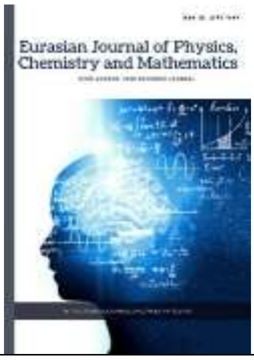

On Solution of Optimization Function Generated by Using Laplace Equation

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ABSTRACT

The Laplace equation generates in each such space the equation of minimizing the residual functional. The existence and uniqueness of optimal splines are proved. For their coefficients and residuals, exact formulas are obtained. It is shown that with increasing N, the minimum of the residual functional is $O(N^{-5})$, and the special sequence consisting of optimal splines is fundamental

Keywords: interpolation; spline ; Chebyshev's polynomials

1- Introduction

Laplace equation $u_{tt} + u_{\xi\xi} = 0$, the value specified in the rectangle is changed to the form by replacing variables $c_1 u_{tt} + c_2 u_{\xi\xi} = 0$, (in terms of new variables from the square $\Pi \doteq [0,1]^2$) Let , $c_1 > 0$, $c_2 > 0$ a continuous functions $f_0, f_1, g_0, g_1 : [0,1] \to R$ such that

$$f_0(0) = g_0(0),\ f_0(1) = g_1(0),$$

$$f_1(0) = g_0(1),\ f_1(1) = g_1(1)$$

there are derivatives $f_0''(0)$, $f_0''(1)$, $f_1''(0)$, $f_1''(1)$, $g_0''(0)$, $g_0''(1)$, $g_1''(0)$, $g_1''(1)$ and the equalities are met

$$c_1 g_0''(0) + c_2 f_0''(0) = 0, \quad c_1 g_1''(0) + c_2 f_0''(1) = 0$$

$$c_1 g_0''(1) + c_2 f_1''(0) = 0 \ ,$$

$$c_1 g_1''(1) + c_2 f_1''(1) = 0$$

Function $u = u(t,\xi)$, $(t,\xi) \in \Pi$ which is the solution to the equation

$$\begin{cases} c_1 u_{tt} + c_2 u_{\xi\xi} = 0, \\ u(0,\xi) = f_0(\xi), u(1,\xi) = f_1(\xi), u(t,0) = g_0(t), u(t,1) = g_1(t) \end{cases}$$

Represent table in the form $u = u^{(0)} + u^{(1)} + u^{(2)}$ where

$$u^0 = u^0(t,\xi) \doteq f_0(0)(1-t)(1-\xi) + f_0(1)(1-t)\xi + f_1(0)t(1-\xi) + f_1(1)t\xi$$

a bilinear function, and functions $u^1 = u^1(t,\xi)$ and $u^2 = u^2(t,\xi)$ solve equations

$$\begin{cases} c_1 u_{tt} + c_2 u_{\xi\xi} = 0, \\ u(0,\xi) = p_0(\xi), u(1,\xi) = p_1(\xi), u(t,0) == \tilde{g}_0(t), u(t,1) = \tilde{g}_1(t) \end{cases}$$

$$\begin{cases} c_1 u_{tt} + c_2 u_{\xi\xi} = 0, \\ u(0,\xi) = \tilde{f}_0(\xi), u(1,\xi) = \tilde{f}_1(\xi), u(t,0) = q_0(t), u(t,1) = q_1(t) \end{cases}$$

respectively. The decomposition uses the following notation:

$$p_i(\xi) \doteq -\frac{1}{12} f_i''(0)(\xi^3 - 3\xi^2 + 2\xi) + \frac{1}{12} f_i''(1)(\xi^3 - \xi), i = 0,1$$

$$q_i(\xi) \doteq -\frac{1}{12} g_i''(0)(t^3 - 3t^2 + 2t) + \frac{1}{12} g_i''(1)(t^3 - t), i = 0,1$$

cubic polynomials,

$$\tilde{f}_i(\xi) \doteq f_i(\xi) - f_i(0)(1-\xi) - f_i(1)\xi - p_i(\xi), i = 0,1$$

$$\tilde{g}_i(t) \doteq g_i(t) - g_i(0)(1-t) - g_i(1)t - q_i(t), i = 0,1$$

These are functions generated by boundary functions of the original problem. In this paper, we discuss a special optimization problem generated by (1), and a similar equation generated by (2) is symmetric, so we only need to perform substitutions $c_1 \leftrightarrow c_2$, $t \leftrightarrow \xi$, $p_i(.) \leftrightarrow q_i(.)$, $g_i(.) \leftrightarrow f_i(.)$

2. Statement of the problem of constructing the optimal spline

From equation (1) generates the equation of finding the optimal spline of the equation

$$J(u) \doteq \left\| c_1 u_{tt} + c_2 u_{\xi\xi} \right\|^2_{L_2(\Pi)} \to \min \qquad u \in \sigma(\Pi)$$

