
# A new method spectral conjugate gradient in unconstrained optimization

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ABSTRACT

In this study, we provide novel spectral conjugate gradient optimization techniques and analyze their convergence. Numerical tests reveal that our approaches can perform better than those already in use. the spectral conjugate gradient methods are commonly used for unconstrained optimization, especially when the dimension is large. Updated spectral techniques for tackling unconstrained optimization issues are developed based on curvature information. The strategies provided meet the descending criteria. The conjugate gradient algorithm is a powerful iterative method based on the parameters' conjugate gradient. We analyze the convergence properties of the algorithm and then give some numerical results which show the modified algorithms are robust and efficient. Furthermore, it is demonstrated that the innovative spectral techniques are globally convergent. The numerical findings show that the suggested techniques are successful when compared to the Fletcher –Reeves method.

**Keywords:** conjugate gradient, spectral type(SCG), Strong Wolfe-Powel Line search(SW), Sufficient decent property, Global Converge, Unconstrained optimization

## 1. Introduction

The CG-method 's quick convergence and minimal storage requirement, it is widely utilized for optimization (Hager and Zhang 2006)[1]. For the continuously differentiable objective function f, we seek to solve the following mathematically optimization(unconstrained) problem $f: R^n \to R$:

$$Minimim\ \{f(x) \text{ where } x \in R^n\}, \qquad (1)$$

Where $g_n = \nabla f(x_n)$ defines the gradient and $f: R^n \to R$ is a continuously differential nonlinear function, and $x_0$ is any beginning approximation to the solution to problem (1).
The CG method iterates formula is provided by following:

$$x_{k+1} = x_k + \alpha_k d_k\ ,\ \forall . k \geq 0 \qquad (2)$$

Where $d_k$ is the search direction & $\alpha_n$ is a positive step length ,[2]. Step size is established by :

$$d_k = \begin{cases} -g_k & , if\ k = 0 \\ -g_k + \beta_k d_{k-1} & , if\ k \geq 1 \end{cases} \qquad (3)$$

Generally , Where $\beta_k$ is a scalar that is calculated using a different formula depending on the conjugate gradient technique being employed , There are six main types of $\beta_k$ , which include :Hestenes & Stiefel method (HS , 1952)[3] , Fletcher & Reeves method ( FR , 1964) [4] , Polka&Ribiere-Polak method (PRP, 1969) [5] , Conjugate Descent [6] (CD) and the Liu & Storey (LS, 1991) [7] , Dai-Yuan (Dy, 1999) method [8] .

$$\beta_K{}^{HS} = \frac{g_k^T (g_k - g_{k-1})}{(g_k - g_{k-1})^T d_{k-1}} ; \ \beta_K{}^{FR} = \frac{g_k^T g_k}{\|g_{k-1}\|^2} ; \quad \beta_K{}^{PRP} = \frac{g_k^T (g_k - g_{k-1})}{\|g_k\|^2} ;$$

$$\beta_K{}^{CD} = -\frac{g_k^T g_k}{g_{k-1}^T d_{k-1}} ; \ \beta_K{}^{LS} = \frac{g_k^T y_{k-1}}{-g_{k-1}^T d_{k-1}} ; \ \beta_K{}^{DY} = \frac{g_k^T g_k}{y_{k-1}^T d_{k-1}}$$

The convergence of CG-method has been researched by several authors where $y_{k-1} = g_k - g_{k-1}$ Under various line-searches, some have estimated the step length using an exact-line-search (ELS), but often $\alpha_k$ is created via an inexact line search, such as the Wolfe line search condition (WL) which is described by :

$$\begin{aligned} f(x_k + \alpha_k d_k) &\le f(x_k) + \delta \alpha_k g_k^T d_k \\ d_k^T g(x_k + \alpha_k d_k) &\ge \sigma\, d_k^T g_k \end{aligned} \qquad (4)$$

Alternative definitions for the Strong Wolfe Line search (SWL) :

$$\begin{aligned} f(x_k + \alpha_k d_k) &\le f(x_k) + \delta \alpha_k g_k^T d_k \\ \left| g(x_k + \alpha_k d_k)^T d_k \right| &\le \sigma \left| g_k^T d_k \right| \end{aligned} \qquad (5)$$

where $\mathbf{0} < \boldsymbol{\delta} < \frac{1}{2} < \boldsymbol{\sigma} < 1$.

