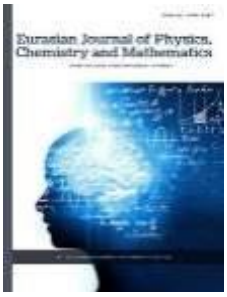

Full- Discrete Petrov Weak Galerkin Finite Element Method for Solving Coupled Burgers' Problem

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ABSTRACT

In this paper, we introduce full- discrete Petrov weak Galerkin finite element method (PWG-FEM) for solving coupled Burgers' equations in two dimensions. The slicing in the full-discrete Petrov weak Galerkin finite element method (FDPWG-FEM) is done for both space and time. The backward Euler method is used to approximate the time derivative method with (PWG-FEM). We proved the optimal order error in L^2 −norm for FDPWG-FEM. We obtained the numerical experiment for confirm the theoretical results obtained.

Keywords: Petrov weak Galerkin finite element, Full-discrete, Coupled Burgers′ equations, Optimal order error

1: Introduction

In this study, we consider the nonlinear time-dependent coupled Burgers׳ problem in two dimensions [1].

$$\frac{\partial u}{\partial t} - \varepsilon\Delta u + uu_x + vu_y = f(x,y,t), \quad (x,y,t) \in \Omega \times (0,T], \qquad (1.1)$$

$$\frac{\partial v}{\partial t} - \varepsilon\Delta v + uv_x + vv_y = g(x,y,t), \quad (x,y,t) \in \Omega \times (0,T], \qquad (1.2)$$

with Dirichlet boundary conditions

$$u(x,y,t) = \zeta(x,y,t), \quad (x,y,t) \in \partial\Omega \; x \; (0,T], \qquad (1.3)$$

$$v(x,y,t) = \eta(x,y,t), \quad (x,y,t) \in \partial\Omega \; x \; (0,T], \qquad (1.4)$$

and initial conditions

$$u(x,y,0) = u^0(x,y), \quad (x,y) \in \Omega, \qquad (1.5)$$

$$v(x,y,0) = v^0(x,y), \quad (x,y) \in \Omega \qquad (1.6)$$

Where $\Omega = \{(x,y), a \le x \le b, c \le y \le d\}$ is the computational domain and $\partial\Omega$ its boundary, $u(x,y,t)$ and $v(x,y,t)$ are the velocity components to be determined, u^0, v^0, ζ and η are known functions, $\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}$ are unsteady terms, uu_x, vv_y are the nonlinear convection terms, $\varepsilon\Delta u$, $\varepsilon\Delta v$ are the diffusion terms, $f, g \in L^2(\Omega, t)$ are source terms (often equal to zero). For the numerical solution of Burgers' equations, several approaches have been developed. These methods mainly include finite difference, finite volume, finite element method, homotopy method, decompstion method, differential transformation method, and boundary element etc., see [2,3,4,5,6,7,8]. It is common knowledge that the direct application of the Galerkin finite element approach to singularly perturbed Burgers' equations may produce spurious oscillation in the approximate solution. Several approaches have been used by researchers to address this oscillation including. Petrov-Galerkin approach [9,10,11] Petrov -discontinuous Galerkin method [12,13]. Weak Galerkin is a finite element method for PDEs where the differential operators (gradient, divergence, curl, Laplacian etc.) in the weak forms are approximated by discrete generalized distributions . These weak differential operators shall serve as building blocks for WG-FEM to partial differential equations. The fundamental distinction between WG-FEM and other techniques is the use of weak functions and weak derivatives (i.e., locally reconstructed differential operators) in the creation of numerical schemes based on known weak forms for the underlying PDEs, see [14,15]. In this paper, we show The FDPWG-FEM for solving two-dimensional coupled Burgers' problem is intended to eliminate the inaccuracies and oscillations obtained using WG-FEM when $h > \varepsilon$ (where ε is the diffusion coefficient and h *is mesh size*).

The rest of the paper is organized as follows. In section 2 we introduce the definition of PWG-FE space. In section 3, we define Petrov weak variational form. In section 4 we introduce the definition of the full-discrete of PWG-FEM and some lemmas which are necessary for error estimate. In section 5 we prove the error analysis of full-discrete PWG-FEM. In section 6, a numerical experiment is given. Finally, in section 7, Discussion and Conclusion.

2: A Petrov Weak Galerkin Spaces

Let U, V be two trial spaces and $\varphi, \emptyset$ be test spaces defined as follows:

Let U, V be two trial spaces and $\varphi, \emptyset$ be test spaces defined as follows:

$$U = \{u = \{u_0, u_b\}: \{u_0, u_b\} \in L^2(\Omega) \times L^2(\partial\Omega),\ \forall K \in T_h\}, \quad (2.1)$$

$$V = \{v = \{v_0,\ v_b\}: \{v_0, v_b\} \in L^2(\Omega) \times L^2(\partial\Omega),\ \forall K \in T_h\}, \quad (2.2)$$

$$\varphi = \{\ \mathcal{M}:\ \mathcal{M} = w_0 + \delta\beta.\nabla_d\, w :\ w \in U\}, \quad (2.3)$$

$$\emptyset = \{\gamma: \gamma = p_0 + \delta\beta.\nabla_d\, p :\ p \in V\}. \quad (2.4)$$

We define PWG – FE spaces,

There are two trial finite element spaces defined as follows:

$$\mathrm{U_h} = \{u = \{u_0, u_b\}: \{u_0, u_b\}\,|_K \in p_l(K) \times p_j(\partial K),\ \forall K \in T_h\}, \quad (2.5)$$

$$\mathrm{V_h} = \{\mathrm{v} = \{v_0, v_b\}: \{v_0,\ v_b\}\,|_K \in p_l(K) \times p_j(\partial K),\ \forall K \in T_h\}. \quad (2.6)$$

Define two test spaces by,

$$\varphi_h = \{\mathcal{M}:\ \mathcal{M} = w_0 + \delta\beta.\nabla_{d,r}\, w : w \in U_h\}, \quad (2.7)$$

$$\emptyset_h = \{\gamma: \gamma = p_0 + \delta\beta.\nabla_{d,r}\, p :\ p \in V_h\}, \quad (2.8)$$

and

$$\mathrm{U_h^0} = \{u = \{u_0, u_b\} \in U_h:\ u_b\,|_{\partial K \cap \partial\Omega} = 0\}, \quad (2.9)$$

$$\varphi_h^0 = \{\mathcal{M} = w_0 + \delta\beta.\nabla_{d,r} w:\ w \in U_h^0\}, \quad (2.10)$$

and

$$V_h^0 = \{v = \{v_0, v_b\} \in V_h:\ v_b\,|_{\partial K \cap \partial\Omega} = 0\}, \quad (2.11)$$

$$\emptyset_h^0 = \{\gamma = p_0 + \delta\beta.\nabla_{d,r} p : p \in V_h^0\}, \quad (2.12)$$

a constant stability parameter is shown here by the symbol δ. The selection will be [16]:

$$\delta = \begin{cases} \eta h & if\ \varepsilon < h \\ 0 & if\ \varepsilon \geq h \end{cases} \quad ;\ 0 < \eta < \frac{1}{4} \quad \text{(small constant),}$$

and dim U, V = dim $\varphi, \emptyset$, respectively.

