Cofinitely Lifting Semi modules

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ABSTRACT

In this paper (cofinitely lifting semimodules) are defined which generalize cofinitely lifting modules. We looking for some properties of these semimodules. We demonstrate all cofinite direct summnd of (cofinitely lifting semimodule) is ("cofinitely lifting"). Furthermore, we prove that if cofinitely lifting subtractive semimodule A as well as fully invariant subsemimodule U of A , $(\frac{A}{U})$ is also ("cofinitely lifting").

Keywords: cofinitely lifting semimodule, cofinite subsemimodule, coclosed subsemimodule, fully invariant subsemimodule.

1. Introduction

In 2010, Wang and Wu [14] defined cofinitely lifting module. We present the idea of (cofinitely lifting semimodules) in this study. Firstly, let use point that, S will indicate an associtione semiring with identity as well as A will indicate a unitary left S-semimodule throughout this article. A (left) S-semimodule A is commutative additive semigroup with zero elements 0_A, jointly with mapping from $S \times A$ into A (sending $(s,\ a)$ to sa) in such a way $(r+s)a = ra + rs$, $r(a+b) = ra + rb$, $r(sa) = (rs)a$ as well as $0_a = s0_a = 0$ for all a, $b \in A$ as well as $r,\ s\ \in S$. Let U be subset of A. We say that U is an S-subsemimodule of A, identified by $U \leq A$, if as well as only if N is itself an S-semimodule concerning the process for A [5]. A subsemimodule $U \leq A$ is known as essential in A, denoted by $U \leq_e A$ (or $U \leq_e A$), if $(U \cap L \neq 0$)for every non-zero subsemimodule $(L \leq A)$ [12]. A subsemimodule $U \leq A$ is said to be small or superfluous in A (writes $U \ll A$), if for every subsemimodule $K \leq A$ with $U + K = A$ assume that $K = A$ [13]. radical of S-semimodule A, denoted by $Rad(A)$, is the total of all small subsemimodules of (A) [13]. An S-semimodule (A) is said to be direct sum of subsemimodules $A_1,\ A_2, \ldots, A_n$ of(A), if each $(a \in A)$ can be written uniquely as $(a = a_1 + a_2 + \cdots + a_n$ where $(a_i \in A_i, 1 \leq i \leq n)$. It is denoted by $(A = A_1 \oplus A_2 \oplus \ldots \oplus A_n$.) Each A_i is known as direct summand of A [8]. A is named simple if it has no nontrivial subsemimodules, as well as A is semisimple if A is direct sum of its simple subsemimodules [8]. A semiring S is named semilocal semiring if $S/Rad(S)$ is semisimple. Socle of A meaning $Soc(A)$, is the total of all simple subsemimodules of (A) [8]. Let $(U$,$K \leq A)$. K is known as supplement of U in A if it is minimal with respect to $(A = U + K)$.(A) subsemimodule (K of A) is supplement of U in A if and only if $A = U + K$ as well as $U \cap$

$K \ll K$ [2]. A is supplemented if each subsemimodule U of A has supplement in A. $(U \leq A)$ has ample supplements in (A) if each subsemimodules of A such that $A = U + K$ contains supplement of U in A.A semimodule (A) is known as amply supplemented if every subsemimodule of (A) has ample supplements in A. Hollow semimodules are clearly amply supplemented [2]. A semimodule (A) is said to be lifting if for every subsemimodule U of A, there is decomposition $(A = A_1 \oplus A_2$)such that $(A_1 \leq U)$ as well as $(A_2 \cap U \ll A)$ [12]. Let (A) be semimodule as well as $(U \leq A.$ $U)$ is said to be cofinite subsemimodule of A if A/U is finitely generated. $U \leq A$ is subtractive subsemimodule of A if a, $a + (b \in U)$ then $(b \in U)$ [8]. If every $(U \leq A)$ is subtractive, then (A) is named subtractive. If U is subtractive subsemimodule, then $(\frac{A}{U})$ is an S-semimodule [5, p. 165]. Section 2 is devoted to introducing the idea that(cofinitely lifting semimodules) , a broadening of the meaning of cofinitely lifting modules. In this section we prove that if semimodule A is duo as well as A as finite direct sum to (cofinitely lifting semimodules) , then A is cofinitely lifting semimodule. Section 3 discusses (cofinitely lifting semimodules) in case (A) is direct sum of two relatively cofinitely small projective semimodules.