The spectral Cojugate gradient method (SCG) , which Barzilai and Borwein that the firstly proposed [9] , a well-known approach, can be applied to address the issue (1). The direction $\mathbf{d}_{k+1}$ is defined by :

$$d_k = \begin{cases} -g_k & , if\ k = 0 \\ -\boldsymbol{\theta}_k\, g_k + \beta_k d_{k-1} & , if\ k \ge 1 \end{cases} \qquad (6)$$

The parameter of spectral gradient is $\boldsymbol{\theta}_k$ , where? Later, Raydan developed the SCG-method in [10] for high dimention unconstrained optimization problems.The SCG-method's interesting characteristic is that it simply requires gradient directions to guarantee global convergence for each line search . The SCG-method performs better than the advanced CG approach in a surprising number of recognized cases . Birgin and Martinez[10] found that their SCG-method is globally convergent under several fair assumptions . The SCG-method cannot , however , be called on to generate the right direction . Because of this , Jiang et al. , Wolfe line search , and Andrei [11] stated the descending SCG-algorithm. Following on an improved CG algorithm put forward by Zhang et al. , numerous writers have developed the SCG-method with appropriate descent condition [1], [12]–[15] .

## 2. The descent property and a New spectral CG-method

We shall compute a new spectral parameter $\boldsymbol{\theta_k}$ in this section. The SCG-method's typical search direction is as follows:

$$d_k = -\theta_k g_k + \beta_k d_{k-1}, \forall k \geq 1 \qquad (7)$$

Second, using the conjugate parameter approach from reference [16] with the form

$$\beta^{JYJ}_k = \frac{\|g_k\|^2 - \frac{(g^T_k d_{k-1})^2}{\|d_{k-1}\|^2}}{\|g_{k-1}\|^2 + \mu |d^T_{k-1} g_k|}, \mu = 2 \qquad (8)$$

Using the hybrid concept of Reference[17] completely , we suggest a new conjugate parameter in the way shown below .

$$\beta^{ALI}_k = \frac{\|g_k\|^2 - \frac{(g^T_k d_{k-1})^2}{\|d_{k-1}\|^2}}{max\{\|g_{k-1}\|^2, d^T_{k-1} (g_k - g_{k-1})\}} \qquad (9)$$

So far, a fundamental understanding of our SCGM has been developed. Consequently, a new SCGM is proposed, and its theoretical characteristics and numerical performance are evaluated and reported. Multiply both sides of the second element of the expression (7) by $\boldsymbol{y_{k-1}}$ , and we will get

$$d^T_k y_{k-1} = -\theta_k g^T_k y_{k-1} + \beta_k d^T_{k-1} y_{k-1} \qquad (10)$$

Actual algorithms, on the other hand, typically use inexact line searches rather than exact line searches. Dai and Liao [18] recently replaced the conjugation criterion.
by the condition:

$$d^T_k y_{k-1} = -t g^T_k s_{k-1} = 0, where\ t = 0 \qquad (11)$$

Substitutes (8) and (11) in (10), we get:

$$0 = -\theta_k g^T_k y_{k-1} + \beta^{ALI}_k d^T_{k-1} y_{k-1}$$

$$0 = -\theta_k g^T_k y_{k-1} + \frac{\left(g^T_k g_k - \frac{(g^T_k d_{k-1})^2}{d^T_{k-1} d_{k-1}}\right)}{max\{\|g_{k-1}\|^2, d^T_{k-1} (g_k - g_{k-1})\}} d^T_{k-1} y_{k-1}$$

$$\theta_k g^T_k y_{k-1} = \frac{g^T_k g_k d^T_{k-1} y_{k-1} - \frac{(g^T_k d_{k-1})^2}{d^T_{k-1} d_{k-1}} d^T_{k-1} y_{k-1}}{max\{\|g_{k-1}\|^2, d^T_{k-1} (g_k - g_{k-1})\}}$$