Here β indicate the convection coefficient and ε represent diffusion coefficient

T_h represent a collection of all triangulation on Ω

$L^2(\Omega)$ indicates space of square-integrable functions

p_l(K) indicates the set of polynomials on K with a degree no more than l

$p_j(\partial K)$ represent the set of polynomials on ∂K with a degree no more than j

∇ represent gradient operator

K indicates a triangle element

∂K indicates the boundary for the polygonal domain

3. Petrov Weak Variational Form

Multiply equations (1.1) and (1.2) by the test functions $(w_0 + \delta\beta.\nabla_d w)$ and $(p_0 + \delta\beta.\nabla_d\, p)$ respectively and integrating by part, we get

$$\left(u_t,\ w_0 + \delta\beta.\nabla_d w\right) + \varepsilon\,(\nabla u, \nabla w) + \left(uu_x,\ w_0 + \delta\beta.\nabla_d w\right) + (vu_y,\ w_0 + \delta\beta.\nabla_d w) = (f, w_0 + \delta\beta.\nabla_d w),\quad \forall w \in U \quad (3.1)$$

$$(v_t,\ p_0 + \delta\beta.\nabla_d\, p) + \varepsilon\,(\nabla v, \nabla p) + (uv_x,\ p_0 + \delta\beta.\nabla_d\, p) + (vv_y,\ p_0 + \delta\beta.\nabla_d\, p) = (g,\ p_0 + \delta\beta.\nabla_d\, p),\quad \forall p \in V \quad (3.2)$$

and
$(u(x,y,0), w_0+\delta\beta.\nabla_d w)=(u^0, w_0+\delta\beta.\nabla_d w)$,
$(v(x,y,0), p_0+\delta\beta.\nabla_d p)=(v^0, p_0+\delta\beta.\nabla_d p)$.
We can write the nonlinear terms uu_x and vv_y in conservation form and integrating by part, we get

$$(uu_x, w_0+\delta\beta.\nabla_d w) = \frac{1}{2}((u^2)_x, w_0+\delta\beta.\nabla_d w) = -\frac{1}{2}(u^2, w_x),$$

$$(vv_y, p_0+\delta\beta.\nabla_d p) = \frac{1}{2}((v^2)_y, p_0+\delta\beta.\nabla_d p) = -\frac{1}{2}(v^2, p_y).$$

Substituting in the equation (3.1) and (3.2) the Petrov weak variational form is find $u \in U$ and $v \in V$, such that

$$\begin{cases} (u_t, w_0+\delta\beta.\nabla_d w) + (\varepsilon\nabla u, \nabla w) - \frac{1}{2}(u^2, w_x) \\ +(vu_y, w_0+\beta.\nabla_d w) = (f, w_0+\delta\beta.\nabla_d w), \quad u(x,y,0) \\ = u^0(x,y) \qquad \forall (x,y) \in \Omega \qquad \forall w \in U, \end{cases} \tag{3.3}$$

$$\begin{cases} (v_t, p_0+\delta\beta.\nabla_d p) + (\varepsilon\nabla v, \nabla p) + (uv_x, p_0+\delta\beta.\nabla_d p) \\ -\frac{1}{2}(v^2, p_y) = (g, p_0+\delta\beta.\nabla_d p), \\ v(x,y,0) = v^0(x,y) \qquad \forall (x,y)\in\Omega \qquad \forall p \in V, \end{cases} \tag{3.4}$$

where $a(u,w) = (\varepsilon\nabla u, \nabla w) - \frac{1}{2}(u^2, w_x) + (vu_y, w_0+\delta\beta.\nabla_d w)$,

$a(v,p) = (\varepsilon\nabla v, \nabla p) + (uv_x, p_0+\delta\beta.\nabla_d p) - \frac{1}{2}(v^2, p_y)$.

And for $\beta_1, \beta_2 > 0$ the property (coercive) hold. i.e, [17]

$$a(u,u) \geq \beta_1 \|\nabla_d u\|^2 \quad \forall u \in U, \tag{3.5}$$
$$a(v,v) \geq \beta_2 \|\nabla_d v\|^2 \quad \forall v \in V.$$

4. The Fill-discrete PWG – FEM

we shall establish the Fill-discrete PWG-FEM for Burgers' equations and derive the error estimation in L^2– norm. Let $0 = t^0 < t^1 < .... < t^n = T$ be a partition for time interval $[0,T]$ and the time level $t = t^n = n\tau$ where n is non-negative integer. The backward Euler method is used to approximate the time derivative method with (PWG-FEM). $\tilde{\partial}_t u_h^n = (u_h^n - u_h^{n-1})/\tau$ and $\tilde{\partial}_t v_h^n = (v_h^n - v_h^{n-1})/\tau$.

The Fill-discrete PWG – FEM is find $u_h^n \in U_h$ and $v_h^n \in V_h$ such that

$$(\tilde{\partial}_t u_h^n, w_0) + (\tilde{\partial}_t u_h^n, \delta\beta.\nabla_{d,r}w) + \varepsilon(\nabla_{d,r}u_h^n, \nabla_{d,r}w) - \frac{1}{2}(u_h^n u_h^n, \frac{\partial_{d,r}w}{\partial x})$$
$$+ (v_h^n \left(\frac{\partial_{d,r}u_h^n}{\partial y}\right), w_0) + (v_h^n \left(\frac{\partial_{d,r}u_h^n}{\partial y}\right), \delta\beta.\nabla_{d,r}w)$$
$$= (f, w_0) + (f, \delta\beta.\nabla_{d,r}w), \qquad \forall w \in U_h^0, \tag{4.1}$$

$$(\tilde{\partial}_t v_h^n, p_0) + (\tilde{\partial}_t v_h^n, \delta\beta.\nabla_{d,r}p) + \varepsilon(\nabla_{d,r}v_h^n, \nabla_{d,r}p) - \frac{1}{2}\left(v_h^n v_h^n, \frac{\partial_{d,r}p}{\partial y}\right)$$
$$+ (u_h^n \left(\frac{\partial_{d,r}v_h^n}{\partial x}\right), p_0) + (u_h^n \left(\frac{\partial_{d,r}v_h^n}{\partial x}\right), \delta\beta.\nabla_{d,r}p)$$
$$= (g, p_0) + (g, \delta\beta.\nabla_{d,r}p), \qquad \forall p \in V_h^0, \tag{4.2}$$

or

$$(\tilde{\partial}_t u_h^n, w_0) + (\tilde{\partial}_t u_h^n, \delta\beta.\nabla_{d,r}w) + a_{PW}(u_h^n, w)$$
$$= (f, w_0) + (f, \delta\beta.\nabla_{d,r}w), \qquad \forall w \in U_h^0, \tag{4.3}$$

$$(\tilde{\partial}_t v_h^n, p_0) + (\tilde{\partial}_t v_h^n, \delta\beta.\nabla_{d,r}p) + a_{PW}(v_h^n, p)$$
$$= (g, p_0) + (g, \delta\beta.\nabla_{d,r}p), \qquad \forall p \in V_h^0, \tag{4.4}$$

where,

$$a_{PW}(u_h^n, w) = \varepsilon(\nabla_{d,r}u_h^n, \nabla_{d,r}w) - \frac{1}{2}(u_h^n u_h^n, \frac{\partial_{d,r}w}{\partial x}) + (v_h^n \left(\frac{\partial_{d,r}u_h^n}{\partial y}\right), w_0)$$
$$+ (v_h^n \left(\frac{\partial_{d,r}u_h^n}{\partial y}\right), \delta\beta.\nabla_{d,r}w),$$

$$a_{PW}(v_h^n, p) = \varepsilon(\nabla_{d,r}v_h^n, \nabla_{d,r}p) - \frac{1}{2}\left(v_h^n v_h^n, \frac{\partial_{d,r}p}{\partial y}\right) + (u_h^n \left(\frac{\partial_{d,r}v_h^n}{\partial x}\right), p_0)$$

