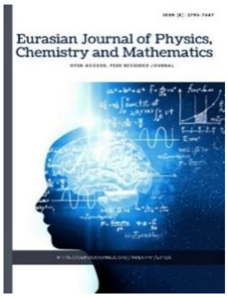


# Conditional Correctness of the Initial-Boundary Value Problem for The System of Second Order Mixed Type Equations

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**ABSTRACT**

In this paper, we investigate the ill-posed initial boundary value problem for the system of mixed type equations. A priori estimate of solution is obtained with the method of logarithmic convexity. The theorems of uniqueness and conditional stability are proved on the set of correctness of the solution. An approximate solution to the problem is constructed by the regularization method. We calculate an estimate for efficiency of the norm of the difference between exact and approximate solutions.

**Keywords:** system of mixed type equations, ill-posed problem, a priori estimate, set of correctness, theorem of uniqueness, conditional stability, regularization, parameter of regularization.

This article is devoted to the study of the ill-posed initial-boundary value problem for the system of second order mixed-type equations.

In the given domain $\Omega=\{(x,t):-l<x<l, x\neq 0, 0<t<T\}$ we consider the system of equations

$$\begin{cases} u_{tt}+\operatorname{sgn}x\cdot u_{xx}+a_1\cdot u+b_1\cdot v=0, \\ v_{tt}+\operatorname{sgn}x\cdot v_{xx}+a_2\cdot v+b_2\cdot u=0, \end{cases} \tag{1}$$

where $a_i, b_i$ - given real numbers, $b_i\neq 0$, $i=1,2$, $(a_1-a_2)^2+4b_1b_2>0$.

**Problem.** Find a pair of functions $(u(x,t), v(x,t))$ that satisfies the system of equations (1) and the following conditions:

the initial

$$\left.\begin{aligned} u|_{t=0}&=\varphi_1(x), & u_t|_{t=0}&=\psi_1(x), \\ v|_{t=0}&=\varphi_2(x), & v_t|_{t=0}&=\psi_2(x), \end{aligned}\right\} \quad -l\leq x\leq l \tag{2}$$

boundary

$$\left.\begin{aligned} u|_{x=l}&=u|_{x=-l}=0, \\ v|_{x=l}&=v|_{x=-l}=0, \end{aligned}\right\} 0\leq t\leq T \tag{3}$$

and gluing conditions

$$\begin{aligned} u|_{x=-0}&=u|_{x=+0}, & u_x|_{x=-0}&=u_x|_{x=+0}, \\ v|_{x=-0}&=v|_{x=+0}, & v_x|_{x=-0}&=v_x|_{x=+0}. \end{aligned} \tag{4}$$

Boundary value problems for mixed type equations were studied among the first in the scientific works of F. Tricomi and S. Gellerstedt [8]. F.I. Frankl, I.N. Vekua showed in his scientific work that mixed type equations are related to important practical issues [7]. Later, various problems for mixed type equations were the subject of research by many mathematicians. Including M.A. Lavrent'ev, A.V. Bitsadze, M.H. Protter, S. Agmon, C.S. Morawetz, K.I. Babenko, S.P. Pulkin, M.M. Smirnov, M.S. Salakhitdinov, T.D. Djuraev, V.N. Vragov, G.D. Karatoprakliev J.M. Rassias, K.B. Sabitov, A.I. Kozhanov, A.P. Soldatov and the scientific works of their scientific school were devoted to the study of this type of equations [1, 13-15].

K.S. Fayazov, I.O. Khajiyev, Ya.K. Khudayberganov researches related to checking the conditional correctness and building a regularized approximate solution of ill-posed problems for the second order mixed type differential equations and the system of equation [2-6, 9, 10].

In this work, the conditional correctness of the ill-posed initial boundary value problem for the system of second order mixed type equations is studied. An a priori estimate of the solution is obtained by the method of logarithmic convexity. The uniqueness and conditional stability theorems of the solution are proved in the set of correctness. An approximate solution is constructed by the regularization method.

We make the following substitution for the problem (1) - (4)

$$u=\frac{a_1-\lambda_2}{b_2\cdot(\lambda_1-\lambda_2)}\cdot\omega-\frac{a_1-\lambda_1}{b_2\cdot(\lambda_1-\lambda_2)}\cdot\vartheta,\ v=\frac{1}{(\lambda_1-\lambda_2)}(\omega-\vartheta) \qquad (5)$$

where $\lambda_1,\lambda_2$ - are the real roots of the quadratic equation

$$\lambda^2-(a_1+a_2)\lambda+a_1a_2-b_1b_2=0.$$

As a result, we come to the following problems depending to the functions $\omega(x,t)$, $\vartheta(x,t)$.