$$\theta_k g^T{}_k y_{k-1} = \frac{g^T{}_k \, g_k \, d^T{}_{k-1} \, y_{k-1}}{max\{\|g_{k-1}\|^2, \; d^T{}_{k-1} \, (g_k - g_{\,k-1})\}} - \frac{\frac{(g^T{}_k \, d_{k-1})^2}{d^T{}_{k-1} d_{k-1}} d^T{}_{k-1} \, y_{k-1}}{max\{\|g_{k-1}\|^2, \; d^T{}_{k-1} \, (g_k - g_{\,k-1})\}}$$

$$\frac{\theta_k g^T{}_k y_{k-1}}{g^T{}_k y_{k-1}} = \frac{g^T{}_k \, g_k \, d^T{}_{k-1} \, y_{k-1}}{max\{\|g_{k-1}\|^2, \; d^T{}_{k-1} \, (g_k - g_{\,k-1})\}} - \frac{\frac{(g^T{}_k \, d_{k-1})(g^T{}_k \, d_{k-1})}{d^T{}_{k-1} d_{k-1}} d^T{}_{k-1} \, y_{k-1}}{max\{\|g_{k-1}\|^2, \; d^T{}_{k-1} \, (g_k - g_{\,k-1})\} \, g^T{}_k y_{k-1}}$$

$$\theta_k^{ALI} = \beta^{ALI}{}_k \, \frac{d^T{}_{k-1} \, y_{k-1}}{g^T{}_k y_{k-1}} \; , \qquad (12)$$

The parameter $\beta^{ALI}{}_k$ **as in 9,** note that, if $1 < \theta_k < 0$, then we put $\theta_k = 1$. i.e (6) decrease to (3)

**New ( SCG) algorithm :**

**Procedure 1:** Assuming a starting point $x_0 \in R^n$, $\epsilon = 10^{-6} > 0$, Let $d_0 = -g_0$, $set\ k = 1$

**Procedure 2:** check $\|g_k\| \le \epsilon$ , terminate. else go to step 2.

**Procedure 3:** Determine a step length $\alpha_k$ by (4).

**Procedure 4:** Generate new points through (2), calculate the gradient $g_{k+1} = g\,(x_{k+1})$ , test $\|g_{k+1}\| \le \epsilon$ , terminate . Otherwise , continue.

**Procedure 5:** Calculate the spectral parameter $\theta_k^{ALI}$ **by** (12), if $0 < \theta_k < 1$ , put $\theta_k = 1$ ; Else, evaluate the conjugate parameter $\beta_k^{ALI}$ represented by (9).

**Procedure 6:** The direction $d_k$ defined in (6).

**Procedure 7:** If the Powel restart requirements are met,

$$|g^T{}_k \, g_{k-1}| \ge 0.2 \, \|g_k\|^2 \qquad (13)$$

Put $d_k = -g_k$ go to 3; otherwise, continue.

**Procedure 8:** Put $k = k + 1$ and go to 4.

Now comes the verification of the algorithm's descent condition for the suggested parameters:

**Theorem (1)**

Suppose the SCG-method with search direction equation (6) and the parameter $\beta_k^{ALI}$ giver in (9), and the step length $\alpha_k$ is obtained by (SWL) . The following sufficient descent property holds :

$$g^T{}_k \, d_k \le -\eta \, \|g_k\|^2 \;\; , \;\; \eta > 0 \, , \forall \, k \ge 0 \qquad (14)$$

**Proof:** To prove this assertion, we will use mathematical induction, if $k = 0$ , $g_0^T d_0 = -\|g_0\|^2$. Therefore, condition (14) holds true. Now we assume that $k \geq 0$ is correct. Condition (14) also hold, now multiply both sides of (6) by ${g^T}_k$ , we get

$$ {g^T}_k\, d_k = -{\theta^{ALI}}_k\, \|g_k\|^2 + {\beta^{AlI}}_k\, {g^T}_k\, d_{k-1} $$

$$ {g^T}_k\, d_k = -{\beta^{ALI}}_k\, \frac{{d^T}_{k-1}\, y_{k-1}}{{g^T}_k\, y_{k-1}}\, \|g_k\|^2 + {\beta^{AlI}}_k\, {g^T}_k\, d_{k-1} $$