$+ \ (u^n{}_h(\frac{\partial_{d,r}\ v_h{}^n}{\partial x}), \delta\beta ? \nabla_{d,r}\text{p}).$

Lemma 4.1. [18] If $u \in H_0^1(\Omega) \cap H^{z+1}(\Omega)$, $Q_h u \in U_h^O$ or $Q_h v \in V_h^O$. Then

$\|Q_h u - u\| \leq \text{C}\, h^z \|u\|_z \qquad 0 \leq z \leq k+1,$ (4.5)

$\|\nabla_d \, \text{Q}_\text{h} u - \nabla u\| \leq \text{C}\, h^z \, \|u\|_{1+z} \qquad 0 \leq z \leq k+1.$ (4.6)

$Q_h u$ *indicate the* L^2*projection of* $H^1(K)$ *on to* $P_l(K) \times P_j(\partial K)$

Lemma 4.2. [19] for $u \in H^{1+z}$ with $z > 0$, we have

$\|u - \prod_h u\| \leq \text{C}\, h^z \|u\|_{1+z},$ (4.7)

$\|\nabla u - \nabla \prod_h u\| \leq \text{C}\, h^z \|u\|_{1+z.}$ (4.8)

H^{1+z} *indicate Hilbert space of order* $z+1$,

Π_h*indicate the projection of* $H(div, \Omega)$

5. The Error Analysis of PWG – FEM

The goal of this section is to e prove the error estimates for full-discrete PWG – FEM, in the L^2 –norm.

Lemma 5.1. (L^2– error in FDPWG) Let u^n, v^n and u_h^n, v_h^n be the solutions of (3.3), (3.4.) and (4.1), (4.2) respectively, then exists a constant C that is independent of h, such that;

$$\|e^n\|^2 \leq \|e^0\|^2 + \ C h^{2k} \sum_{j=1}^{n} \|u^j\|_{1+k}^2 + \tau^2 \int_{t_{j-1}}^{t_j} \|u_{tt}\|^2 \, dt, \tag{5.1}$$

$$\|e^n\|^2 \leq \|e^0\|^2 + \ C h^{2k} \sum_{j=1}^{n} \|v^j\|_{1+k}^2 + \tau^2 \int_{t_{j-1}}^{t_j} \|v_{tt}\|^2 \, dt. \tag{5.2}$$

Proof. Let $t = t^n$ *in* equation (3.3), and applying the fact

$(\Pi_h u^n, w_0) = (u^n, w_0)$, we get

$$(u_t\,(t^n), w_0) + \left(u_t\,(t^n), \delta\beta.\nabla_{d,r} w\right) + \varepsilon\left(\prod\nolimits_\text{h} \nabla u\,(t^n), \nabla_{d,r} w\right) - \frac{1}{2}\left(\prod\nolimits_\text{h} u^2(t^n), \frac{\partial_{d,r} w}{\partial x}\right) + \left(v\,(t^n)\, u_y\,(t^n), w_0\right) + \left(v\,(t^n) u_y\,(t^n), \delta\beta.\nabla_{d,r} w\right) = (f, w_0) + (f, \delta\beta.\nabla_{d,r} w). \tag{5.3}$$

By adding and subtracting

$(\partial_t u^n, w_0) + (\partial_t u^n, \delta\beta.\nabla_{d,r} w) + a_{PW}(u^n, w)$

to equation (5.3), and using the fact $(Q_h\, u^n) = (u^n)$, *we obtain*

$$(u_t\,(t^n), w_0) + (u_t\,(t^n), \delta\beta.\nabla_{d,r} w) + \varepsilon\,(\Pi_h \nabla u\,(t^n), \nabla_{d,r} w) - \frac{1}{2}\,\left(\Pi_h u^2(t^n), \frac{\partial_{d,r} W}{\partial x}\right) + (v\,(t^n)\, u_y\,(t^n), w_0) + (v(t^n) u_y(t^n), \delta\beta.\nabla_{d,r} w) + (\partial_t(Q_h u^n), w_0) + (\partial_t(Q_h u^n), \delta\beta.\nabla_{d,r} w) - (\partial_t(Q_h u^n), w_0) - (\partial_t(Q_h u^n), \delta\beta.\nabla_{d,r} w) + a_{PW}(Q_h u^n, w) - a_{PW}(Q_h u^n, w) = (f, w_0) + (f, \delta\beta.\nabla_{d,r} w). \tag{5.4}$$

Where,

$$a_{PW}(Q_h u^n, w) = \varepsilon(\nabla_{d,r}\, Q_h u^n, \nabla_{d,r} w) - \frac{1}{2}\left((Q_h u^n)^2, \frac{\partial_{d,r}\, w}{\partial x}\right) + \left(v^n \left(\frac{\partial_{d,r}(Q_h u^n)}{\partial \text{y}}\right), w_0\right) + \left(v^n \left(\frac{\partial_{d,r}(Q_h u^n)}{\partial \text{y}}\right), \delta\beta.\nabla_{d,r} w\right).$$

Subtract equation (5.4) from (4.3), we get

$$\left(\tilde{\partial}_t\,(Q_h u^n - u_h^n), w_0\right) + \left(\tilde{\partial}_t\,(Q_h u^n - u_h^n), \delta\beta.\nabla_{d,r} w\right) + a_{PW}(Q_h u^n - u_h^n, w) = \left(\tilde{\partial}_t u^n - u_t^n, w_0\right) + \left(\tilde{\partial}_t u^n - u_t^n, \delta\beta.\nabla_{d,r} w\right) + \varepsilon\left(\nabla_{d,r}\, u^n, \nabla_{d,r}\, \text{w}\right) - \varepsilon\,(\prod\nolimits_h \nabla u^n, \nabla_{d,r} \text{w}) + \frac{1}{2}\left(\prod\nolimits_\text{h}(u^n)^2, \frac{\partial_{d,r} w}{\partial X}\right) - \frac{1}{2}\left((u^n)^2, \frac{\partial_{d,r} w}{\partial x}\right) + \left(v^n\left(\frac{\partial_{d,r}\, u^n}{\partial \text{y}}\right) - v^n\, u_y^n, w_0\right) + \left(v^n\left(\frac{\partial_{d,r}\, u^n}{\partial \text{y}}\right) - v^n\, u_y^n, \delta\beta.\nabla_{d,r} w\right), + \left(v^n\left(\frac{\partial_{d,r}\, u^n}{\partial \text{y}}\right) - v^n\, u_y^n, w_0\right) + \left(v^n\left(\frac{\partial_{d,r}\, u^n}{\partial \text{y}}\right) - v^n\, u_y^n, \delta\beta.\nabla_{d,r} w\right), \tag{5.5}$$

where,

$$a_{PW}(Q_h u^n - u_h^n, w) = \left(\varepsilon\, \nabla_{d,r}\,(Q_h u^n - u_h^n), \nabla_{d,r} w\right) - \frac{1}{2}\left(\left((Q_h u^n)^2 - u_h^n\, u_h^n\right), \frac{\partial_{d,r} w}{\partial x}\right) + \left(v^n\left(\frac{\partial_{d,r}(Q_h u^n)}{\partial \text{y}}\right) - v_h^n\left(\frac{\partial_{d,r}\, u_h^n}{\partial \text{y}}\right), w_0\right)$$

