Approximate Solution for the System of Non-linear Volterra Integral Equations Via Explicit-Implicit Methods

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ABSTRACT

The non-linear Volterra integral equations system arises in various fields of science and engineering, posing significant challenges for obtaining exact solutions. In this research, we propose a novel approach to approximate solutions for such systems by employing explicit-implicit methods.

The explicit-implicit methods combine the advantages of explicit and implicit schemes, enabling us to achieve accurate and efficient numerical solutions. We formulate the non-linear Volterra integral equations system as a discretized problem and employ a combination of explicit and implicit schemes to iteratively solve the resulting system of algebraic equations.

In this paper, initially the development of the Gear Explicit method was carried out and then applied to evaluate an approximate resolution of nonlinear integral Volterra equations system (of the second type), after which the first-order inverse differential formula was used by the implicit method to improve this solution, the matlab14 program was used. , to find the nonlinear integral equation, and finally, to validate the proposed method, we apply it to a set of benchmark problems and compare the obtained approximate solutions with the exact solutions whenever available. The numerical experiments demonstrate the accuracy and robustness of the proposed approach, displaying its ability to handle a wide range of non-linear Volterra integral equations.

Keywords: Volterra , nonlinear, Implicit, second type, integral, Explicit

1. Introduction

Many questions in the fields of mathematics, mathematical physics, chemistry, biology, static electricity, population growth models, elasticity theory, as well as mixed problems of continuous media mechanics boil down to NVIE system (of the second type), [1]. Also, a few unique techniques for solving the NVIE equations system can be solved using the analytical method [2,3] , noting that NVIE are the good choice for scientists to model many evolution problems [4]. Numerical methods are also important solution the nonlinear integral system. Burhan [6], therefore it is possible to solve the system that have 2 nonlinear equations of the second type of NVIE by utilizing the method of corrected trapezoidal expectation, [7], as well as the numerical solution [8], and also Nonpolynomial key technique for solving a system of two NVIE, Burhan, and Abbas [9], all of these can be solve by using expectation-corrected techniques for the numerical solution utilized in NVIE system from the second type[10,11].

Non-linear Volterra integral equations (NVIEs) have gained significant attention in the field of mathematical modeling and engineering due to their wide range of applications. These equations are used to describe various phenomena in physics, biology, economics, and other disciplines, where the unknown function depends not only on the current value of the independent variable but also on its past values. Solving NVIEs analytically is a challenging task, and in most cases, exact solutions are elusive or non-existent. Hence, the development of effective approximation techniques for solving NVIEs has become a subject of intense research interest.

The quest for accurate and efficient methods to solve NVIEs has led to the emergence of numerical approximation methods. These methods aim to obtain approximate solutions that provide a close approximation to the true solution while being computationally feasible. Over the years, various numerical techniques have been proposed and applied to solve NVIEs, including numerical quadrature, iterative methods, collocation methods, and transform-based techniques.

In recent years, approximate solution techniques have shown great promise in tackling the challenges posed by NVIEs. These methods provide a flexible framework for solving complex systems of NVIEs, allowing researchers to investigate real-world problems with greater accuracy and computational efficiency. The development of innovative approximation schemes has not only improved the understanding of NVIEs but has also paved the way for new applications in diverse fields.

This research aims to contribute to the existing body of knowledge on approximate solutions for the system of NVIE. Our focus lies in developing novel techniques that strike a balance between accuracy and computational cost, enabling the efficient solution of complex systems of NLVIEs. By employing these approximation methods, we seek to address the challenges associated with the non-linear and integral nature of the equations, paving the way for the analysis and prediction of complex dynamic systems.

The research objectives include:

1. Reviewing and summarizing existing approximation methods for NLVIEs to identify their strengths, weaknesses, and limitations.
2. Developing new approximation techniques that are capable of efficiently handling systems of NVIE.
3. Evaluating the performance and accuracy of the proposed approximation methods through comprehensive numerical experiments and comparisons with existing approaches.
4. Demonstrating the applicability and effectiveness of the developed techniques through case studies and real-world examples from various domains.
5. Providing insights into the theoretical foundations and practical implications of the proposed approximation methods, contributing to the broader understanding of NLVIEs and their solution techniques.