**Problem 1.** Find a function $\omega(x,t)$ in the domain $\Omega=\{-l<x<l,x\neq 0,0<t<T\}$, that satisfies the equation

$$\omega_{tt}+\operatorname{sgn}x\cdot\omega_{xx}+\lambda_1\cdot\omega=0$$

and the next conditions

$$\omega|_{t=0}=\bar{\varphi}_1(x),\quad \omega_t|_{t=0}=\bar{\varphi}_2(x),\ -l\le x\le l,$$

$$\omega(-l,t)=\omega(l,t)=0,\ 0\le t\le T,$$

$$\omega|_{x=-0}=\omega|_{x=+0},\quad \omega_x|_{x=-0}=\omega_x|_{x=+0},\ 0\le t\le T$$

where $\bar{\varphi}_1(x)=b_2\cdot\varphi_1(x)+(\lambda_1-a_1)\cdot\psi_1(x)$, $\bar{\varphi}_2(x)=b_2\cdot\varphi_2(x)+(\lambda_1-a_1)\cdot\psi_2(x)$.

**Problem 2.** Find a function $\vartheta(x,t)$ in the domain $\Omega=\{-l<x<l,x\neq 0,0<t<T\}$ that satisfies the equation

$$\vartheta_{tt}+\operatorname{sgn}x\cdot\vartheta_{xx}+\lambda_2\cdot\vartheta=0$$

and the next conditions,

$$\vartheta|_{t=0}=\bar{\psi}_1(x),\ \vartheta_t|_{t=0}=\bar{\psi}_2(x),\ -l\le x\le l$$

$$\vartheta(-l,t)=\vartheta(l,t)=0,\ 0\le t\le T,$$

$$\vartheta|_{x=-0}=\vartheta|_{x=+0},\quad \vartheta_x|_{x=-0}=\vartheta_x|_{x=+0},\ 0\le t\le T$$

where $\bar{\psi}_1(x)=b_2\cdot\varphi_1(x)+(\lambda_2-a_1)\cdot\psi_1(x)$, $\bar{\psi}_2(x)=b_2\cdot\varphi_2(x)+(\lambda_2-a_1)\cdot\psi_2(x)$.

**Lemma 1.** Let the function $\omega(x,t)$ in the domain $\Omega=\{-l<x<l,0<t<T\}$ satisfies the equation

$$\omega_{tt}+\operatorname{sgn}x\cdot\omega_{xx}+a\cdot\omega=0 \qquad (6)$$

and next conditions

$$\omega(-l,t)=\omega(+l,t)=0,$$

$$\omega(-0,t)=\omega(+0,t),\ \omega_x(-0,t)=\omega_x(+0,t).$$

Then the inequality

$$\int_{-l}^{l}\omega^2dx \le 4l^2\left(\int_{-l}^{l}\omega_x^2\Big|_{t=0}dx+|\alpha|\right)^{1-\frac{t}{T}}\left(\int_{-l}^{l}\omega_x^2\Big|_{t=T}dx+|\alpha|\right)^{\frac{t}{T}}e^{2t(T-t)}$$

is valid for $\forall(x,t)\in\Omega$, $a\in R$, where $\alpha=\frac{1}{2}\cdot\left(\int_{-l}^{l}\left(\operatorname{sgn}x\cdot\omega_{xx}^2+a\cdot\omega_x^2-\omega_{xt}^2\right)\Big|_{t=0}dx\right)$.

**Proof.** For the solution of the problem, we consider the function $f(t)$ as following:

$$f(t)=\int_{-l}^{l}\omega_x^2dx.$$

Has a continuous the first and second derivatives, then

$$f'(t)=2\int_{-l}^{l}\omega_x\cdot\omega_{xt}dx,$$

$$f''(t)=2\int_{-l}^{l}\omega_{xt}\cdot\omega_{xt}dx+2\int_{-l}^{l}\omega_x\cdot\omega_{xtt}dx=2\int_{-l}^{l}\omega_{xt}^2dx-2\int_{-l}^{l}\omega_{xx}\cdot\omega_{tt}dx.$$

We change the second term of the expression $f''(t)$ using equation (6)

$$f''(t)=2\int_{-l}^{l}\omega_{xt}^2dx+2\int_{-l}^{l}\omega_{xx}\left(\operatorname{sgn}x\cdot\omega_{xx}+a\cdot\omega\right)dx.$$

Now we consider the following differential

$$\frac{d}{dt}\left(\int_{-l}^{l}\left(\operatorname{sgn}x\,\omega_{xx}^2+a\,\omega\,\omega_{xx}\right)dx\right)=\int_{-l}^{l}\left(2\operatorname{sgn}x\,\omega_{xx}\omega_{xxt}+a\,\omega_t\,\omega_{xx}+a\,\omega\,\omega_{xxt}\right)dx=$$

$$\int_{-l}^{l}\left(2\operatorname{sgn}x\,\omega_{xx}\omega_{xxt}+2a\,\omega\,\omega_{xxt}\right)dx=$$

$$2\int_{-l}^{l}\omega_{xxt}\left(\operatorname{sgn}x\,\omega_{xx}+a\,\omega\right)dx=-2\int_{-l}^{l}\omega_{xxt}\omega_{tt}dx=2\int_{-l}^{l}\omega_{xt}\omega_{xtt}dx.$$