$+\left(v^n\left(\frac{\partial_{d,r}(Q_h u^n)}{\partial y}\right) - v_h^n\left(\frac{\partial_{d,r} u_h^n}{\partial y}\right), \delta\beta.\nabla_{d,r} w\right).$ (5.6)

Using $e^n = Q_h u^n - u_h^n$ and $w = e^n$ in equation (5.5), we get

$$(\tilde{\partial}_t e^n, e^n) + (\tilde{\partial}_t e^n, \delta\beta.\nabla_{d,r} e^n) + a_{PW}(e^n, e^n) = (\tilde{\partial}_t u^n - u_t^n, e^n)$$
$$+ (\tilde{\partial}_t u^n - u_t^n, \delta\beta.\nabla_{d,r} e^n) + \varepsilon(\nabla_{d,r} u^n, \nabla_{d,r} e^n) - \varepsilon(\prod_h \nabla u^n, \nabla_{d,r} e^n)$$
$$+ \frac{1}{2}(\prod_h (u^n)^2, \frac{\partial_{d,r} e^n}{\partial x}) - \frac{1}{2}((u^n)^2, \frac{\partial_{d,r} e^n}{\partial x}) + \left(v^n\left(\frac{\partial_{d,r} u^n}{\partial y}\right) - v^n u_y^n, e^n\right)$$
$$+\left(v^n\left(\frac{\partial_{d,r} u^n}{\partial y}\right) - v^n u_y^n, \delta\beta.\nabla_{d,r} e^n\right). \quad (5.7)$$

Hence

$$\left(\frac{e^n - e^{n-1}}{\tau}, e^n\right) + \left(\frac{e^n - e^{n-1}}{\tau}, \delta\beta.\nabla_{d,r} e^n\right) + a_{PW}(e^n, e^n) = (\tilde{\partial}_t u^n - u_t^n, e^n) +$$
$$(\tilde{\partial}_t u^n - u_t^n, \delta\beta.\nabla_{d,r} e^n) + \varepsilon(\nabla_{d,r} u^n, \nabla_{d,r} e^n) - \varepsilon(\prod_h \nabla u^n, \nabla_{d,r} e^n)$$
$$+ \frac{1}{2}(\prod_h (u^n)^2, \frac{\partial_{d,r} e^n}{\partial x}) - \frac{1}{2}\left((u^n)^2, \frac{\partial_{d,r} e^n}{\partial x}\right) + \left(v^n\left(\frac{\partial_{d,r} u^n}{\partial y}\right) - v^n u_y^n, e^n\right)$$
$$+\left(v^n\left(\frac{\partial_{d,r} u^n}{\partial y}\right) - v^n u_y^n, \delta\beta.\nabla_{d,r} e^n\right). \quad (5.8)$$

By using Property (3.5) and the Cauchy-Schwartz inequality, we get

$$\|e^n\|^2 - (e^{n-1}, e^n) + (e^n, \delta\beta.\nabla_{d,r} e^n) - (e^{n-1}, \delta\beta.\nabla_{d,r} e^n) + \tau\|e^n\|_{PW}^2$$
$$= \tau(\tilde{\partial}_t u^n - u_t^n, e^n) + \tau(\tilde{\partial}_t u^n - u_t^n, \delta\beta.\nabla_{d,r} e^n) + \tau\varepsilon(\nabla_{d,r} u^n, \nabla_{d,r} e^n)$$
$$- \tau\varepsilon(\prod_h \nabla u^n, \nabla_{d,r} e^n) + \frac{\tau}{2}(\prod_h (u^n)^2, \frac{\partial_{d,r} e^n}{\partial x}) - \frac{\tau}{2}((u^n)^2, \frac{\partial_{d,r} e^n}{\partial x})$$
$$+ \tau\left(v^n\left(\frac{\partial_{d,r} u^n}{\partial y}\right) - v^n u_y^n, e^n\right) + \tau\left(v^n\left(\frac{\partial_{d,r} u^n}{\partial y}\right) - v^n u_y^n, \delta\beta.\nabla_{d,r} e^n\right).$$

Using Cauchy-Schwartz inequality and Young's- inequality, we get

$$\|e^n\|^2 + \tau\|e^n\|_{PW}^2 - \|e^{n-1}\|^2 = \tau\sum_{i=1}^{6} A_i^n, \quad (5.9)$$

where,

$A_1^n = (\tilde{\partial}_t u^n - u_t^n, e^n),$

$A_2^n = (\tilde{\partial}_t u^n - u_t^n, \delta\beta.\nabla_{d,r} e^n),$

$A_3^n = \varepsilon(\nabla_{d,r} u^n, \nabla_{d,r} e^n) - \varepsilon(\prod_h \nabla u^n, \nabla_{d,r} e^n),$

$A_4^n = \frac{1}{2}(\prod_h (u^n)^2, \frac{\partial_{d,r} e^n}{\partial x}) - \frac{1}{2}\left((u^n)^2, \frac{\partial_{d,r} e^n}{\partial x}\right),$

$A_5^n = \left(v^n\left(\frac{\partial_{d,r} u^n}{\partial y}\right) - v^n u_y^n, e^n\right),$

$A_6^n = \left(v^n\left(\frac{\partial_{d,r} u^n}{\partial y}\right) - v^n u_y^n, \delta\beta.\nabla_{d,r} e^n\right).$

To estimate A_1^n of equation (5.9)

$A_1^n = (\tilde{\partial}_t u^n - u_t^n, e^n),$

Cauchy-Schwartz inequality, Young's -inequality and Poincare inequality [20] provide the following results

$$|A_1^n| \le \frac{1}{2}\left\|(\tilde{\partial}_t u^n - u_t^n)\right\|^2 + \frac{C}{2}\left\|\nabla_{d,r} e^n\right\|^2. \quad (5.10)$$

To estimate A_2^n of equation (5.9)

$A_2^n = (\tilde{\partial}_t u^n - u_t^n, \delta\beta.\nabla_{d,r} e^n),$

by Cauchy – Schwartz inequality and Young's-inequality, we obtain

$$|A_2^n| \le \frac{1}{2}\left\|(\tilde{\partial}_t u^n - u_t^n)\right\|^2 + \frac{1}{2}\left\|(\delta\beta.\nabla_{d,r} e^n)\right\|^2. \quad (5.11)$$

To estimate A_3^n of equation (5.9), we add and subtract $\varepsilon(\nabla u^n, \nabla_{d,r} e^n)$, we get

$A_3^n = \varepsilon(\nabla_{d,r} u^n - \nabla u^n, \nabla_{d,r} e^n) + \varepsilon(\nabla u^n - \prod_h \nabla u^n, \nabla_{d,r} e^n),$

again by Cauchy – Schwartz inequality and Young's-inequality, we get

$$|A_3^n| \le \frac{\varepsilon}{2}\left\|(\nabla_{d,r} u^n - \nabla u^n)\right\|^2 + \frac{\varepsilon}{2}\|\nabla u^n - \prod_h \nabla u^n\|^2 + \varepsilon\left\|\nabla_{d,r} e^n\right\|^2. \quad (5.12)$$