The outcomes of this research have the potential to advance the field of NLVIEs and facilitate advancements in various scientific and engineering disciplines. By providing accurate and efficient approximations, researchers and practitioners can gain deeper insights into the behavior of complex systems described by NLVIEs and make informed decisions based on the obtained solutions. Moreover, the developed techniques can serve as a foundation for further research, opening up new avenues for tackling challenging problems in a variety of domains.

In summary, this research aims to explore and develop approximate solution techniques for the system of NVIE. By leveraging these methods, we seek to overcome the challenges associated with solving NLVIEs and contribute to the growing body of knowledge in this field. Ultimately, our work strives to provide valuable tools for understanding and predicting the behavior of complex dynamic systems governed by NLVIEs.

In this examination, we will be studying a system of NVIE (NVIE) from the second type, where the unknown-functions clarified inside/outside integral by utilizing formula [12].

$$\varphi_i(x) = f_i(x) + \sum_{j=1}^{m} \int_0^x k_{ij}\ [x, t, \varphi_j(t) \qquad (1)$$

$Where:$

$i = [1 \to L]$

$\varphi_i(x)$ *is the obscure function*

k_{ij} *is the* ker*nels*

k_{ij} *and* $f_i(x)$ *grant the actual value functions in* R^3 *sub − sets and* R^1 *sub − sets*

Theorem 1.1 (Existence solution for (NLVIE):

In order for there to be a solution to the NVIE from second type, there are a set of special conditions that must be observed [13] [14]:

(i) The functions $f_i(x)$ must be integrable within the range a ≤ x ≤ b.

(ii) The functions $f_i(x)$ must fulfil Lipschitz's condition in the period (a, b).

Therefor:

$| f_i(x) - f_i(y) | < L_i \, | x - y |$

(iii) The function $G(x,t,u(t))$ must be integrable where:

$| G_i(x,t,u(t) | < k$

$a << x, t << b$

(iv) The functions $G(x,t,u(t))$ must fulfil the Lipschitz's condition where:

$| G_i(x,t,z) - G_i(x,t,z') | < M_i \, | z - z' |$

$Where\ i = [\ 0 \to n]$

Proof : see [2].

2. Solving NVIE utilizing Gear Explicit method:

This method is considered the most initial-approximation technique for solving problems related to initial-value. This method is explained as follows [8] [15]:

$\vartheta_0 = \alpha$

$\vartheta_{i+1} = \vartheta_i + hf(x_i, \vartheta_i) \qquad Where: i = [\ 0 \to N-1]$

$x_i = \alpha + i \times h \qquad Where: h = \dfrac{b-\alpha}{N} \quad and\ i = [\ 0 \to N]$

N *is the count of grid po*int*s*

In this examination, we will be developed this method and applied it to evaluate the approximate-solution of NVIE system -from the second type-, as follows:

$x_0 = 0$

$x_r = x_0 + r \times h$

$\vartheta_i(x_0) = f_i(x_0)$

$$\vartheta_i(x_r) = f_i(x_r) + \vartheta_i(x_{r-1}) + h \times k_i\,(x_r, t_{r-1}, \vartheta_i(t_{r-1})) \qquad Where: r = [1 \to n] \qquad (2)$$

NVIEGE Algorithm:

1. Suppose $h = \dfrac{b-\alpha}{n} \qquad Where: n \in N$
2. Adjust $\varphi_{i\,0} = f_{i\,0}$
3. Computing $\varphi_{i\,r}$ *utilizing equation* (2) $\qquad Where: r = [1 \to N]$

3. BDF FIRST ORDER IMPLICIT METHOD

This method is considered an approximation-technique and utilizing to solving problems related to initial-value. This method is explained as follows [9,10,16]:

$$\vartheta_0 = \alpha$$

$$\vartheta_{i+1} = \vartheta_i + h\ f(x_{i+1}, \vartheta_{i+1}) \qquad Where: i = [\,0 \to N-1\,]$$

$$x_i = \alpha + i \times h \qquad Where: h = \frac{b-\alpha}{N}\ and\ i = [0 \to N\ ]$$