Here integration by parts and boundary conditions $\omega(-l,t)=\omega(+l,t)=0$ are used. We have following equation from the latest equation

$$\frac{d}{dt}\left(\int_{-l}^{l}\left(\operatorname{sgn}x\cdot\omega_{xx}^2+a\cdot\omega\cdot\omega_{xx}\right)dx\right)=\frac{d}{dt}\left(\int_{-l}^{l}\omega_{xt}^2dx\right).$$

We integrate this over the interval $(0,t)$ and

$$\int_{-l}^{l}\left(\operatorname{sgn}x\cdot\omega_{xx}^2+a\cdot\omega\cdot\omega_{xx}\right)dx=\int_{-l}^{l}\omega_{xt}^2dx+2\alpha,$$

where

$$\alpha=\frac{1}{2}\cdot\left(\int_{-l}^{l}\left(\operatorname{sgn}x\cdot\omega_{xx}^{2}+a\omega\omega_{xx}-\omega_{xt}^{2}\right)\Big|_{t=0}dx\right) \text{ or } \alpha=\frac{1}{2}\cdot\left(\int_{-l}^{l}\left(\operatorname{sgn}x\cdot\omega_{xx}^{2}-a\omega_{x}^{2}-\omega_{xt}^{2}\right)\Big|_{t=0}dx\right).$$

As a result,

$$f''(t)=2\cdot\int_{-l}^{l}\omega_{xt}^{2}dx+2\left(\int_{-l}^{l}\omega_{xt}^{2}dx+2\alpha\right)=\int_{-l}^{l}\omega_{xt}^{2}dx+4\alpha .$$

We enter the new function $g(t)=\ln\left(f(t)+|\alpha|\right)$. In that case

$$g'(t)=\frac{f'(t)}{f(t)+|\alpha|},\ g''(t)=\frac{f''(t)\cdot\left(f(t)+|\alpha|\right)-\left(f'(t)\right)^{2}}{\left(f(t)+|\alpha|\right)^{2}}.$$

We bring the above expressions $f(t)$, $f'(t)$ and $f''(t)$ to the expression $g''(t)$ and get

$$g''(t)=\frac{\left(4\int_{-l}^{l}\omega_{xt}^{2}dx+4\alpha\right)\cdot\left(\int_{-l}^{l}\omega_{x}^{2}dx+|\alpha|\right)-\left(2\int_{-l}^{l}\omega_{x}\cdot\omega_{xt}dx\right)^{2}}{\left(f(t)+|\alpha|\right)^{2}}=$$

$$=\frac{4\cdot\int_{-l}^{l}\omega_{xt}^{2}dx\cdot\int_{-l}^{l}\omega_{x}^{2}dx+4|\alpha|\int_{-l}^{l}\omega_{xt}^{2}dx+4\alpha\cdot\left(f(t)+|\alpha|\right)-4\cdot\left(\int_{-l}^{l}\omega_{x}\cdot\omega_{xt}dx\right)^{2}}{\left(f(t)+|\alpha|\right)^{2}}\geq$$

$$\geq\frac{4\alpha\left(f(t)+|\alpha|\right)}{\left(f(t)+|\alpha|\right)^{2}}=\frac{4\alpha}{\left(f(t)+|\alpha|\right)}\geq\frac{-4|\alpha|}{\left(f(t)+|\alpha|\right)}\geq\frac{-4|\alpha|-4f(t)}{\left(f(t)+|\alpha|\right)}=-4.$$

In deriving the last inequality, we used the Cauchy–Schwarz inequality.
From the differential inequality $g''(t)\geq-4$ follows

$$g(t)\leq g(0)\frac{T-t}{T}+g(T)\frac{t}{T}+2t(T-t)$$

or

$$f(t)+|\alpha|\leq\left(f(0)+|\alpha|\right)^{\frac{T-t}{T}}\left(f(T)+|\alpha|\right)^{\frac{t}{T}}e^{2t(T-t)}.$$

From this, substituting the expression of the function $f(t)$, we get the inequality

$$\int_{-l}^{l}\omega_{x}^{2}dx\leq\left(\int_{-l}^{l}\omega_{x}^{2}\Big|_{t=0}dx+|\alpha|\right)^{1-\frac{t}{T}}\cdot\left(\int_{-l}^{l}\omega_{x}^{2}\Big|_{t=T}dx+|\alpha|\right)^{\frac{t}{T}}\cdot e^{2t(T-t)}.$$

Now, considering the inequality $\int_{-l}^{l}\omega^{2}dx\leq4l^{2}\int_{-l}^{l}\omega_{x}^{2}dx$, we obtain the required inequality

$$\int_{-l}^{l}\omega^{2}dx\leq4l^{2}\left(\int_{-l}^{l}\omega_{x}^{2}\Big|_{t=0}dx+|\alpha|\right)^{1-\frac{t}{T}}\left(\int_{-l}^{l}\omega_{x}^{2}\Big|_{t=T}dx+|\alpha|\right)^{\frac{t}{T}}e^{2t(T-t)}.$$