To estimate A_4^n of equation (5.9), we add and subtract $\left((u^n)^2, \frac{\partial_{d,r} e^n}{\partial x}\right)$, we get

$$A_4^n = \frac{1}{2}\left(\Pi_h (u^n)^2 - (u^n)^2, \frac{\partial_{d,r} e^n}{\partial x}\right) - \frac{1}{2}\left((u^n)^2 - (Q_h u^n)^2, \frac{\partial_{d,r} e^n}{\partial x}\right),$$

by Cauchy – Schwartz inequality and Young's-inequality, we obtain

$$|A_4^n| \le \frac{1}{8}\left\|\left(\Pi_h (u^n)^2 - (u^n)^2\right)\right\|^2 + \frac{1}{8}\left\|(u^n)^2 - (Q_h u^n)^2\right\|^2 + \left\|\frac{\partial_{d,r} e^n}{\partial x}\right\|^2. \quad (5.13)$$

To estimate A_5^n, we add and subtract $(Q_h v^n\, Q_h\, u_y^n, e^n)$, we get

$$A_5^n = \left(v^n\left(Q_h\, u_y^n - u_y^n\right), e^n\right) - \left(Q_h v^n Q_h\, u_y^n - v^n\, \frac{\partial_{d,r} u^n}{\partial y}, e^n\right),$$

again by Cauchy-Schwartz inequality, Young's-inequality and Poincare inequality [20], we obtain

$$|A_5^n| \le \|v^n\|_\infty^2 \left\|\left(Q_h\, u_y^n - u_y^n\right)\right\|^2 + \left\|\left(Q_h v^n Q_h\, u_y^n - v^n\, \frac{\partial_{d,r} u^n}{\partial y}\right)\right\|^2 + \frac{C}{2}\left\|\nabla_{d,r} e^n\right\|^2. \quad (5.14)$$

To estimate A_6^n, we add and subtract $(Q_h v^n\, Q_h\, u_y^n, \delta\beta.\nabla_{d,r} e^n)$, we get

$$A_6^n = \left(v^n\left(Q_h\, u_y^n - u_y^n\right), \delta\beta.\nabla_{d,r} e^n\right) - \left(Q_h v^n Q_h\, u_y^n - v^n\, \frac{\partial_{d,r} u^n}{\partial y}, \delta\beta.\nabla_{d,r} e^n\right),$$

using Cauchy-Schwartz inequality and Young's-inequality once more, we arrive to

$$|A_6^n| \le \|v^n\|_\infty^2 \left\|\left(Q_h\, u_y^n - u_y^n\right)\right\|^2 + \left\|\left(Q_h v^n Q_h\, u_y^n - v^n\, \frac{\partial_{d,r} u^n}{\partial y}\right)\right\|^2 + \frac{1}{2}\left\|(\delta\beta.\nabla_{d,r} e^n)\right\|^2. \quad (5.15)$$

Substituting (5.10), (5.11), (5.12), (5.13), (5.14) and (5.15) in equation (5.9) with noting that $\|(u^n)^2 - (Q_h u^n)^2\|^2$ and $\left\|\left(Q_h v^n Q_h\, u_y^n - v^n\, \frac{\partial_{d,r} u^n}{\partial y}\right)\right\|^2$ are non-negative terms, we get

$$\|e^n\|^2 + \tau\|e^n\|_{PW}^2 - \|e^{n-1}\|^2 \le \tau\left\|\left(\tilde{\partial}_t u^n - u_t^n\right)\right\|^2 + \tau\,\frac{\varepsilon}{2}\left\|\left(\nabla_{d,r}\, u^n - \nabla u^n\right)\right\|^2 + \tau\,\frac{\varepsilon}{2}\|\nabla u^n - \Pi_h \nabla u^n\|^2 + \frac{\tau}{8}\|(\Pi_h (u^n)^2 - (u^n)^2)\|^2 + 2\tau\,\|v^n\|_\infty^2\left\|\left(Q_h\, u_y^n - u_y^n\right)\right\|^2 + \tau\left\|(\delta\beta.\nabla_{d,r} e^n)\right\|^2 + M\,\tau\left\|\nabla_{d,r} e^n\right\|^2 + \tau\left\|\frac{\partial_{d,r} e^n}{\partial x}\right\|^2. \quad (5.16)$$

Where $M = C + \varepsilon$, *and since*

$$\tau\left\|(\delta\beta.\nabla_{d,r} e^n)\right\|^2 + M\tau\left\|\nabla_{d,r} e^u\right\|^2 + \tau\left\|\frac{\partial_{d,r} e^u}{\partial x}\right\|^2 \le \tau\|e^n\|_{PW}^2.$$

This will lead to

$$\|e^n\|^2 - \|e^{n-1}\|^2 + \tau\|e^n\|_{PW}^2 - \tau\|e^n\|_{PW}^2 \le \tau\left\|\left(\tilde{\partial}_t u^n - u_t^n\right)\right\|^2 + \tau\frac{\varepsilon}{2}\left\|\left(\nabla_{d,r}\, u^n - \nabla u^n\right)\right\|^2 + \tau\frac{\varepsilon}{2}\|\nabla u^n - \Pi_h \nabla u^n\|^2 + \frac{\tau}{8}\|(\Pi_h (u^n)^2 - (u^n)^2)\|^2 + 2\,\tau\,\|v^n\|_\infty^2\left\|\left(Q_h\, u_y^n - u_y^n\right)\right\|^2. \quad (5.17)$$

equation (5.17) can be expressed simply as follows.

$$\|e^n\|^2 \le \|e^{n-1}\|^2 + \tau\left\|\left(\tilde{\partial}_t u^n - u_t^n\right)\right\|^2 + L_1^n + L_2^n + L_3^n. \quad (5.18)$$

Where, $L_1^n = \tau\frac{\varepsilon}{2}\left\|\left(\nabla_{d,r}\, u^n - \nabla u^n\right)\right\|^2 + \tau\,\frac{\varepsilon}{2}\|\nabla u^n - \Pi_h \nabla u^n\|^2$

$L_2^n = \frac{\tau}{8}\left\|\left(\Pi_h (u^n)^2 - (u^n)^2\right)\right\|^2$

$L_3^n = 2\tau\,\|v^n\|_\infty^2\left\|\left(Q_h\, u_y^n - u_y^n\right)\right\|^2,$

by taking summation from (1 to N) of equation (5.18), we arrive to

$$\|e^n\|^2 \le \|e^0\|^2 + \tau\sum_{j=1}^n\left\|\left(\tilde{\partial}_t u^j - u_t^j\right)\right\|^2 + \sum_{j=1}^n L_1^j + \sum_{j=1}^n L_2^j + \sum_{j=1}^n L_3^j. \quad (5.19)$$

In the first term for the right-hand side of equation (5.19), we can

$$\xi^j = u_t^j - \tilde{\partial}_t(u^j) = u_t^j - \frac{1}{\tau}\left(u^j - u^{j-1}\right),$$