(3)

Where $\sigma(\Pi)$ this is a space consisting of acceptable splines that depend on the coefficients $u_1^i, u_2^i, i = 0,1,\ldots\ldots,3N-1$ (where N the parameter responsible for the number of nodes of the difference diagram), and defined in the square of Π. Let, then, be the parameter responsible for the number of nodes of the difference diagram), and defined in the square of Π. Let, then , $n \doteq N-1$, $\tau \doteq \frac{1}{3N}$, $h \doteq \frac{1}{3}$, $\theta \doteq \frac{b}{aN^2}$ and the points $(\tau_i, h_j) \in \Pi$ such that $\tau_i \doteq i\tau$, $i = 0,1,\ldots..,3N$, $h_j \doteq jh$ $j = 0,1,2,3$. set (u_j^i) , $i = 0,1,\ldots\ldots,3N$, $j = 0,1,2,3$ it is called valid if:

1) $u_0^i = \tilde{g}_0(\tau_i)$, $u_3^i = \tilde{g}_1(\tau_i)$, , $i = 0,1,\ldots\ldots,3N$;
2) $u_j^0 = p_0(h_j)$, $u_j^{3N} = p_1(h_j)$, , $j = 0,1,2,3$;

One-dimensional Lagrange interpolation polynomials

$$\omega_k(\xi) \doteq \prod_{\alpha=0,1,2,3:\alpha\neq k} \frac{\zeta - \alpha}{k - \alpha} \quad ,\zeta \in R\ ,\ ,k = 0,1,2,3$$

(such that $\omega_k(\mu) = \delta_{k\mu}$ for all $k, \mu = 0,1,2,3$ where $\delta_{k\mu}$ Kronecker symbol) and a valid array (u_j^i) , $i = 0,1,\ldots\ldots,3N$, $j = 0,1,2,3$ generate a family of two-dimensional polynomials

$$Q^k(s,\eta) \doteq \sum_{i=0}^{3}\sum_{j=0}^{3} u_j^{3k-3+i}\omega_i(s)\omega_j(\eta) \quad , \qquad s,\eta \in R$$

, $k = 0,1,\ldots\ldots,N$

Let,, next, $P^k(t,\xi) = Q^k(s,\eta)$ where $s \doteq \frac{t}{\tau} - 3k + 3$ $\eta \doteq \frac{\xi}{h} = 3\xi$. Then $P^k(\tau_{3k-3+i}, h_j) = u_j^{3k-3+i}$ for all , $k = 0,1,\ldots\ldots,N$ and , $i, j = 0,1,2,3$. Hence, the polynomial $P^k(.,.)$ is a two-dimensional Lagrange interpolation polynomial defined at 16 nodes of the band $\Pi^k \doteq \{(t,\xi) \in \Pi : \tau_{3k-3} \le t \le \tau_{3k}, 0 \le \xi \le 1\}$.

Thus, a continuous function is defined $u : \Pi \to R$ such that $u(t,\xi) = P^k(t,\xi)$ by $(t,\xi) \in \Pi^k$.In other words, every valid array generates a bicubic spline, which we call an approximating spline. The variety of such splines is determined only by sets of numbers $u_1^i, u_2^i, i = 0,1,\ldots\ldots,3N-1$. This means that the approximating splines form a finite-dimensional space. Let's denote it $\sigma(\Pi) = \sigma_N(\Pi)$.

3. The finite difference approximating splines

Any valid array (u_j^i) , $i = 0,1,\ldots\ldots,3N$, $j = 0,1,2,3$ generates a term

$x^i = u_0^i - u_1^i + u_2^i + u_3^i$, $y^i = u_0^i - 3u_1^i + 3u_2^i - u_3^i$, $i = 0,1,\ldots\ldots,3N$

$$X_0^k \doteq x^{3k-3} - x^{3k-2} - x^{3k-1} + x^{3k}$$
$$Y_0^k \doteq x^{3k-3} - 3x^{3k-2} + 3x^{3k-1} - x^{3k}$$
$$X_1^k \doteq y^{3k-3} - y^{3k-2} - y^{3k-1} + y^{3k}$$