2. On Cofinitely Lifting Semimodules

This part provides an introduction to the idea of (cofinitely lifting semimodules) . Several properties of cofinitely lifting semimodulels are given. We begin with definition of cofinitely supplemented.

Similar to [15] or [1], we have the next explanation:

Definition 2.1: A semimodule (A) is named (cofinitely supplemented) if to all cofinite subsemimodule of (A) contains supplement in A.

Similar to [10] or [6], we have the next explanation:

Definition 2.2: A semimodule (A) is named ($\oplus$-supplemented) if to all subsemimodule of A include supplement which is direct summand of A.

Similar to [14], we have the next explanation:

Definition 2.3: A semimodule (A) is known as ($\oplus$-cofinitely supplemented) if every cofinite subsemimodule of A has supplement which is direct summand of A.

Similar to [4], we have the next explanation:

Definition 2.4: A semimodule (A) is known as (cofinitely lifting semimodule) if, for any cofinite subsemimodule U of A, there is direct summand F of A together with $F \leq U$ and $U/F \ll A/F$, equivalently, for each cofinite subsemimodule U of A, there is several subsemimodules $(F, F'$)of (A) together with $A = F \oplus F'$, $(F \leq U)$ as well as $U \cap F' \ll F'$.

Remark 2.5: Clearly, each lifting semimodule is (cofinitely lifting). Instead, all finitely generated (cofinitely lifting semimodule) is lifting as well as all (cofinitely lifting semimodule) is $\oplus$-cofinitely supplemented. Morever, (cofinitely lifting semimodules) are not required to be lifting, as well as $\oplus$-cofinitely supplemented semimodules are not required to be (cofinitely lifting).

The next examples explain that (cofinitely lifting semimodules) lie totally between lifting semimodules as well as $\oplus$-cofinitely supplemented semimodules.

Example 2.6: Z-semimodule (Q) of rational numbers is cofinitely lifting while is no lifting.

Example 2.7: Take S be commutative semidomain as well as not semifield, as well as Q semifield of fraction of S. Let I be each no non-empty index set as well as (A) the S-semimodule $Q^{(I)}$. Then A contains no maximal subsemimodule as well as so A is only cofinite subsemimodule of A. Therefore, A is (cofinitely lifting). Consider S now is Dedekind semidomain as well as I is an infinit set. Then A no supplemented as in modules [16, Theorems 2.4 as well as 3.1].

Example 2.8: Let p be prime integer with A be Z-semimodule $(Z/pZ) \oplus (Z/p^3Z)$. Then (A) is ($\oplus$-cofinitely supplemented) while (A) not (cofinitely lifting).

Definition 2.9: [4] Let $K \leq H \leq A$. If $H/K \ll A/K$, then K is known as coessential subsemimodule of H in A.

Definition 2.10: [7] A subsemimodule H of A is named coclosed (written $H \leq_{CC} A$) if H contains no proper coessential subsemimodule. We also, say K coclosure (s-closure) of H in A if K is coessential subsemimodule of H as well as K is coclosed in A.

We will give the next lemma for later use in other proofs.

Lemma 2.11: Take A be subtractive semimodule as well as H, F any subsemimodules of A. Then the following statements are holds:

(1) If $F/H \ll A/H$ as well as $N \ll A$, then $(F+N)/(H+N) \ll A/(H+N)$. The converse is true if $N \ll A$.

(2) If $F/H \ll A/H$ as well as $L/N \ll A/N$, then $(F+L)/(H+N) \ll A/(H+N)$.