After that, from equality (5) we get the following:

$$\omega(x,t)=b_{2}u(x,t)+(\lambda_{1}-a_{1})v(x,t),\ \vartheta(x,t)=b_{2}u(x,t)+(\lambda_{2}-a_{1})v(x,t). \quad (7)$$

In order to obtain a priori estimate of the solution of the problem (1)-(4), we apply lemma 1 to problem 1 and problem 2 and obtain the inequalities

$$\int_{-l}^{l}\omega^2 dx \le 4l^2\left(\int_{-l}^{l}\omega_x^2\Big|_{t=0}dx+\alpha_1\right)^{1-\frac{t}{T}}\left(\int_{-l}^{l}\omega_x^2\Big|_{t=T}dx+\alpha_1\right)^{\frac{t}{T}}e^{2t(T-t)},$$

$$\int_{-l}^{l}\vartheta^2 dx \le 4l^2\left(\int_{-l}^{l}\vartheta_x^2\Big|_{t=0}dx+\alpha_2\right)^{1-\frac{t}{T}}\left(\int_{-l}^{l}\vartheta_x^2\Big|_{t=T}dx+\alpha_2\right)^{\frac{t}{T}}e^{2t(T-t)},$$

$$\alpha_1=\frac{1}{2}\cdot\left(\int_{-l}^{l}\left(\operatorname{sgn}x\cdot\omega_{xx}^2-\lambda_1\omega_x^2-\omega_{xt}^2\right)\Big|_{t=0}dx\right),\ \alpha_2=\frac{1}{2}\cdot\left(\int_{-l}^{l}\left(\operatorname{sgn}x\cdot\vartheta_{xx}^2-\lambda_2\vartheta_x^2-\vartheta_{xt}^2\right)\Big|_{t=0}dx\right).$$

These inequalities, based on equalities (7) and (5) notations we generate estimates

$$\begin{aligned}\|u(x,t)\|^2 \le\ & C_1^2\left(\|\bar\varphi_1'(x)\|^2+\bar\alpha_1\right)^{1-\frac{t}{T}}\left(\|b_2u_x(x,T)+(\lambda_1-a_1)v_x(x,T)\|^2+\bar\alpha_1\right)^{\frac{t}{T}}e^{2t(T-t)}+\\ & C_2^2\left(\|\bar\psi_1'(x)\|^2+\bar\alpha_2\right)^{1-\frac{t}{T}}\left(\|b_2u_x(x,T)+(\lambda_2-a_1)v_x(x,T)\|^2+\bar\alpha_2\right)^{\frac{t}{T}}e^{2t(T-t)},\end{aligned} \tag{8}$$

$$\begin{aligned}\|v(x,t)\|^2 \le\ & C_3^2\left(\|\bar\varphi_1'(x)\|^2+\bar\alpha_1\right)^{1-\frac{t}{T}}\left(\|b_2u_x(x,T)+(\lambda_1-a_1)v_x(x,T)\|^2+\bar\alpha_1\right)^{\frac{t}{T}}e^{2t(T-t)}+\\ & C_3^2\left(\|\bar\psi_1'(x)\|^2+\bar\alpha_2\right)^{1-\frac{t}{T}}\left(\|b_2u_x(x,T)+(\lambda_2-a_1)v_x(x,T)\|^2+\bar\alpha_2\right)^{\frac{t}{T}}e^{2t(T-t)},\end{aligned} \tag{9}$$

where

$$C_1=\frac{2\sqrt{2}l(a_1-\lambda_2)}{b_2\cdot(\lambda_1-\lambda_2)},\ C_2=\frac{2\sqrt{2}l(a_1-\lambda_2)}{b_2\cdot(\lambda_1-\lambda_2)},\ C_3=\frac{2\sqrt{2}l}{(\lambda_1-\lambda_2)},$$

$$\bar\alpha_1=\frac{1}{2}\|\bar\varphi_1''(x)\|^2+\frac{|\lambda_1|}{2}\|\bar\varphi_1'(x)\|^2+\frac{1}{2}\|\bar\varphi_2'(x)\|^2,\ \ \bar\alpha_2=\frac{1}{2}\|\bar\psi_1''(x)\|^2+\frac{|\lambda_2|}{2}\|\bar\psi_1'(x)\|^2+\frac{1}{2}\|\bar\psi_2'(x)\|^2.$$

For the problem (1)-(4), we introduce the set of correctness in the form

$$M=\left\{(u(x,t),v(x,t)):\|u_x(x,T)\|+\|v_x(x,T)\|<m,\ m<\infty\right\}.$$

**Theorem 1.** *Suppose the solution of the problem (1) - (4) exist and* $(u(x,t),v(x,t))\in M$ *. In this case the solution of the problem (1)-(4) is unique.*

**Proof.** We assume that the solution of the problem (1)-(4) is not unique, let $(u_1(x,t),v_1(x,t))$ and $(u_2(x,t),v_2(x,t))$ be solutions of the problem (1)-(4). We denote $u(x,t)=u_1(x,t)-u_2(x,t)$, $v(x,t)=v_1(x,t)-v_2(x,t)$.

