this space in the mapping of T_X corresponds to the interior of the Euclidean sphere of space $R_3 \subset {}^1R_4$ [13].

Theorem 2. In space ${}^2S_4$, planes $x_4 = C$ form a one-dimensional foliation, the layer of which is an expanding Lobachevskii space.

Proof. The sphere of real radius of space ${}^1R_4$ given by equation (8) is a hyperbolic type surface. In space ${}^1R_4$, the plane of space ${}^2S_4$ given by equation $x_4 = 0$ corresponds to plane $u_4 = 0$, which is the Euclidean space R_3. In a ball bounded by a sphere of this space, the planes of spaces ${}^2S_4$, which are the Lobachevskii spaces ${}^1S_3$ as well, are interpreted. Then follows the interpretation.

At $t = a \neq 0$, the corresponding subspace metric has the Lobachevskii form

$$ch\delta = \frac{1+a^2-u_1v_1-u_2v_2-u_3v_3}{\sqrt{1+a^2-u_1^2-u_2^2-u_3^2}\cdot\sqrt{1+a^2-v_1^2-v_2^2-v_3^2}}$$

.

But this is a Lobachevskii space metric with curvature $K = \frac{1}{1+a^2}$, that is, it is realized in the interior of a ball of radius $r^2 = 1+a^2$. Therefore, the radius of the sphere increases. Since the radius of the ball, inside which the Lobachevskii space is realized, increases, we can conclude that the Lobachevskii space is expanding.

The fact that set $x_4 = a$ forms a foliation follows from the definition of a foliation [14], [15], because each a corresponds to the interior of a ball of real radius of space ${}^1R_4$ which is a compact manifold with a given metric.

A sphere of space ${}^1R_4$ with a real radius given by equation (8) can be represented as a one-sheet hyperbolic surface, where the belt is a unit ball of three-dimensional space R_3, and for the belt $u_4 = 0$. For $u_4 = a \neq 0$, the section of sphere ${}^1R_4$ is a Euclidean ball with radius $r^2 = 1 + a^2$.

The lines and planes of space ${}^2S_4$ are represented by a part of the lines and planes of space ${}^1R_4$ corresponding to the sections of the ball by these lines and planes. This is an analog to the Cayley–Klein interpretation of the Lobachevskii plane for a four-dimensional hyperbolic space.

Since ${}^1R_4$ is an affine space, then with the lines and planes of this space, one can define convex polyhedra of space ${}^2S_4$. Then the convex polyhedra of space ${}^2S_4$ are expressed by the convex polyhedra of space ${}^1R_4$ contained inside the sphere of real radius of space ${}^1R_4$. The theorem is proven.

At that, finite polyhedra correspond to polyhedra strictly contained inside the ball, and infinite polyhedra are polyhedra that have common points with the sphere of real radius of space ${}^1R_4$.

Conclusion

Studies have confirmed that in subspaces of space ${}^2R_5$, in addition to elliptic spaces, there is a geometry of three-dimensional spaces with projective metrics. De Sitter space of the second kind is also realized in the sphere of imaginary radius. De Sitter space is a geodesic mapping in four-dimensional Minkowski space.

References:

[1]. Dubrov B. A, Novikov S. P, Fomenko A. T. Modern geometry. Methods and Applications. Vol 1, Moscow, Nauka, 920 pp (2013).

[2]. Rosenfeld B. A. Non-Euclidean spaces. Moscow, Nauka, 548 pp (1969). [in Russian]

[3]. Vazgin V. P. To the history of the “Erlangen Programme” by F. Klein. Historical and Mathematical Studies. Moscow, Nauka, 218-248 (1973). [in Russian]

[4]. Scott P. The Geometries of 3-manifolds. Vol. 15, London mathematical society, 168 pp (1986)

[5]. Artykbaev A., Sokolov D. D. Geometry as a Whole in Flat space-time. Tashkent, Fan, 179 pp. (1991). [in Russian]

[6]. Sultonov B. M., Ismoilov Sh. Sh. Cyclic surfaces formed in pseudo-Euclidean space. International Journal of Statistics and Applied Mathematics. Vol. 5. Issue 1, 28-31 (2020)

[7]. Bang Yen Chen. Marginally trapped surfaces and surfaces with parallel mean curvature vector in lorentzian space forms. Conference: Differential Geometry, Seminar at, Michigan State University, East Lansing, MI, USA, September 2010.

[8]. Bang Yen Chen, Joeri Van der Veken. Marginally trapped surfaces in Lorentzian space forms with positive relative nullity. Institute of Physics Publishing, Class. Quanum Grav. 24 551-563 (2007)

[9]. Hawking J, Ellis. Large-scale structure of space-time. Mir Publishing House, Moscow, (1977). [in Russian]

[10]. Pogorelov A. V. Exterior Geometry of Convex Surfaces. Moscow, Nauka, 760 pp (1969). [in Russian]

[11]. Mamadaliev B.M. Geometry of complete two-dimensional surfaces in a five-dimensional pseudo-euclidean space of index two // Bulletin of the Institute of Mathematics, Vol. 6, No. 2, –P 31-43. (2023). (01.00.00 №6)

[12]. Artikbaev A, Mamadaliev B. M. Geometry on subspaces of space ${}^2R_5$. Bulletin of the Institute of Mathematics, Vol. 3, 42-46 (2021). [in Russian]

[13]. Josef Mikes, Elena Stepanova, Alena Vanzurova et al. Differential Geometry of Special Mappings. Olomouc 570 pp (2019)

[14]. Narmanov A. Ya. Geometry of orbits of vector fields and singular foliations. Modern mathematics, Fundamental directions. Vol. 65, Issue 1, 54-71 (2019). [in Russian]

[15]. Narmanov A. Ya, Saripov A. S. On the diffeomorphism groups of foliated manifolds. Modern mathematics and its applications. Subject reviews. Vol. 181, 74-83 (2020). [in Russian]

[16]. Artykbaev A., Mamadaliev B.M., Interpretation of de sitter space of second kind // Bulletin of National University of Uzbekistan: Mathematics and Natural Sciences, Vol. 4, No. 3, 2021, pp 131-141. (01.00.00 №2)

[17]. Artykbaev A., Mamadaliev B.M., Geometry of two-dimensional surfaces in the five-dimensional pseudo-euclidean space of index two // AIP Conference Proceedings, Volume 11. Issue 3. – P. 020068-1-020068-4 (2023) (3. Scopus, Cite Score IF=0.16).
[18]. Artykbaev A., Mamadaliev B.M., Geometry of Two-Dimensional Surfaces in Space ${}^{2}R_{5}$ // Lobachevskii Journal of Mathematics, Volume 44, Issue 4, 2023, pp 1243-1247. (3. Scopus Q2, Cite Score IF=0.42).