$\tau\xi^j = \tau\, u_t^j - \int_{t_{j-1}}^{t_j} u_t(t)\ dt = (t_j - t_{j-1})u_t^j - \int_{t_{j-1}}^{t_j} u_t(t)\ dt,$

adding and subtracting $t_{j-1}u_t^{j-1}$, we get

$\tau\xi^j = t_j\, u_t^j - t_{j-1}u_t^{j-1} - \left(t_{j-1}u_t^j - t_{j-1}u_t^{j-1}\right) - \int_{t_{j-1}}^{t_j} u_t(t)\, dt\,,$

$= \int_{t_{j-1}}^{t_j} t\, u_{tt}\ dt - t_{j-1}\int_{t_{j-1}}^{t_j} u_{tt}\,(t)\, dt = \int_{t_{j-1}}^{t_j} (t - t_{j-1})\, u_{tt}\ dt.$

$\tau\,\xi^j = \int_{t_{j-1}}^{t_j} \tau\, u_{tt}\ dt.$

$\tau\left\|\xi^j\right\| \le \tau\int_{t_{j-1}}^{t_j} \|u_{tt}\|\ dt.$

Using Jensen's inequality, we get

$\left\|\xi^j\right\|^2 \le \left(\int_{t_{j-1}}^{t_j} \|u_{tt}\|\ dt\right)^2 = \tau^2\left(\int_{t_{j-1}}^{t_j} \|u_{tt}\|\ \frac{dt}{\tau}\right)^2,$

$\left\|\xi^j\right\|^2 \le \tau^2\int_{t_{j-1}}^{t_j} \|u_{tt}\|^2\ \frac{dt}{\tau} = \tau\int_{t_{j-1}}^{t_j} \|u_{tt}\|^2\, dt.$ (5.20)

To approximation L_1^j, by Lemma (4.1) and Lemma (4.2), we have

$\sum_{j=1}^{n} L_1^j \le Ch^{2k}\sum_{j=1}^{n}\left\|u^j\right\|_{1+k}^{2}.$ (5.21)

To approximation L_2^j, by Lemma (4.2), we have

$\sum_{j=1}^{n} L_2^j \le Ch^{2k}\sum_{j=1}^{n}\left\|u^j\right\|_{1+k}^{2}.$ (5.22)

To approximation L_3^j, *by* Lemma (4.1), we have

$\sum_{j=1}^{n} L_3^j \le Ch^{2k}\sum_{j=1}^{n}\left\|u^j\right\|_{1+k}^{2}.$ (5.23)

Substituting (5.20), (5.21), (5.22) and (5.23) in equation (5.19), we obtain

$\|e^n\|^2 \le \|e^0\|^2 + Ch^{2k}\sum_{j=1}^{n}\left\|u^j\right\|_{1+k}^{2} + \tau^2\int_{t_{j-1}}^{t_j}\|u_{tt}\|^2\, dt.$ (5.24)

In the same way, the proof for equation (5.2).

6. Numerical experiment

In this section, we compute the error $u - u_h$ of L^2- norm of the PWG-FEM in the case of FDPWG-FEM FEM by using Matlab R2014a software**.** We take into account the system of coupled Burgers' equations in two dimensions (1.1)and (1.2) over the square domain $\Omega: [0,1]\times[0,1]$. Burgers' equation in two dimensions linked has the following precise solutions [21]:

$$u(x,y,t) = -2\epsilon\frac{2\pi e^{-5\pi^2\epsilon t}\cos(2\pi x)\sin(\pi y)}{2 + e^{-5\pi^2\epsilon t}\sin(2\pi x)\sin(\pi y)}\ ,$$

$$v(x,y,t) = -2\epsilon\frac{\pi e^{-5\pi^2\epsilon t}\sin(2\pi x)\cos(\pi y)}{2 + e^{-5\pi^2\epsilon t}\sin(2\pi x)\sin(\pi y)}\ .$$

Various computational meshes are utilized, and the computation's time step is satisfactory.

$\tau = cfl * min\,(h^2)$,

where the shortest length of all the triangles is $min(h)$, and cfl is a parameter that depends on the issue. The exact solution is utilized to determine the boundary and initial conditions. In the test ϵ = 0.01 and 0.1 are employed to determine if the time step size τ and mesh size h have converged.

The L^2 and H^1 - errors for the velocities u and v are displayed in Tables 1,2,5 and 6 in the WG - FEM when $\delta = 0$, and Tables 3,4,7 and 8 in the PWG - FEM when $\delta = \frac{h}{32}, \beta = [1,1]$. The PWG and WG methods use a linear element and mesh size $h = \frac{1}{n}, n = 2,4,8,16,32,$ with $T = 1$, and $clf = 0.05$. Figures 1 and 3 show the numerical and exactly solutions concerning u and v in the WG -FEM, and Figures 2 and 4 show the numerical and exactly solutions concerning u and v in the PWG -FEM.

Table 1: L^2 and H^1error for u in case $T = 1,\ \epsilon = 0.01$ and $clf = 0.05$ in WG -FEM.

h	$\|\nabla_d e^u\|$	Order $\|\nabla_d e^u\|$	$\|e^u\|_{\{L^2,k\}}$	Order $\|e^u\|_{\{L^2,k\}}$	$\|e^u\|_{\{L^2,\partial k\}}$	Order $\|e^u\|_{\{L^2,\partial k\}}$
$\frac{1}{2}$	4.4311e-03		2.3154e-03		1.4754e-02	
$\frac{1}{4}$	3.7353e-03	0.2464	1.9237e-03	0.2674	1.0224e-02	0.5291
$\frac{1}{8}$	2.6215e-03	0.5107	1.5077 e-03	0.3515	9.0776e-03	0.1715
$\frac{1}{16}$	2.1152e-03	0.3096	1.4791e-03	0.0276	7.2465e-03	0.3250
$\frac{1}{32}$	1.8042e-03	0.2294	1.0042e-03	0.5586	5.9241e-03	0.2906

Table 2: L^2 and H^1error for v in case $T = 1,\ \epsilon = 0.01$ and $clf = 0.05$ in WG -FEM.

h	$\|\nabla_d e^v\|$	Order $\|\nabla_d e^v\|$	$\|e^v\|_{\{L^2,k\}}$	Order $\|e^v\|_{\{L^2,k\}}$	$\|e^v\|_{\{L^2,\partial k\}}$	Order $\|e^v\|_{\{L^2,\partial k\}}$
$\frac{1}{2}$	2.7814e-03		2.0142e-03		9.2089e-03	
$\frac{1}{4}$	1.7820e-03	0.6422	1.0772e-03	0.9028	7.7274e-03	1.5076
$\frac{1}{8}$	1.1796e-03	0.5952	9.3677e-04	0.2015	5.5780e-03	1.1631
$\frac{1}{16}$	9.4680e-04	0.3171	9.3645e-04	0.0004	5.3625 e-03	1.2280
$\frac{1}{32}$	7.1400e-04	0.4071	9.3610e-04	0.0005	2.1549e-03	0.0128

Table 3: L^2 and H^1error for u in case $T = 1,\ \epsilon = 0.01$ and $clf = 0.05$ in PWG -FEM.

h	$\|\nabla_d e^u\|$	Order $\|\nabla_d e^u\|$	$\|e^u\|_{\{L^2,k\}}$	Order $\|e^u\|_{\{L^2,k\}}$	$\|e^u\|_{\{L^2,\partial k\}}$	Order $\|e^u\|_{\{L^2,\partial k\}}$
$\frac{1}{2}$	2.2155e-03		1.1577e-03		7.3770e-03	
$\frac{1}{4}$	1.2451e-03	0.8313	6.4123e-04	0.8523	4.4080e-03	0.7429
$\frac{1}{8}$	8.7383e-04	0.5108	4.0256e-04	0.6716	3.0258e-03	0.5427
$\frac{1}{16}$	7.0506e-04	0.4659	3.9303e-04	0.0345	2.1145e-03	0.5170