Then the pair of functions $(u(x,t),v(x,t))$ satisfies the system of equations (1), the initial condition

$$\left.\begin{aligned}u\big|_{t=0}=0,\quad u_t\big|_{t=0}=0,\\ v\big|_{t=0}=0,\quad v_t\big|_{t=0}=0,\end{aligned}\right\}\quad -l\le x\le l, \tag{10}$$

and the (3), (4) conditions. From here, based on condition (10), it follows that $\bar\varphi_1(x)=0$, $\bar\varphi_2(x)=0$, $\bar\psi_1(x)=0$, $\bar\psi_2(x)=0$. Therefore, $\bar\alpha_1=0$, $\bar\alpha_2=0$. As a result, from inequalities (8) and (9), we find that $\|u(x,t)\| \le 0$, $\|v(x,t)\| \le 0$. It appears from these inequalities that only $u \equiv 0, v \equiv 0$ or $u_1 \equiv u_2$, $v_1 \equiv v_2$. So, the solution of the problem (1)-(4) is unique.

Now we show the conditional correctness of the problem (1)-(4). Let the pair of functions $(u(x,t), v(x,t))$ be the solution of the problem (1)-(4) corresponding to the exact data $\varphi_i(x), \psi_i(x)$, and the pair of functions $(u_\varepsilon(x,t), v_\varepsilon(x,t))$ be the solution of the problem (1) - (4) corresponding to the approximate data $\varphi_{i\varepsilon}(x)$, $\psi_{i\varepsilon}(x)$, $i = 1,2$.

**Theorem 2.** Let the solution of the problem (1) - (4) exist, $(u,v) \in M$, $(u_\varepsilon, v_\varepsilon) \in M$, $\|\varphi_i(x) - \varphi_{i\varepsilon}(x)\|_{W_2^{3-i}[-l,l]} \le \varepsilon$, $\|\psi_i(x) - \psi_{i\varepsilon}(x)\|_{W_2^{3-i}[-l,l]} \le \varepsilon$, $i = 1,2$. Then for the solution of the problem (1)-(4) inequalities

$$\|u(x,t) - u_\varepsilon(x,t)\|^2 \le C_1^2 \delta_\varepsilon(m,\lambda_1,t) + C_2^2 \delta_\varepsilon(m,\lambda_2,t),$$

$$\|v(x,t) - v_\varepsilon(x,t)\|^2 \le C_3^2 \delta_\varepsilon(m,\lambda_1,t) + C_3^2 \delta_\varepsilon(m,\lambda_2,t)$$

appropirate, where

$$\delta_\varepsilon(m,\lambda,t) = \left(|b_2| + |\lambda - a_1|\right)^2 \left(\left(2 + 0.5|\lambda|\right)\varepsilon^2\right)^{1-\frac{t}{T}} \left(\left(m^2 + 0.5|\lambda|\varepsilon^2 + \varepsilon^2\right)\right)^{\frac{t}{T}} e^{2t(T-t)}.$$

**Proof.** We introduce denotations $\bar{u} = u - u_\varepsilon$, $\bar{v} = v - v_\varepsilon$. Then the pair of functions $(\bar{u}, \bar{v})$ satisfies the system of equations (1), the initial condition

$$\left.\begin{aligned} \bar{u}\big|_{t=0} &= \varphi_1(x) - \varphi_{1\varepsilon}(x), \left.\frac{\partial \bar{u}}{\partial t}\right|_{t=0} = \varphi_2(x) - \varphi_{2\varepsilon}(x), \\ \bar{v}\big|_{t=0} &= \psi_1(x) - \psi_{1\varepsilon}(x), \left.\frac{\partial \bar{v}}{\partial t}\right|_{t=0} = \psi_2(x) - \psi_{2\varepsilon}(x), \end{aligned}\right\} -l \le x \le l$$

and conditions (3), (4). So, for the pair of functions $(\bar{u}, \bar{v})$, estimates (8) and (9) are appropriate. Here

$$\bar{\varphi}_i(x) = b_2 \cdot \left(\varphi_i(x) - \varphi_{i\varepsilon}(x)\right) + (\lambda_1 - a_1) \cdot \left(\psi_i(x) - \psi_{i\varepsilon}(x)\right),$$

$$\bar{\psi}_i(x) = b_2 \cdot \left(\varphi_i(x) - \varphi_{i\varepsilon}(x)\right) + (\lambda_2 - a_1) \cdot \left(\psi_i(x) - \psi_{i\varepsilon}(x)\right).$$