We have,

$$ {g^T}_k\, d_k = -{\beta^{ALI}}_k \left( \frac{{d^T}_{k-1}\, y_{k-1}}{{g^T}_k\, y_{k-1}}\, \|g_k\|^2 - {g^T}_k\, d_{k-1} \right) \quad (15) $$

Used one side of (13) in (16), we get

$$ \begin{aligned} g_k^T\, y_{k-1} &= g_k^T (g_k - g_{k-1}) \\ &= \|g_k\|^2 - g_k^T\, g_{k-1} \end{aligned} \quad (16) $$

$$ \begin{aligned} g_k^T\, y_{k-1} &\leq \|g_k\|^2 + 0.2\, \|g_k\|^2 \\ &= 1.2\, \|g_k\|^2 \end{aligned} \quad (17) $$

And used another sides of (13) in (16) , we get

$$ \begin{aligned} g_k^T\, y_{k-1} &\geq \|g_k\|^2 - 0.2\, \|g_k\|^2 \\ &= 0.8\, \|g_k\|^2 \end{aligned} \quad (18) $$

by assumption, $d_{k-1}^T\, g_{k-1} < 0$ , it follows that

$$ d_{k-1}^T\, g_k = y_{k-1}^T\, d_{k-1} + d_{k-1}^T\, g_{k-1} < y_{k-1}^T\, d_{k-1} \; . $$

$$ i.e. \; d_{k-1}^T\, g_k < y_{k-1}^T\, d_{k-1} \quad (19) $$

Follows by (19) , $y_{k-1}^T\, d_{k-1} = (g_k - g_{k-1})^T d_{k-1}$ and **SWL , we get**

$$ -(1-\sigma)\; g_k^T\, d_{k-1} \leq y_{k-1}^T\, d_{k-1} \leq -(1+\sigma)\; g_k^T\, d_{k-1} \quad (20) $$

Setting equation (13), (17-20) in equation (15)

$$ {g^T}_k\, d_k \leq -{\beta^{ALI}}_k \left( \frac{{d^T}_{k-1}\, y_{k-1}}{{g^T}_k\, y_{k-1}}\, \|g_k\|^2 - y_{k-1}^T\, d_{k-1} \right) $$

$$ \begin{aligned} &= -{\beta^{ALI}}_k \left(1 - \frac{1}{1.2}\right) {d^T}_{k-1}\, y_{k-1} \\ &\leq -{\beta^{ALI}}_k \left(1 - \frac{1}{1.2}\right) {d^T}_{k-1}\, g_k \end{aligned} $$

Also using the reality $\mathbf{A^T B} \leq \frac{1}{2}(\mathbf{A}^2 + \mathbf{B}^2)$

$$ {g^T}_k\, d_k \leq -{\beta^{ALI}}_k \left(1 - \frac{1}{1.2}\right) (\|{d^T}_{k-1}\|^2 + \|g_k\|^2) $$

Then

${g^T}_k\, d_k \leq -C\|g_k\|^2$ where the $C = {\beta^{ALI}}_k \left(1 - \frac{1}{1.2}\right)$

**3. Convergence analysis:**

We assume that $f(x)$ meets the following conditions in what follows:

i. $f(x)$ is bounded on the set $\Psi = \{x \in R^n : f(x) \le f(x_0)\}$.

ii. $g$ is Lipschitz continuous, i.e. there exists a constant $L > 0$. That is:

$$\|g(z) - g(u)\| \le L\,\|z - u\|, \ \forall\ z, u \in R^n . \tag{23}$$

With a constant ¥ $\ge 0$ such that $\|\nabla f(x)\| \le$ ¥ , see [19].

**Lemma 1.1** : Assuming that the assumptions are correct, think about any recurrence expression (2) with search direction (3). The Zoutendijk criterion (16) is satisfied in that case [20] .