$\frac{1}{32}$	5.0140e-04	0.4917	2.3473e-04	0.7436	1.4747e-03	0.5198

Table 4: L^2 and H^1error for v in case $T = 1,\ \epsilon = 0.01$ and $cl\ f = 0.05$ in PWG -FEM.

h	$\|\nabla_d e^v\|$	Order $\|\nabla_d e^v\|$	$\|e^v\|_{\{L^2,k\}}$	Order $\|e^v\|_{\{L^2,k}$	$\|e^v\|_{\{L^2,^{\partial}k\}}$	Order $\|e^v\|_{\{L^2,\partial k\}}$
$\frac{1}{2}$	1.3907e-03		1.0071e-03		3.0696e-03	
$\frac{1}{4}$	7.9400e-04	0.8086	4.5906e-04	1.1334	1.9318e-03	0.6680
$\frac{1}{8}$	4.9320 e-04	0.6869	2.1025e-04	1.1265	9.9450e-04	0.9579
$\frac{1}{16}$	3.4720e-04	0.5064	1.1215e-04	0.9067	5.9375e-04	1.2280
$\frac{1}{32}$	2.3800 e-04	0.5448	7.8506e-05	0.5145	3.5915e-04	0.7252

Table 5: L^2 and H^1error for u in case $T = 1,\ \epsilon = 0.1$ and $cl\ f = 0.05$ in WG -FEM.

h	$\|\nabla_d e^u\|$	Order $\|\nabla_d e^u\|$	$\|e^u\|_{\{L^2,k\}}$	Order $\|e^u\|_{\{L^2,k\}}$	$\|e^u\|_{\{L^2,\partial k\}}$	Order $\|e^u\|_{\{L^2,\partial k\}}$
$\frac{1}{2}$	1.3629e-02		9.3221e-03		2.7260e-02	
$\frac{1}{4}$	8.6415e-03	0.6573	5.7382e-03	0.7000	2.3664e-02	0.2041
$\frac{1}{8}$	4.5166e-03	0.9360	2.8956 e-03	0.9867	1.8933e-02	0.3217
$\frac{1}{16}$	3.7038e-03	0.2862	2.2979e-03	0.3335	1.6873e-02	0.1661
$\frac{1}{32}$	2.8042e-03	0.4014	2.0042e-03	0.1972	1.5146e-02	0.1557

Table 6: L^2 and H^1error for v in case $T = 1,\ \epsilon = 0.1$ and $cl\ f = 0.05$ in WG -FEM.

h	$\|\nabla_d e^v\|$	Order $\|\nabla_d e^v\|$	$\|e^v\|_{\{L^2,k\}}$	Order $\|e^v\|_{\{L^2,k\}}$	$\|e^v\|_{\{L^2,\partial k\}}$	Order $\|e^v\|_{\{L^2,\partial k\}}$
$\frac{1}{2}$	4.3700e-03		2.6924e-03		7.6324e-03	
$\frac{1}{4}$	1.6399e-03	1.4140	1.0009e-03	1.4275	4.0860e-03	0.9014
$\frac{1}{8}$	1.0282e-03	0.6734	6.0258e-04	0.7321	3.9348e-03	0.0543
$\frac{1}{16}$	8.5023e-04	0.2742	4.7830e-04	0.3332	3.4942 e-03	0.1713
$\frac{1}{32}$	6.7241e-04	0.3385	3.4319e-04	0.4789	2.8546e-03	0.2916

Table 7: L^2 and H^1error for u in case $T = 1,\ \epsilon = 0.1$ and $cl\ f = 0.05$ in PWG -FEM.

h	$\|\nabla_d e^u\|$	Order $\|\nabla_d e^u\|$	$\|e^u\|_{\{L^2,k\}}$	Order $\|e^u\|_{\{L^2,k\}}$	$\|e^u\|_{\{L^2,\partial k\}}$	Order $\|e^u\|_{\{L^2,\partial k\}}$
$\frac{1}{2}$	6.8145e-03		4.6610e-03		1.3630e-02	
$\frac{1}{4}$	3.8805e-03	0.8123	1.9127e-03	1.2850	7.8880e-03	0.7890
$\frac{1}{8}$	2.5055e-03	0.6311	9.6519e-04	0.9867	5.3110e-03	0.5706
$\frac{1}{16}$	2.1346e-03	0.2311	5.7447e-04	0.7485	3.6243e-03	0.5512
$\frac{1}{32}$	1.3473e-03	0.6638	3.6806e-04	0.6422	3.0486e-03	0.2495

Table 8: L^2 and H^1error for v in case $T = 1,\ \epsilon = 0.1$ and $cl\ f = 0.05$ in PWG -FEM.

h	$\|\nabla_d e^v\|$	Order $\|\nabla_d e^v\|$	$\|e^v\|_{\{L^2,k\}}$	Order $\|e^v\|_{\{L^2,k\}}$	$\|e^v\|_{\{L^2,\partial k\}}$	Order $\|e^v\|_{\{L^2,\partial k\}}$
$\frac{1}{2}$	2.1850e-03		1.3462e-03		3.8162e-03	
$\frac{1}{4}$	9.4663e-04	1.2067	5.3363e-04	1.3349	23620e-03	0.6921
$\frac{1}{8}$	5.4273e-04	0.8025	3.0086e-04	0.8267	1.3116e-03	0.8486
$\frac{1}{16}$	2.8341e-04	0.9373	1.5943e-04	0.9161	9.6473 e-04	0.4431
$\frac{1}{32}$	2.2413e-04	0.3385	1.1439e-04	0.4789	7.5153e-04	0.3602

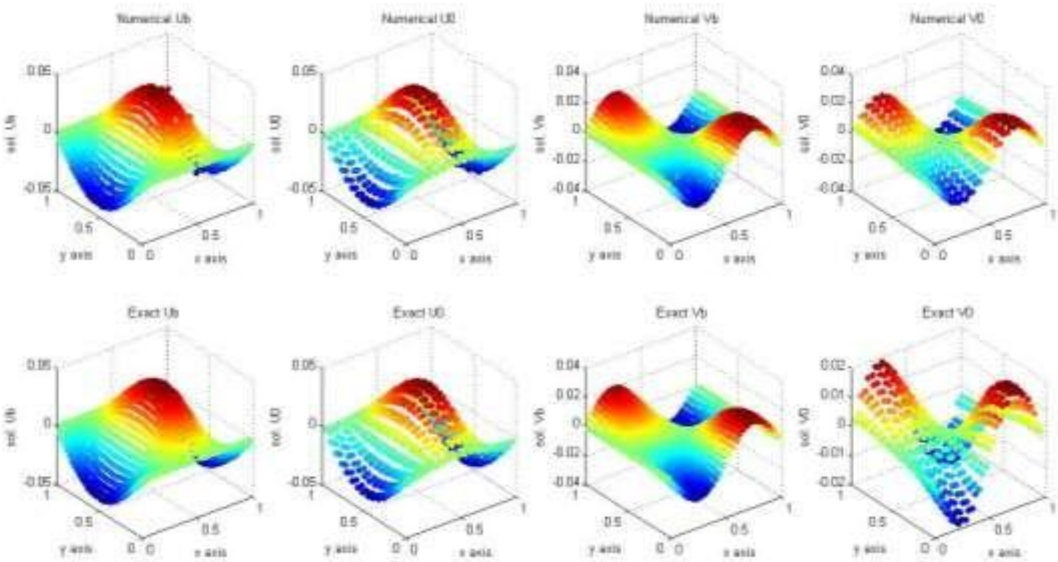

Figure 1: Numerical and exact solutions for u and v in case $(T = 1,\ cl\ f = 0.05,\ \epsilon = 0.01)$ for the WG $-$ FEM.