Based on these and the fact that $(u,v), (u_\varepsilon, v_\varepsilon) \in M$, we estimate the following:

$$\|\bar{\varphi}_1'(x)\| \le |b_2| \cdot \|\varphi_1'(x) - \varphi_{1\varepsilon}'(x)\| + |\lambda_1 - a_1| \cdot \|\psi_1'(x) - \psi_{1\varepsilon}'(x)\| \le \left(|b_2| + |\lambda_1 - a_1|\right)\varepsilon,$$

$$\|\bar{\psi}_1'(x)\| \le |b_2| \cdot \|\varphi_1'(x) - \varphi_{1\varepsilon}'(x)\| + |\lambda_2 - a_1| \cdot \|\psi_1'(x) - \psi_{1\varepsilon}'(x)\| \le \left(|b_2| + |\lambda_2 - a_1|\right)\varepsilon,$$

$$\|b_2 u_x(x,T) + (\lambda_1 - a_1) v_x(x,T)\| \le$$
$$|b_2| \|u_x(x,T)\| + |\lambda_1 - a_1| \|v_x(x,T)\| \le \left(|b_2| + |\lambda_1 - a_1|\right) m,$$

$$\|b_2 u_x(x,T) + (\lambda_2 - a_1) v_x(x,T)\| \le$$
$$|b_2| \|u_x(x,T)\| + |\lambda_2 - a_1| \|v_x(x,T)\| \le \left(|b_2| + |\lambda_2 - a_1|\right) m.$$

Similarly,

$\|\bar{\varphi}_2'(x)\| \le \left(|b_2| + |\lambda_1 - a_1|\right)\varepsilon$, $\|\bar{\psi}_2'(x)\| \le \left(|b_2| + |\lambda_2 - a_1|\right)\varepsilon$

estimates are valid. And from these we get

$$\bar{\alpha}_1 = \frac{1}{2}\|\bar{\varphi}_1''(x)\|^2 + \frac{|\lambda_1|}{2}\|\bar{\varphi}_1'(x)\|^2 + \frac{1}{2}\|\bar{\varphi}_2'(x)\|^2 \le \left(1 + 0.5|\lambda_1|\right)\left(|b_2| + |\lambda_1 - a_1|\right)^2 \varepsilon^2,$$

$$\bar{\alpha}_2 = \frac{1}{2}\|\bar{\psi}_1''(x)\|^2 + \frac{|\lambda_2|}{2}\|\bar{\psi}_1'(x)\|^2 + \frac{1}{2}\|\bar{\psi}_2'(x)\|^2 \le \left(1+0.5|\lambda_2|\right)\left(|b_2| + |\lambda_2 - a_1|\right)^2 \varepsilon^2 .$$

As a result, from (8) and (9) we have inequalities

$$\|\bar{u}(x,t)\|^2 \le C_1^2 \delta_\varepsilon(m,\lambda_1,t) + C_2^2 \delta_\varepsilon(m,\lambda_2,t),$$

$$\|\bar{v}(x,t)\|^2 \le C_3^2 \delta_\varepsilon(m,\lambda_1,t) + C_3^2 \delta_\varepsilon(m,\lambda_2,t),$$

where $\delta_\varepsilon(m,\lambda,t) = \left(|b_2| + |\lambda - a_1|\right)^2 \left(\left(2+0.5|\lambda|\right)\varepsilon^2\right)^{1-\frac{t}{T}} \left(\left(m^2 + 0.5|\lambda|\varepsilon^2 + \varepsilon^2\right)\right)^{\frac{t}{T}} e^{2t(T-t)}$. Taking into account the denotations $\bar{u} = u - u_\varepsilon$, $\bar{v} = v - v_\varepsilon$, the required inequalities derived.

Let $\{X_k^+(x)\}, \{X_k^-(x)\}$ be eigenfunctions of the spectral problem corresponding to the problem (1)-(4), $\mu_k^+$, $\mu_k^-$ be the eigenvalues ($\mu_k^+ > 0$, $\mu_k^- < 0$, $\forall k \in N$).

The numbers $\mu_k^+$, $-\mu_k^-$ form non-decreasing sequences and are solutions of the transcendental equation $tg\sqrt{|\mu_k^\pm|}l + th\sqrt{|\mu_k^\pm|}l = 0$.

Let $(\varphi,\psi) = \int_{-l}^{l} \varphi \cdot \psi dx$ be the scalar product in $L_2[-l;l]$, then [12], we have

$$\|u(x,t)\|_0^2 = \sum_{n=1}^{\infty}\left(\operatorname{sgn} x u(x,t), X_k^+\right)^2 + \sum_{n=1}^{\infty}\left(\operatorname{sgn} x u(x,t), X_k^-\right)^2 . \qquad (11)$$

where

$$X_k^+(x) = \begin{cases} \dfrac{\sin\sqrt{\mu_k^+}(x-l)}{\sqrt{l}\cos\sqrt{\mu_k^+}l}, 0 < x \le l, \\ \dfrac{sh\sqrt{\mu_k^+}(x+l)}{\sqrt{l}ch\sqrt{\mu_k^+}l}, -l \le x < 0, \end{cases} . X_k^-(x) = \begin{cases} \dfrac{sh\sqrt{|\mu_k^-|}(x-l)}{\sqrt{l}ch\sqrt{-\mu_k^-}\pi}, 0 < x \le l, \\ \dfrac{\sin\sqrt{|\mu_k^-|}(x+l)}{\sqrt{l}\cos\sqrt{|\mu_k^-|}l}, -l \le x < 0, \end{cases}$$

From the results of [12], the eigenfunctions $\{X_k^+(x)\}, \{X_k^-(x)\}$ form a Riesz basis in $H_0$ and the norm in the space $L_2[-l;l]$ defined by equality (11) is equivalent to the original one.