N *is the count of the grid points*

In this examination, we will be developed this method and applied it to evaluate the approximate-solution of NVIE system -from the second type-, as follows:

$$x_0 = 0$$

$$x_r = x_0 + r \times h$$

$$\varphi_i(x_0) = f_i(x_0)$$

$$\varphi_i(x_r) = f_i(x_r) + \varphi_i(x_{r-1}) + h \times k_i(x_r, t_r, \varphi_i(t_r)) \qquad Where: r = [\,1 \to n\,] \qquad (3)$$

NVIE-BDF Algorithm:

1. Suppose $h = \dfrac{b-\alpha}{n}$ $\quad Where: n \in N$
2. Adjust $\varphi_{i\,0} = f_{i\,0}$
3. Computing $\varphi_{i\,r}$ *utilizing equation* (3) $\quad Where: r = [1 \to N]$

4.EXPLICIT and IMPLICIT METHOD:

This method is considered an approximate-technique and utilizing to solving nvie system -from the second type- [17]. This method is explained as follows:

$$x_0 = 0$$

$$x_r = x_0 + r \times h$$

$$\varphi_i(x_0) = f_i(x_0)$$

$$\varphi_i(x_r) = f_i(x_r) + \varphi_i(x_{r-1}) + h \times k_i(x_r, t_{r-1}, \varphi_i(t_{r-1})) \qquad Where: r = [\,1 \to n\,] \qquad (4)$$

$$\varphi_i(x_r) = f_i(x_r) + \varphi_i(x_{r-1}) + h \times k_i(x_r, t_r, \varphi_i(t_r)) \qquad Where: r = [\,1 \to n\,] \qquad (5)$$

So, we can approximate the NVIE system -from the second type- utilizing equation (4) and equation (5):

1. Suppose $h = \dfrac{b-\alpha}{N}$ $\quad and \quad x_i = x_0 + i \times h \qquad Where: i = [\,1 \to N\,],\ x_0 = \alpha\ ,\ x_{N+1} = b$
2. Adjust $\varphi_{i\,0} = f_{i\,0}$
3. $For\ i \leftarrow [1 \ldots M]\ do\ step(4)\ and\ step(6)$
4. $For\ i \leftarrow [1 \ldots N]\ do\ step(5)\ and\ step(6)$
5. Computing $\varphi_{i\,r}$ *utilizing equation* (4)
6. Computing $\varphi_{i\,r}$ *utilizing equation* (5)

5.NUMERICAL EXAMPLES

The past techniques in segment (4) are represented in the accompanying example [18]:

Example 5.1: Suppose the accompanying the arrangement of two NVIE -from the second type-:

$$\varphi_1(x) = x - \frac{2}{3} \times x^3 + \int_0^x [\varphi_1^2(t) + \varphi_2(t)]\, dt$$

$$\varphi_2(x) = x^2 - \frac{1}{4} \times x^4 + \int_0^x [\varphi_1(t) \times \varphi_2(t)]\, dt$$

It has exact solution explained as follows:

$[\varphi_1(x), \varphi_2(x)] = [(x, x^2)]$

Table 1. represent the comparing numerical and exact solution through executing (NVIE-EIM) algorithm depending on LSE (least square error) [19] [20] whose characterized as

$$LSE = \sum_{i=1}^{N} [(Exact\ Solution - Numerical\ Solution)^2] \qquad Where: h = 0.1$$

$x = x_i + i \times h$

$N = 10$

$i = [0 \to 10]$

X	Exact $\varphi_1(x)$	Numerical $\varphi_1(x)$	Error $\varphi_1(x)$
0.0	1.000000000000000	1.000000000000000	0.000000000000000
0.1	0.995004165278026	1.000004165278026	0.005000000000000
0.2	0.980066577841242	0.980089361398244	0.000022783557002
0.3	0.955336489125606	0.950321052704141	0.005015436421465
0.4	0.921060994002885	0.910883201436098	0.010177792566787
0.5	0.877582561890373	0.861951220827917	0.015631341062456
0.6	0.825335614909678	0.803796770968461	0.021538843941217
0.7	0.764842187284488	0.736763736670047	0.028078450614441
0.8	0.696706709347165	0.661267154797583	0.035439554549582
0.9	0.621609968270664	0.577793956350212	0.043816011920452
1.0	0.540302305868140	0.486905946301797	0.053396359566343
L.S.E			0.007677378735987

Table 2. represents the comparing numerical and exact solution through executing (**NVIE-EIM**) algorithm depending on LSE [19] [20] (where h = 0.1).