(3) For $H \leq F \leq A$, $N/F \ll A/F$ if and only if $(\frac{N}{H})/(\frac{F}{H}) \ll (\frac{A}{H})/(\frac{F}{H})$.

(4) For $H \leq F \leq A$, if $H \ll A$ as well as F is coclosed in A, then $H \ll F$.

(5) If $V \leq U$ as well as $U \leq_{cc} A$, then $U/V \leq_{cc} A/V$.

The next lemma is evident:

Lemma 2.12: Let K be cofinite subsemimodule of A. If $(H \leq A)$ as well as $(K \leq H)$, then H is cofinite subsemimodule from to A.

Proposition 2.13: Let (U) be supplement for (V) in (A). If both (U) as well as (V) are cofinite subsemimodules from to A, then A is finitely generated.

Proof: Suppose, $A = U + V$ as well as $U \cap V \ll A$, when U as well as V are cofinite subsemimodules of A. Since $\frac{A}{V} = (U+V)/V \cong U/(U \cap V)$, U finitely generated as well as therefore A is finitely generated. □

Proposition 2.14: If (A) is cofinitely lifting semimodule, then A is cofinitely supplemented as well as all cofinitely supplemented subsemimodule from A is direct summand from A.

Proof: Take U be cofinite subsemimodule from A. Then there is subsemimodules V_1, V_2 of A together with $A = V_1 \oplus V_2$, with $V_1 \leq U$ as well as $(V_2 \cap U \ll V_2.)$ Hence, $U = V_1 \oplus (U \cap V_2)$. It is evidently $A = V_2 + U$. Not that $V_2 \cap U \ll V_2$, as well as therefor V_2 is supplement to U in A. So, A is cofinitely supplemented. □

Proposition 2.15: Each cofinite direct summand from cofinitely lifting semimodule is (cofinitely lifting).

Proof: Similar to proof of [14, Proposition 2.5]. □

Definition 2.16: [3] A semimodule (A) is known as to have summand sum property (or SSP) if sum for two direct summands of A is again direct summand of (A).

Proposition 2.17: Take A be cofinitely lifting subtractive semimodule as well as U direct summand of A. If A contains SSP, then A/U is (cofinitely lifting).

Proof: Consider U/V be cofinite subsemimodule of A/V. Then U is finite subsemimodule of A. Because A is (cofinitely lifting), there is direct summand F of A together with $F \leq U$, $U/F \ll A/F$ as well as $A = F \oplus F'$, $F' \leq A$. By Lemma 2.11(1), we have $(U+V)/(F+V) \ll A/(F+V)$, i.e., $U/(F+V) \ll A/(F+V)$. By Lemma 2.11(3), $(\frac{U}{V})/(\frac{(F+V)}{V}) \ll (\frac{A}{V})/(\frac{(F+V)}{V})$. Since A contains SSP, $F+N$ is direct summand of A. Evident, $(F+V)/V$ is direct summand of A/V. Hence, A/V is (cofinitely lifting). □

Corollary 2.18: Let A is cofinitely lifting semimodule as well as SSP, then each direct summand from A is (cofinitely lifting).

Definition 2.19: ([3], [11]) A subsemimodule U is known as fully invariant if to any endomorphism f of A, $f(U) \leq U$. An S-semimodule A is known as duo semimodule provided every subsemimodule of A is fully invariant.

Theorem 2.20: If A is subtractive cofinitely lifting semimodule with U is fully invariant subsemimodule from A, then A/U is (cofinitely lifting).

Proof: Similar to proof of [14, Theorem 2.8]. □

Corollary 2.21: If A is subtractive cofinitely lifting semimodule. Then both $A/Rad(A)$ as well as $A/Soc(A)$ are (cofinitely lifting).

The next example explains that finite direct sum of ("cofinitely lifting semimodules") is not required to be (cofinitely lifting).

Example 2.22: If $A=(Z/2Z)\oplus(Z/8Z)$ be Z-semimodule. Then $Z/2Z$ with $Z/8Z$ are (cofinitely lifting). While A is not (cofinitely lifting).