(4)

$$Y_1^k \doteq y^{3k-3} - 3y^{3k-2} + 3y^{3k-1} - y^{3k} \quad ,k = 1,\ldots.,N$$

and boundary elements

$$z_0^k \doteq \frac{1}{2}\theta^{-1}[u_0^{3k-3} - u_0^{3k-2} - u_0^{3k-1} + u_0^{3k} + u_3^{3k-3} - u_3^{3k-2} - u_3^{3k-1} + u_3^{3k}]$$

$$w_0^k \doteq \frac{3}{2}\theta^{-1}[u_0^{3k-3} - 3u_0^{3k-2} + 3u_0^{3k-1} - u_0^{3k} + u_3^{3k-3} - 3u_3^{3k-2} + 3u_3^{3k-1} - u_3^{3k}]$$

$$z_1^k \doteq \frac{1}{6}\theta^{-1}[u_0^{3k-3} - u_0^{3k-2} - u_0^{3k-1} + u_0^{3k} - u_3^{3k-3} + u_3^{3k-2} + u_3^{3k-1} - u_3^{3k}]$$

$$w_1^k \doteq \frac{1}{2}\theta^{-1}[u_0^{3k-3} - 3u_0^{3k-2} + 3u_0^{3k-1} - u_0^{3k} - u_3^{3k-3} + 3u_3^{3k-2} - 3u_3^{3k-1} + u_3^{3k}]$$

$$p_0^k \doteq z_0^k + w_0^k,\quad q_0^k \doteq z_0^k - w_0^k,$$
$$p_1^k \doteq z_1^k + w_1^k, \qquad q_1^k \doteq z_1^k - w_1^k,\ ,k = 1,\ldots.,N$$

(5)

(recall that $\theta = \frac{b}{aN^2}$; see comments on (4))

$\xi^0 \doteq p_0^1 + x^0$, $\xi^k \doteq p_0^{k+1} - q_0^k$, $\xi^N \doteq -x^{3N} - q_0^N$

$\eta^0 \doteq p_1^1 + y^0, \eta^k \doteq p_1^{k+1} - q_1^k, \eta^N \doteq -y^{3N} - q_1^N$
$, k = 1,\dots,n$ (6)
$\xi \doteq col(\xi^0, \xi^1, \dots, \xi^N)$, $\eta \doteq col(\eta^0, \eta^1, \dots, \eta^N)$

4. The results

1. For any $u \in \sigma_N(\Pi)$, the equality holds

$$j(u) = \frac{3a^2}{64\tau^3} \sum_{k=1}^{N} I^k$$

$$I^k \doteq 4\theta^2[x^{3k-3} + x^{3k} + 2z_0^k]^2 - 6\theta(1+\theta)[x^{3k-3} + x^{3k} + 2z_0^k]^2 X_0^k$$

$$+\frac{9}{10}(3+5\theta+3\theta^2)[X_0^k]^2 + \frac{4}{3}\theta^2[x^{3k-3} - x^{3k} + 2w_0^k]^2$$

$$-\frac{6}{5}\theta(5+\theta)[x^{3k-3} - x^{3k} + 2w_0^k]Y_0^k + \frac{27}{70}(21+7\theta+\theta^2)[Y_0^k]^2$$

$$12\theta^2[y^{3k-3} + y^{3k} + 2z_1^k]^2 - \frac{18}{5}\theta(1+5\theta)[y^{3k-3} + y^{3k} + 2z_1^k]^2 X_1^k$$

$$+\frac{27}{70}(1+7\theta+21\theta^2)[X_1^k]^2 + 4\theta^2[y^{3k-3} - y^{3k} + 2w_1^k]^2$$

$$-\frac{18}{5}\theta(1+\theta)[y^{3k-3} - y^{3k} + 2w_1^k]Y_1^k + \frac{81}{350}(5+7\theta+5\theta^2)[Y_1^k]^2$$

2. The Coefficients $u_1^i, u_2^i, i = 0,1,\dots\dots,3N-1$ of the optimal approximating spline $u \in \sigma_N(\Pi)$ are computable by formulas (6) in terms of the values x^{3k} , y^{3k} , $k = 0,1,\dots.,N$, X_0^k, y_0^k , X_1^k, Y_1^k , $k = 1,\dots.,N$ satisfying the system of equations

$$\begin{cases} x^{3k-3} + 2yx^{3k} + x^{3k+3} + v_0^k = 0, k = 1,\dots,n \\ X_0^k = \gamma_0[x^{3k-3} + x^{3k} + 2z_0^k], k = 1,\dots\dots.N \\ Y_0^k = \delta_0[x^{3k-3} - x^{3k} + 2w_0^k], k = 1,\dots\dots.N \end{cases}$$
(7)
$$\begin{cases} y^{3k-3} + 2xy^{3k} + y^{3k+3} + v_1^k = 0, k = 1,\dots,n \\ X_1^k = \gamma_1[y^{3k-3} + y^{3k} + 2z_1^k], k = 1,\dots\dots.N \\ Y_1^k = \delta_1[y^{3k-3} - y^{3k} + 2w_1^k], k = 1,\dots\dots.N \end{cases}$$