**Theorem 3.2.** : Assume that the direction $d_{k+1}$ was produced using new algorithms. Then:

$$\lim_{k \to \infty} \inf \|g_k\| = 0 . \tag{24}$$

**Proof :**

Let's assume that the conclusion is false because of contradiction. Then a positive constant $\gamma > 0$ exists. By contradiction,
such that

$$\|\boldsymbol{g_k}\|^2 \ge \gamma^2 \tag{25}$$

Again, from (7), we obtain combining this equation and Lemma 1, we have to prove that the direction $d_k$ is bounded, now taking the norm function both sides
Firstly $\|\boldsymbol{d_0}\| = \|\boldsymbol{g_0}\|$, which is bounded
Suppose that
$\|\boldsymbol{d_{k-1}}\|$is also bounded for $k-1$

(26)

$$\|\boldsymbol{d_k}\| = \left\|-\boldsymbol{\theta^{ALI}}_{\boldsymbol{k}}\ \boldsymbol{g_k} + \boldsymbol{\beta^{AlI}}_{\boldsymbol{k}}\boldsymbol{d_{k-1}}\right\|$$

And distributed the norm in right side

$$\|\boldsymbol{d_k}\| \le \left\|\boldsymbol{\theta^{ALI}}_{\boldsymbol{k}}\ \boldsymbol{g_k}\right\| + \left\|\boldsymbol{\beta^{AlI}}_{\boldsymbol{k}}\boldsymbol{d_{k-1}}\right\|$$

$$\le \left|\boldsymbol{\theta^{ALI}}_{\boldsymbol{k}}\right| \|\boldsymbol{g_k}\| + \left|\boldsymbol{\beta^{AlI}}_{\boldsymbol{k}}\right| \|\boldsymbol{d_{k-1}}\|$$

As we now the conjugacy parameters are scale parameter and belong the (0,1) and the spectral parameter practically are in range (0,1) also and in when this parameter take value outside of these range which reset to value 1. So that

$$\|\boldsymbol{d_k}\| \le \|\boldsymbol{g_k}\| + \|\boldsymbol{d_{k-1}}\|$$

Consequence to eq (24-25) that the directions $\boldsymbol{d_k}$ are bounded then the convergence is inevitable.

**4. Numerical results**

This section includes the outcomes of novel techniques on a number of test problems. The codes are created in Fortran, and double precision calculations are used. Computers are used to conduct every exam. Our tests are based on a set of 25 nonlinear situations that a transducer is capable of producing. These problems are discussed in Andrei[21] and are part of the CUTE test. The standard stop, s.t. $\|g_{k+1}\| \le 10^{-6}$ , is used in all algorithms. The total number of function evaluations (NOF) and the total number of iterations are both taken into account while evaluating the algorithms' performance (NOI) and (CPU). Tables 1and 2 contains the results. There are important papers in the field of optimization as [22]–[25] .