Theorem 2.23: Take (A=$A_1\oplus A_2$)be direct sum from (cofinitely lifting semimodules) A_1 as well as A_2. If each cofinite subsemimodule to A is fully invariant, then A is (cofinitely lifting).

Proof: Let U be cofinite subsemimodule of A. Then U is fully invariant by assume. Therefore, $U=(U\cap A_1)\oplus(U\cap A_2)$ as well as $\frac{A}{U}=\left[\frac{(U+A_1)}{U}\right]\oplus[U+A_2)/U]$. Then $A_1/(U\cap A_1)\cong(U+A_1)/U$ as well as $A_2/(U\cap A_2)\cong(U+A_2)/U$ are finitely generated. Here $U\cap A_1$ is cofinite subsemimodule of A_1 as well as $U\cap A_2$ is cofinite subsemimodule of A_2. Since A_1 is (cofinitely lifting), there exists direct summand F_1 of A_1 such that $F_1\leq U\cap A_1$ as well as $(U\cap A_1)/F_1\ll A_1/F_1$. Since A_2 is (cofinitely lifting), there exists direct summand F_2 of A_2 such that $F_2\leq U\cap A_2$ as well as $(U\cap A_2)/F_2\ll A_2/F_2$. By Lemma 2.11(2), $[(U\cap A_1)\oplus(U\cap A_2)]/(F_1\oplus F_2)\ll A/(F_1\oplus F_2)$, i.e., $U/(F_1\oplus F_2)\ll A/(F_1\oplus F_2)$. Clearly, $F_1\oplus F_2$ is direct summand to A. Hence, A is (cofinitely lifting).

The following corollaries are direct consequence of Theorem 2.23.

Corollary 2.24: Let $(A=A_1\oplus A_2)$ be direct sum of ("cofinitely lifting semimodules) A_1as well as A_2. If A is duo semimodule, then A is (cofinitely lifting).

Corollary 2.25: If $A=(A_1\oplus A_2\oplus\cdots\oplus A_n)$ be direct sum of (cofinitely lifting semimodules) A_i $(i=1,2,\ldots..,n)$. When A is duo semimodule, then A is (cofinitely lifting).

Corollary 2.26: If $A=(A_1\oplus A_2\oplus\cdots\oplus A_n)$ be direct sum of lifting semimodules A_i $(i=1,2,\ldots..,n)$. When A is duo semimodule, then A is (cofinitely lifting).

Remark 2.27: (A) is amply supplemented as well as all coclosed subsemimodule from A is direct summand if and only if A is lifting.

Proposition 2.28: Take (A) be subtractive semimodule together with all cofinite subsemimodule from A contains coclosure in A. Then A is (cofinitely lifting) if and only if all cofinitely coclosed subsemimodule from A is direct summand to (A).

Proof: Similar to proof of [14, Theorem 3.6]. ▯

Remark 2.29: Every subsemimodule from an amply supplemented semimodule A contains a coclosure in A.

Corollary 2.30: If (A) be an amply supplemented semimodule. Then all cofinitely coclosed subsemimodule from A is direct summand to A if and only if A is (cofinitely lifting).

The next lemma can be thought of as generalization of [9, Lemma 1.4].

Lemma 2.31: Take A be weakly supplemented semimodule with $V\leq U$ be subsemimodules from A together with U/V is coclosed in A/V as well as V is coclosed in A. Then U is coclosed in A.

Let $A=\bigoplus_{i\in I}A_i$ be direct sum to semimodules A_i to several index set I. to all $i\in I$, A_{-i} will denote $\bigoplus_{j\in I\setminus\{i\}}A_j$.

Theorem 2.32: Take $A=\bigoplus_{i\in I}A_i$ be direct sum to semimodules A_i to several index set I as well as $|I|\geq 2$. If A is an amply supplemented semimodule, then the next stetements are equivalent:

(1) A (cofinitely lifting).