The system includes constants x^0 , x^{3N} , y^0 , y^{3N} (according to definitions (4), they are known a priori), and numbers

$$\alpha_0 \doteq \frac{1}{2}(1+\theta^2)/(3+5\theta+3\theta^2),$$
$$\beta_0 \doteq \frac{1}{30}(35+3\theta^2)/(21+7\theta+\theta^2),$$
$$\gamma_0 \doteq \frac{10}{3}\theta(1+\theta)/(3+5\theta+3\theta^2),$$
$$\delta_0 \doteq \frac{14}{9}\theta(5+\theta)/(21+7\theta+\theta^2),$$
$$\alpha_1 \doteq \frac{1}{2}(3+35\theta^2)/(1+7\theta+21\theta^2),$$
$$\beta_1 \doteq \frac{3}{2}(1+\theta^2)/(5+7\theta+5\theta^2),$$
$$\gamma_1 \doteq \frac{14}{3}\theta(5+\theta)/(1+7\theta+21\theta^2),$$
$$\delta_1 \doteq \frac{70}{9}\theta(1+\theta)/(5+7\theta+5\theta^2),$$
$$y \doteq (\alpha_0 + \beta_0)/(\alpha_0 - \beta_0),$$
$$x \doteq (\alpha_1 + \beta_1)/(\alpha_1 - \beta_1)$$

(true $\alpha_0 > \beta_0$, $\alpha_1 > \beta_1$, $y > 2, x > 2$) and boundary elements

$$\upsilon_0^k \doteq (1+y)[z_0^k + z_0^{k+1}] + (1-y)[\omega_0^k - \omega_0^{k+1}]$$
$$\upsilon_1^k \doteq (1+x)[z_1^k + z_1^{k+1}] + (1-x)[\omega_1^k - \omega_1^{k+1}]$$

The first vector equation in (7) has an independent character, that is, it is a system containing only unknown x^{3m}. The matrix of this system has a three diagonal form with the dominant main diagonal (since y > 2), so the system has a unique solution (which is easy to find by the run-through method). After determining the unknown x^{3m} from the second and third vector equations in (7), all the values of X_0^k and y_0^k are explicitly calculated . Similarly, the fourth (where x > 2), the fifth and sixth vector equations in (7). the Obtained values ultimately allow us to find the values $u_1^i, u_2^i, i = 0,1,\dots\dots,3N-1$, that generate the optimal approximating spline.

3. The only solution of the first and fourth subsystems in the system (*7*) are numbers

$$x^{3k} = -\frac{1}{U_n(y)}[B_{k1}(y)x^0 + B_{kn}(y)x^{3N} + \sum_{i=1}^{n} B_{ki}(y)\upsilon_0^i] \quad , k = 1,\dots,n$$

$$y^{3k} = -\frac{1}{U_n(x)}[B_{k1}(x)y^0 + B_{kn}(x)y^{3N} + \sum_{i=1}^{n} B_{ki}(x)\upsilon_1^i] \quad , k = 1,\dots,n$$

The representation uses Chebyshev polynomials of the 2nd kind $U_n(.)$, (according to [5], p. 96, inequalities y > 2, x > 2 entail the inequality $U_n(y) \neq 0$, $U_n(x) \neq 0$) . They generate functional matrices $B(.) = (B_{ki}(.))$,

$$B_{ki}(.) = (-1)^{k+i}[\delta^{\geq}_{i-1,k} U_{k-1}(.) U_{n-k}(.) + \delta^{\geq}_{ki} U_{n-k}(.) U_{i-1}(.)] \quad , k, i = 1, \dots, n$$

The symbol $\delta^{\geq}_{ki}$ is used such that $\delta^{\geq}_{ki} = 0$ for $k < i$ and $\delta^{\geq}_{ki} = 1$ for $k \geq i$.

4. For the minimum $J_N \doteq \min J(.)$ of the functional (4), the exact formula holds

$$J_N = \frac{81b^2}{16N}\left[\frac{\alpha_0 - \beta_0}{U_n(y)}\left\langle \xi, \tilde{B}(y)\xi \right\rangle + \frac{\alpha_1 - \beta_1}{U_n(x)}\left\langle \xi, \tilde{B}(x)\eta \right\rangle\right]$$

The representation uses Chebyshev polynomials of the 1st kind $T_n(.)$, that generate functional matrices $\tilde{B}(.) = (\tilde{B}_{ki}(.))$ such that

$$\tilde{B}_{ki}(.) = (-1)^{k+i}[\delta^{\geq}_{i-1,k} T_k(.) T_{N-i}(.) + \delta^{\geq}_{ki} T_{N-k}(.) T_i(.)] \quad , k, i = 1, \dots, N$$

The vectors ξ and η are boundary elements (6), and the scalar product $\langle .,. \rangle$ of the space R^{1+N} is used to write quadratic forms

5. If $g_0, g_1 \in C^5[0,1]$, then the sequence $\{\bar{u}_m\}$ is fundamental by the norm of the space $L_2(\Pi)$.

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