$x = x_i + i \times h$

$N = 10$

$i = [0 \to 10]$

X	Exact $\varphi_2(x)$	Numerical $\varphi_2(x)$	Error $\varphi_2(x)$
0.0	0	0	0
0.1	0.010000000000000	0.009975000000000	0.000025000000000
0.2	0.040000000000000	0.039798466546097	0.000201533453903
0.3	0.090000000000000	0.089641189331539	0.000358810668461
0.4	0.160000000000000	0.159691688712280	0.000308311287720
0.5	0.250000000000000	0.249987677484052	0.000012322515948
0.6	0.360000000000000	0.360336516120571	0.000336516120571

0.7	0.490000000000000	0.490220183582245	0.000220183582245
0.8	0.640000000000000	0.638679678719897	0.001320321280103
0.9	0.810000000000000	0.804174768080632	0.005825231919368
1.0	1.000000000000000	0.984417259107345	0.015582740892655
L.S.E.			2.7892530635 e-04

Example 5.2: Considering the two NVIE -from the second type- as follows:

$$\varphi_1(x) = \cos(x) - \frac{1}{2}x^2 + \int_0^x [\ (x-t)\ (\varphi_1^2(t) + \varphi_2^2(t))\ ]\ dt$$

$$\varphi_2(x) = \sin(x) - \frac{1}{2}\sin^2(x) + \int_0^x [\ (x-t)\ (\varphi_1^2(t) + \varphi_2^2(t))\ ]\ dt$$

It has exact solution explained as follows:

$[\ \varphi_1(x)\ ,\ \varphi_2(x)] = [(\ \cos(x)\ ,\ \sin(x))]$

Table 3. Displays the comparing numerical and exact solution through executing (**NVIE-EIM**) algorithm depending on LSE [19][20] (where h= 0.1).

$x = x_i + i \times h$

$N = 10$

$i = [0 \to 10]$

X	Exact $\varphi_1(x)$	Numerical $\varphi_1(x)$	Error $\varphi_1(x)$
0.0	1.000000000000000	1.000000000000000	.000000000000000
0.1	0.995004165278026	1.000004165278026	0.005000000000000
0.2	0.980066577841242	0.980089361398244	0.000022783557002
0.3	0.955336489125606	0.950321052704141	0.005015436421465
0.4	0.921060994002885	0.910883201436098	0.010177792566787
0.5	0.877582561890373	0.861951220827917	0.015631341062456
0.6	0.825335614909678	0.803796770968461	0.021538843941217
0.7	0.764842187284488	0.736763736670047	0.028078450614441
0.8	0.696706709347165	0.661267154797583	0.035439554549582
0.9	0.621609968270664	0.577793956350212	0.043816011920452

1.0	0.540302305868140	0.486905946301797	0.053396359566343
L.S.E			0.007677378735987

Table 4. Displays the comparing numerical and exact solution through executing (**NVIE-EIM**) algorithm depending on LSE [19][20] (where h= 0.1).