**Table 1 : The numerical results of the FR and New method with n=100.**

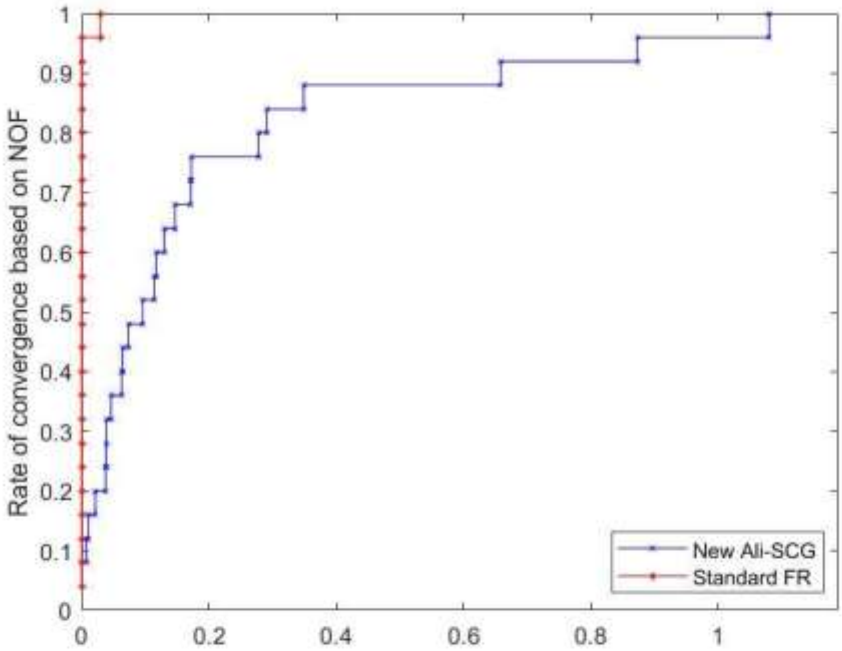
| Functions | FR Algorithm | | | New Algorithm | | |
|---|---|---|---|---|---|---|
| | NOI | NOFG | CPU | NOI | NOFG | CPU |
| Freudenstein & Roth - FREUROTH (CUTE) | 907 | 1047 | 2 | 426 | 495 | 0 |
| Trigonometric | 21 | 47 | 0 | 22 | 45 | 0 |
| Extended Rosenbrock SROSENBR (CUTE) | 1715 | 1888 | 4 | 1035 | 1196 | 3 |
| Extended White & Holst | 2001 | 2187 | 3 | 2001 | 2092 | 4 |
| Penalty | 28 | 67 | 0 | 28 | 66 | 0 |
| Diagonal 1 | 210 | 350 | 1 | 185 | 339 | 1 |
| Generalized Tridiagonal 1 | 44 | 76 | 0 | 41 | 74 | 0 |
| Extended Tridiagonal 1 | 1067 | 1143 | 2 | 896 | 1015 | 2 |
| Extended Three Expo Terms | 33 | 57 | 0 | 22 | 47 | 0 |
| Diagonal 4 | 58 | 116 | 0 | 53 | 106 | 0 |
| Extended Himmelblau | 26 | 60 | 0 | 24 | 57 | 0 |
| Extended Maratos | 1766 | 2025 | 4 | 1608 | 1871 | 3 |
| Extended Wood WOODS (CUTE) | 1875 | 2154 | 5 | 1823 | 2098 | 4 |
| Quadratic QF1 | 191 | 337 | 1 | 194 | 328 | 0 |
| Extended Tridiagonal 2 | 81 | 149 | 0 | 65 | 117 | 2 |
| TRIDIA (CUTE) | 571 | 766 | 1 | 523 | 782 | 1 |
| ARWHEAD (CUTE) | 106 | 197 | 0 | 103 | 196 | 0 |
| NONDQUAR (CUTE) | 2001 | 2251 | 4 | 2001 | 2033 | 0 |
| EG2 (CUTE) | 2001 | 2136 | 10 | 2001 | 2121 | 8 |
| EDENSCH (CUTE) | 44 | 93 | 0 | 46 | 87 | 0 |
| ENGVAL1 (CUTE)(64) | 50 | 93 | 0 | 39 | 76 | 0 |
| DENSCHNA (CUTE) | 37 | 64 | 0 | 31 | 59 | 1 |
| Extended Block-Diagonal BD2 | 85 | 151 | 2 | 68 | 134 | 0 |
| ARGLINB (CUTE) | 1 | 3 | 0 | 1 | 3 | 0 |
| HIMMELBG (CUTE) | 1047 | 907 | 5 | 426 | 495 | 1 |