Let $\varphi_1(x) = \frac{a_1 - \lambda_2}{b_2}\psi_1(x)$, $\varphi_2(x) = \frac{a_1 - \lambda_1}{b_2}\psi_2(x)$ in the problem (1)-(4), $\psi_1(x), \psi_2(x)$ be given functions. Then $\bar{\varphi}_1(x) = d\cdot\psi_1(x)$, $\bar{\varphi}_2(x) = 0$, $\psi_1(x) = 0$, $\bar{\psi}_2(x) = -d\cdot\psi_2(x)$, where $d = \lambda_1 - \lambda_2$.

Let the solution of the problem (1)-(4) exist, then it can be represented as

$$u = A_1\cdot\omega - A_2\cdot\vartheta,\ v = A_3\left(\omega - \vartheta\right) \qquad (12)$$

where

$$\omega(x,t) = \sum_{k=1}^{\infty}\omega_k^+(t)\cdot X_k^+(x) + \sum_{k=1}^{\infty}\omega_k^-(t)\cdot X_k^-(x),\ \vartheta(x,t) = \sum_{k=1}^{\infty}\vartheta_k^+(t)\cdot X_k^+(x) + \sum_{k=1}^{\infty}\vartheta_k^-(t)\cdot X_k^-(x),$$

$$\omega_k^{\pm}(t)=\begin{cases}\bar{\varphi}_{1k}^{\pm}\cos\left(\sqrt{\left|\mu_k^{\pm}+\lambda_1\right|}t\right), & \mu_k^{\pm}+\lambda_1<0,\\ \bar{\varphi}_{1k}^{\pm}, & \mu_k^{\pm}+\lambda_1=0,\\ \bar{\varphi}_{1k}^{\pm}\cosh\left(\sqrt{\left|\mu_k^{\pm}+\lambda_1\right|}t\right), & \mu_k^{\pm}+\lambda_1>0,\end{cases}$$

$$\vartheta_k^{\pm}(t)=\begin{cases}\bar{\psi}_{2k}^{\pm}\sin\left(\sqrt{\left|\mu_k^{\pm}+\lambda_2\right|}t\right)/\sqrt{\left|\mu_k^{\pm}+\lambda_2\right|}, & \mu_k^{\pm}+\lambda_2<0,\\ \bar{\psi}_{2k}^{\pm}t, & \mu_k^{\pm}+\lambda_2=0,\\ \bar{\psi}_{2k}^{\pm}\sinh\left(\sqrt{\left|\mu_k^{\pm}+\lambda_2\right|}t\right)/\sqrt{\left|\mu_k^{\pm}+\lambda_2\right|}, & \mu_k^{\pm}+\lambda_2>0,\end{cases}$$

$$\bar{\varphi}_{1k}^{\pm}=\pm\int_{-1}^{1}\operatorname{sgn}(x)\bar{\varphi}_1(x)X_k^{\pm}(x)dx,\ \bar{\psi}_{2k}^{\pm}=\pm\int_{-1}^{1}\operatorname{sgn}(x)\bar{\psi}_2(x)X_k^{\pm}(x)dx,\quad A_1=\frac{a_1-\lambda_2}{b_2\cdot d},\quad A_2=\frac{a_1-\lambda_1}{b_2\cdot d},$$

$$A_3=\frac{1}{d}.$$

We define the approximate solution $(u_N, v_N)$ according to the exact data as follows

$$u_N=A_1\cdot\omega_N-A_2\cdot\vartheta_N,\ v_N=A_3\left(\omega_N-\vartheta_N\right) \qquad (13)$$

where

$$\omega_N(x,t)=\sum_{k=1}^{N}\omega_k^{+}(t)\cdot X_k^{+}(x)+\sum_{k=1}^{\infty}\omega_k^{-}(t)\cdot X_k^{-}(x),\ \vartheta_N(x,t)=\sum_{k=1}^{N}\vartheta_k^{+}(t)\cdot X_k^{+}(x)+\sum_{k=1}^{\infty}\vartheta_k^{-}(t)\cdot X_k^{-}(x),$$

where $N$ is the integer regularization parameter. The approximate solution $(u_{N\varepsilon}, v_{N\varepsilon})$ according to the approximate data, we define

$$u_{N\varepsilon}=A_1\cdot\omega_{N\varepsilon}-A_2\cdot\vartheta_{N\varepsilon},\ v_{N\varepsilon}=A_3\left(\omega_{N\varepsilon}-\vartheta_{N\varepsilon}\right) \qquad (14)$$

where

$$\omega_{N\varepsilon}=\sum_{k=1}^{N}\omega_{\varepsilon k}^{+}(t)\cdot X_k^{+}(x)+\sum_{k=1}^{\infty}\omega_{\varepsilon k}^{-}(t)\cdot X_k^{-}(x),\ \vartheta_{N\varepsilon}=\sum_{k=1}^{N}\vartheta_{\varepsilon k}^{+}(t)\cdot X_k^{+}(x)+\sum_{k=1}^{\infty}\vartheta_{\varepsilon k}^{-}(t)\cdot X_k^{-}(x),$$