(2) There is $i\in I$ together with all cofinitely coclosed subsemimodule F of A with $A=F+A_i$ or $A=F+A_{-i}$ is direct summand to A.

(3) There is $i\in I$ together with all cofinitely coclosed subsemimodule F of A as well as $(F+A_i)/F\ll A/F$ or $(F+A_{-i})/F\ll A/F$ or $A=F+A_i=F+A_{-i}$ is direct summand to A.

Proof: Similar to proof of [14, Theorem 3.9]. ▯

Corollary 2.33: The next are equivalent for $A=A_1\oplus A_2$ be an amply supplemented semimodule.

(1) A is (cofinitely lifting).

(2) All cofinitely closed subsemimodule F of A as well as $(A=F+A_1)$ or($A=F+A_2$) is direct summand to A.

(3) All cofinitely closed subsemimodule F of A as well as $(F+A_1)/F \ll A/F$ or $(F+A_2)/F \ll A/F$ or $A=F+A_1=F+A_2$ is direct summand to A.

3. Cofinitely Projectivity as well as Cofinitely Lifting Property

This part explores the idea of ("cofinitely lifting semimodules") when S-semimodules are direct sum of S-semimodules. We begin by the following definitions.

Similar to [14] we give the following three definitions.

Definition 3.1: Let A_1 as well as A_2 be semimodules. Semimodule A_1 is cofinitely A_2-projective if every homomorphism $f: A_1 \longrightarrow A_2/K$, where K is cofinite subsemimodule from A_2, can be lifted to homomorphism $\psi: A_1 \longrightarrow A_2$.

Definition 3.2: Let A_1 as well as A_2 be semimodules. Semimodule A_1 is cofinitely small A_2-projective if each homomorphism $f: A_1 \longrightarrow A_2/K$, where K is cofinite subsemimodule of A_2 as well as $\mathrm{Im}f \ll A_2/K$, can be lifted to homomorphism $\psi: A_1 \longrightarrow A_2$.

Definition 3.3: A_1 is cofinitely pseudo-A_2-projective if each epimorphism $f: A_1 \longrightarrow A_2/K$, where K is cofinite subsemimodule from A_2, can be lifted homomorphism $\psi: A_1 \longrightarrow A_2$.

Remark 3.4: you can learn about Z-semimodule (Q) of rational numbers is cofinitely Q_Z-projective although it is not quasi-projective.

Lemma 3.5: The next are equivalent for S be semiring as well as let $A = A_1 \oplus A_2$, where A_1 is S-semimodule as well as A_2 is S-module.

(1) A_1 is cofinite pseudo-A_2-projective.

(2) To all cofinite subsemimodule U of A together with $A = U + A_1 = U + A_2$, there is subsemimodule U' of U together $A = U' \oplus A_2$.

Proof: (1) $\Longrightarrow$ (2) Assume the epimorphisms $\psi: A_1 \longrightarrow A/U, a_1 \longmapsto a_1 + U$ as well as $\xi: A_2 \longrightarrow A/U, a_2 \longmapsto a_2 + U$. There exists homomorphism $\gamma: A_1 \longrightarrow A_2$ in such a way $\xi\gamma = \psi$. That much is clear to see $U' = \{x - \gamma(x) \mid x \in A_1\} \leq U$ as well as $A = U' \oplus A_2$.

Let K be cofinite subsemimodule of A_2, $\beta: A_1 \longrightarrow \frac{A_2}{K}$ an epimorphism as well as $\phi: A_2 \longrightarrow \frac{A_2}{K}$ (2) $\Longrightarrow$ (1) natural epimorphism. Define $U = \{x + y \in A_1 \oplus A_2 | x \in A_1, y \in A_2, \beta(x) = -\phi(y)\}$. Clearly, $K \leq U$ as well as $A = U + A_1 = U + A_2$. Since $\frac{A}{U} = (U + A_2)\ /\ U \cong A_2/(U \cap A_2)$, U is cofinite subsemimodule of A through Lemma 2.12. Therefore, there is subsemimodule U' of U together with $A = U' \oplus A_2$. Consider canonical projection $\varphi: U' \oplus A_2 \longrightarrow A_2$. So, β can be lifted to $\varphi|A_1: A_1 \longrightarrow A_2$. This indicates that A_1 is cofinitely pseudo-A_2-projecative. □