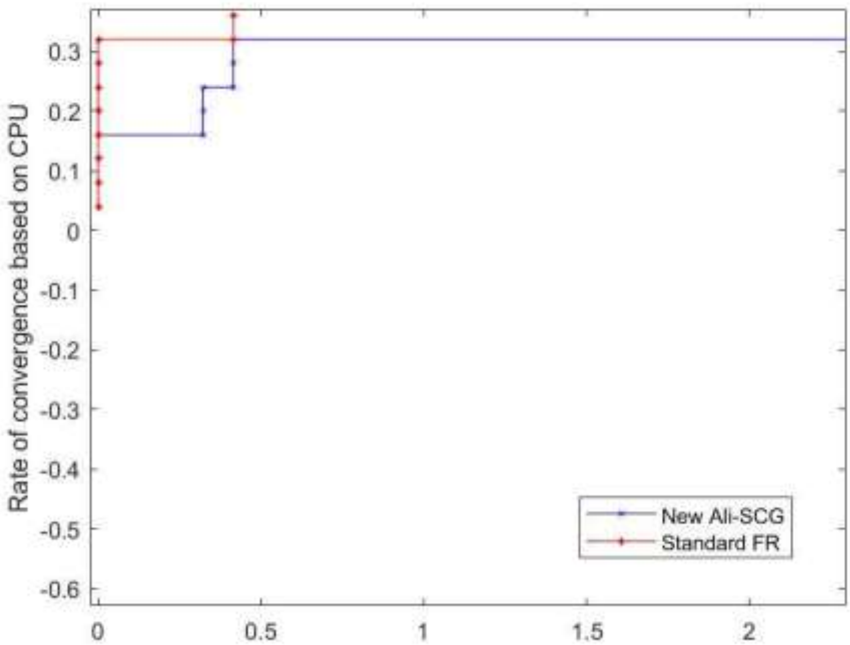
According to this data, the BK1, BK2 and BK3 approaches have a clear advantage over the FR technique since they save around (38-42)% in NOF but only (42-59)% in NOI.

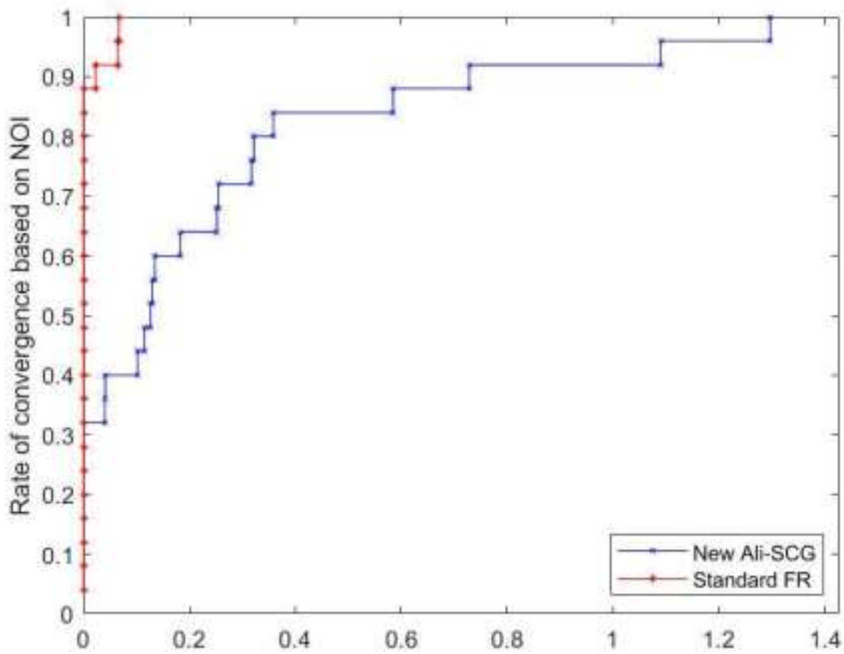
| | NOI | NOFG | CPU |
|---|---|---|---|
| FR algorithm | 100 % | 100 % | 100 % |
| New algorithm | 88% | 83% | 71% |

## 5. Conclusions

Finally, we presented the spectral conjugate gradient methods with parameters denoted by $\boldsymbol{\theta_k^{ALI}}$ and $\boldsymbol{\beta_k^{ALI}}$, as well as a novel modified conjugate gradient formula. Wolfe Line search settings allowed us to identify its global convergence. Simulations have demonstrated that the new algorithm may decrease function evaluations and iterations. While the conventional secant relation only employs gradient values, the modified conjugate gradient approach uses both gradient and function values.



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