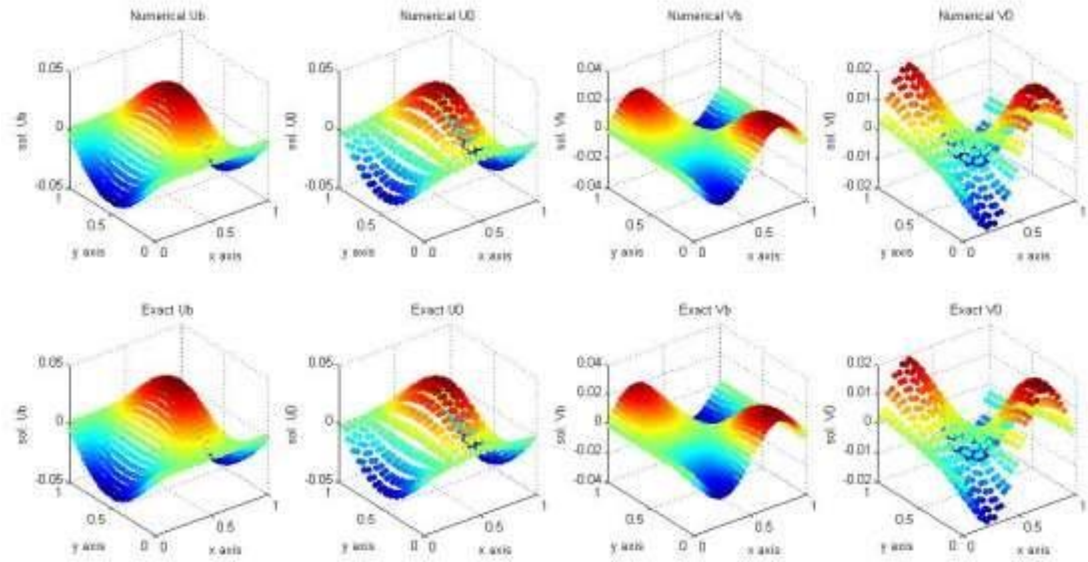

Figure 2: Numerical and exact solutions for u and v in case$(T = 1,\ cl\ f\ = 0.05,\ \epsilon =\ 0.01)$ for the PWG - FEM.

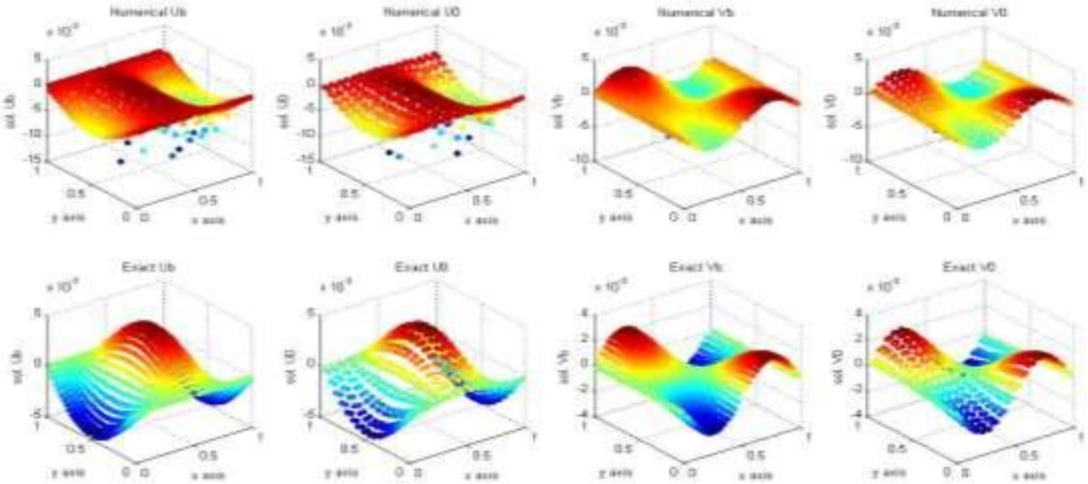

Figure 3: Numerical and exact solutions for u and v in case $(T = 1, cl\ f\ =\ 0.05, \epsilon = 0.1)$ for the WG -FEM.

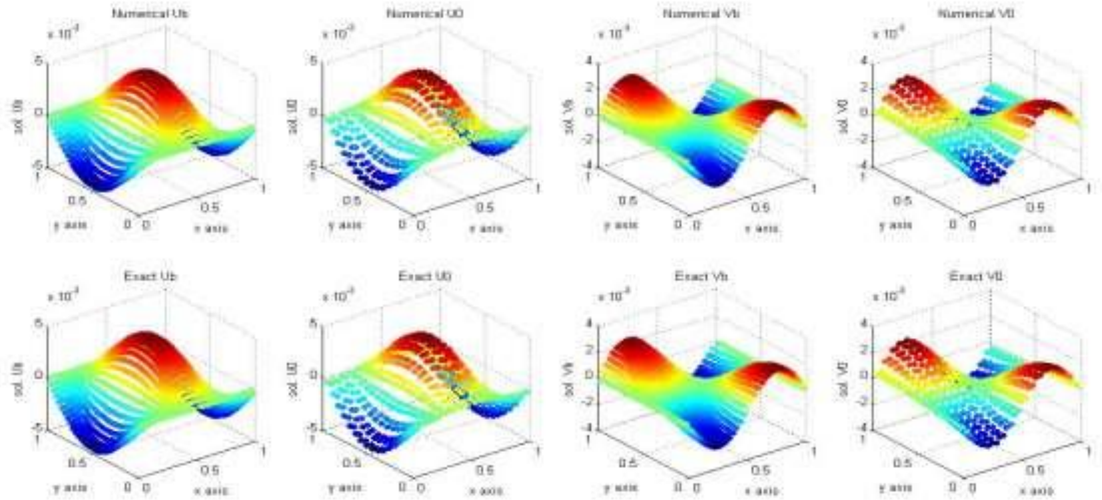

Figure 4: Numerical and exact solutions for u and v in case$(T = 1, cl\,f = 0.05, \epsilon = 0.1)$ For the PWG-FEM.

7. Discussion and Conclusion

In this paper, we consider the full- discrete PWG-FEM for solving coupled Burgers' equations in two dimensions. When comparing Tables (1)-(8) for the PWG-FEM a significant improvement and regularity were observed in the numerical results of the PWG-FEM compared to the numerical results for WG-FEM. Ourfindings demonstrate that the PWG-FEM is significantly more accurate than the WG-FEM, see Tables (1)-(8) and see Figures (1) -(4), we demonstrated consistency between the exact solutions and numerical outcomes for unsteady-state in PWG-FEM.

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