Lemma 3.6: Take A_1 be a semimodule, A_2 cofinitely lifting semimodule as well as $A = A_1 \oplus A_2$. If A_1 is cofinitely pseudo-A_2-projecateve, then every cofinitely coclosed subsemimodule U of A such that $A = U + A_1 = U + A_2$ is direct summand to A.

Proof: Through Lemma 3.5, there is subsemimodule U' of U together with $(A = U' \oplus A_2)$. Clearly, $(\frac{A}{U'})$ is (cofinitely lifting). because U is coclosed in A, $(\frac{U}{U'})$ is coclosed in $(\frac{A}{U'})$ through Lemma 2.11(5). As U is cofinite subsemimodule of A, $(\frac{U}{U'})$ is cofinite subsemimodule of A/U'. Using proposition 2.28,(U/U') is direct summand to $(\frac{A}{U'})$. Therefore, U is direct summand to A. □

Similarly, we have:

Lemma 3.7: Let $(A = A_1 \oplus A_2)$. Then the next are equivalent:

(1) A_1 is cofinitely A_2-projecative.

(2) For every cofinite subsemimodule U of A such that $A = U + A_2$, there exists subsemimodule U' of U such that $A = U' \oplus A_2$.

Lemma 3.8: Let A_1 be a semimodule, A_2 cofinitely lifting semimodule as well as $A = A_1 \oplus A_2$. If A_2 is cofinitely A_1-projecative, then every cofinitely coclosed subsemimodule U of A such that $A = U + A_1$ is direct summand of A.

Lemma 3.9: The next are equivalent for $A = A_1 \oplus A_2$

(1) A_1 is cofinitely small A_2-projecative.
(2) To all cofinite subsemimodule U of A such that $(U + A_1)/U \ll A/U$, there is subsemimodule U' of U together with $A = U' \oplus A_2$.
Lemma 3.10: Let A_1 be a semimodule, A_2 cofinitely lifting semimodule as well as $A = A_1 \oplus A_2$. If A_1 is cofinitely small A_2-projective, then each cofinitely coclosed subsemimodule U from A together with $(U + A_1)/U \ll A/U$ is direct summand to A.
Theorem 3.11: Take A_1 as well as A_2 be ("cofinitely lifting semimodules") as well as $A = A_1 \oplus A_2$ an amply supplemented semimodule. Then A is (cofinitely lifting) when one of the next conditions holds:
(1) A_1 is cofinitely small A_2-prpjecteve as well as each cofinitely coclosed subsemimodule U of A together with $A = U + A_1$ is direct summand.
(2) A_1 as well as A_2 are relatively cofinitely small projective as well as each cofinite coclosed subsemimodule U of A together with $A = U + A_1$ or $A = U + A_2$ is direct summand to(A).
(3) A_2 is cofinitely A_1-projective as well as A_1 is cofinitely small A_2-projective.
(4) A_1 as well as A_2 are relatively cofinitely small projective with A_1 is cofinitely pseudo-A_2-projective.
(5) A_1 as well as A_2 are relatively cofinitely small projective with A_2 is cofinitely pseudo-A_1-projective.
Proof: Using Corollary 2.33, Lemmas 3.6, 3.8 as well as 3.10. □
Theorem 4.3.12: The next are equivalent for semiring S.
(1) (S) is semiperfect.
(2) Each finitely generated quasi-projective semimodule is (cofinitely lifting).
(3) Each cyclic free semimodule is (cofinitely lifting).
(4)($_SS$) is (cofinitely lifting).

Proof: Similar to proof of [10, Theorem 4.41 as well as Corollary 4.42]. □

Acknowledgment: I would like to appreciate the referee for their insightful comments that enhanced the paper's presentation.

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