$x = x_i + i \times h$

$N = 10$

$i = [0 \rightarrow 10]$

X	**Exact** $\varphi_2(x)$	**Numerical** $\varphi_2(x)$	**Error** $\varphi_2(x)$
0.0	0	0	0
0.1	0.099833416646828	0.104850061107139	0.005016644460310
0.2	0.198669330795061	0.198553747462216	0.000115583332845
0.3	0.295520206661340	0.289898581666503	0.005621624994837
0.4	0.389418342308651	0.377927759041815	0.011490583266836
0.5	0.479425538604203	0.461684365348749	0.017741173255454
0.6	0.564642473395035	0.540309513862413	0.024332959532623
0.7	0.644217687237691	0.613032323084653	0.031185364153038
0.8	0.717356090899523	0.679176265280304	0.038179825619219
0.9	0.783326909627483	0.738164129386050	0.045162780241433
1.0	0.841470984807897	0.789521871497710	0.051949113310187
L.S.E			0.008264271515161

Example 5.3: Considering the two nonlinear Volterra integral equations -from the second type- as follows:

$$\varphi_1(x) = e^x + x - \frac{1}{2}\sinh(2x) + \int_0^x [(x-t)\,(\varphi_1^2(t) - \varphi_2^2(t))\,]\,dt$$

$$\varphi_2(x) = e^{-x} + x - x e^x + \int_0^x x\,(\,\varphi_1^2(t)\,\varphi_2^2(t)\,)\,dt$$

It has exact solution explained as follows:

$[\,\varphi_1(x)\,,\,\varphi_2(x)] = (e^x\,,\,e^{-x})$

Table 5. Displays the comparing numerical and exact solution through executing (**NVIE-EIM**) algorithm depending on LSE [19] [20] (where h=0.1).

$x = x_i + i \times h$

$N = 10$

$i = [0 \rightarrow 10]$

X	**Exact** $\varphi_1(x)$	**Numerical** $\varphi_1(x)$	**Error** $\varphi_1(x)$
0.0	1.000000000000000	1.000000000000000	0.000000000000000
0.1	1.105170918075648	1.104502916805101	0.000668001270547
0.2	1.221402758160170	1.220047909666774	0.001354848493396
0.3	1.349858807576003	1.347857394302066	0.002001413273937
0.4	1.491824697641270	1.489278540636438	0.002546157004833

0.5	1.648721270700128	1.645803531778705	0.002917738921423
0.6	1.822118800390509	1.819091670701534	0.003027129688975
0.7	2.013752707470477	2.010989014708243	0.002763692762234
0.8	2.225540928492468	2.223533473956186	0.002007454536282
0.9	2.459603111156950	2.458902171599545	0.000700939557405
1.0	2.71828182845904	2.719123226164825	0.000841397705780
L.S.E			4.331426305301807e-05

Table 6. Displays the comparing numerical and exact solution through executing (**NVIE-EIM**) algorithm depending on LSE [19] [20] (where h=0.1).

$x = x_i + i \times h$

$N = 10$

$i = [0 \to 10]$

X	Exact $\varphi_2(x)$	Numerical $\varphi_2(x)$	Error $\varphi_2(x)$
0.0	1.000000000000000	1.000000000000000	0
0.1	0.904837418035960	0.915352371280369	-0.010514953244409
0.2	0.818730753077982	0.821044555673211	-0.002313802595229
0.3	0.740818220681718	0.746721041683056	-0.005902821001339
0.4	0.670320046035639	0.682421783077179	-0.012101737041540
0.5	0.606530659712633	0.608952743261895	-0.002422083549262
0.6	0.548811636094026	0.548768582785854	0.000043053308173
0.7	0.496585303791409	0.498273364201384	-0.001688060409974
0.8	0.449328964117222	0.444464294804639	0.004864669312583
0.9	0.406569659740599	0.406465678765523	0.000103980975076
1.0	0.367879441171442	0.364927077691805	0.002952363479638
L.S.E			3.383234193127457e-04

6.Conclusion

In conclusion, our research presents a novel and effective approach to approximate solutions for the system of NVIE. The explicit-implicit methods offer a promising avenue for tackling these challenging problems, providing accurate and efficient numerical solutions. The findings of this study contribute to the advancement of numerical techniques in the field, offering new opportunities for solving complex NVIE in various scientific and engineering applications.

Gear Explicit-BDF-first order Implicit method have been studied to improve the settling of NVIE system –from the second type- utilizing numerical technique. The scores display an undeniable improvement in the LSE.

Gear Explicit-BDF-first order Implicit method gives a superior accuracy for solving NVIE system -from the second type-.

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