$$\omega_{\varepsilon k}^{\pm}(t)=\begin{cases}\bar{\varphi}_{1\varepsilon k}^{\pm}\cos\left(\sqrt{\left|\mu_k^{\pm}+\lambda_1\right|}t\right), & \mu_k^{\pm}+\lambda_1<0,\\ \bar{\varphi}_{1\varepsilon k}^{\pm}, & \mu_k^{\pm}+\lambda_1=0,\\ \bar{\varphi}_{1\varepsilon k}^{\pm}\cosh\left(\sqrt{\left|\mu_k^{\pm}+\lambda_1\right|}t\right), & \mu_k^{\pm}+\lambda_1>0,\end{cases}$$

$$\vartheta_{\varepsilon k}^{\pm}(t)=\begin{cases}\bar{\psi}_{2\varepsilon k}^{\pm}\sin\left(\sqrt{\left|\mu_k^{\pm}+\lambda_2\right|}t\right)/\sqrt{\left|\mu_k^{\pm}+\lambda_2\right|}, & \mu_k^{\pm}+\lambda_2<0,\\ \bar{\psi}_{2\varepsilon k}^{\pm}t, & \mu_k^{\pm}+\lambda_2=0,\\ \bar{\psi}_{2\varepsilon k}^{\pm}\sinh\left(\sqrt{\left|\mu_k^{\pm}+\lambda_2\right|}t\right)/\sqrt{\left|\mu_k^{\pm}+\lambda_2\right|}, & \mu_k^{\pm}+\lambda_2>0,\end{cases}$$

$\overline{\varphi}_{1\varepsilon k}^{\pm} = \pm\int_{-1}^{1} \operatorname{sgn}(x)\overline{\varphi}_{1\varepsilon}(x)X_k^{\pm}(x)dx$, $\overline{\psi}_{2\varepsilon k}^{\pm} = \pm\int_{-1}^{1} \operatorname{sgn}(x)\overline{\psi}_{2\varepsilon}(x)X_k^{\pm}(x)dx$, $\overline{\varphi}_{1\varepsilon}(x) = d\cdot\psi_{1\varepsilon}(x)$, $\overline{\psi}_{2\varepsilon}(x) = -d\cdot\psi_{2\varepsilon}(x)$, since $\psi_{1\varepsilon}(x)$, $\psi_{2\varepsilon}(x)$ are approximate data.

Let $\|\psi_1(x)-\psi_{1\varepsilon}(x)\|\le\varepsilon,\ \|\psi_2(x)-\psi_{2\varepsilon}(x)\|\le\varepsilon$ and $(u(x,t),v(x,t))\in M$. Then for the norm of the difference between the exact and approximate solutions we have

$$\|u-u_{N\varepsilon}\|\le\|u-u_N\|+\|u_N-u_{N\varepsilon}\|, \qquad (15)$$

$$\|v-v_{N\varepsilon}\|\le\|v-v_N\|+\|v_N-v_{N\varepsilon}\|. \qquad (16)$$

Then

$$\|u_N-u_{N\varepsilon}\|\le|A_1|\|\omega_N-\omega_{N\varepsilon}\|+|A_2|\|\vartheta_N-\vartheta_{N\varepsilon}\|. \qquad (17)$$

Based on equalities (13) and (14), the estimate of the first expression on the right side of (17) has the form

$$\|\omega_N-\omega_{N\varepsilon}\|^2=\sum_{k=1}^{N}\left(\omega_k^+(t)-\omega_{\varepsilon k}^+(t)\right)^2+\sum_{k=1}^{\infty}\left(\omega_k^-(t)-\omega_{\varepsilon k}^-(t)\right)^2\le d^2\cosh^2\left(\sqrt{\lambda_1+\mu_N^+}\,t\right)\varepsilon^2.$$

Now we evaluate the expression $\left\|\vartheta^N-\vartheta_\varepsilon^N\right\|_0$ in a similar way and have

$$\|\vartheta_N-\vartheta_{N\varepsilon}\|^2\le d^2C_4^2\sinh^2\left(\sqrt{\lambda_2+\mu_N^+}\,t\right)\varepsilon^2,$$

where $C_4^2=\max_k\left\{\left|\lambda_2+\mu_k^{\pm}\right|^{-1}\right\}$. After some simplifications, from (17) we obtain

$$\|u_N-u_{N\varepsilon}\|\le d\left(|A_1|\cosh\left(\sqrt{\lambda_1+\mu_N^+}\,t\right)+C_4|A_2|\sinh\left(\sqrt{\lambda_2+\mu_N^+}\,t\right)\right)\varepsilon. \qquad (18)$$

Let us pass to the estimate of the first term on the right side of inequality (15)

$$\|u-u_N\|\le|A_1|\|\omega-\omega_N\|+|A_2|\|\vartheta-\vartheta_N\|$$

under the condition $(u,v)\in M$. Then we come to the evaluation of the expression

$$\|\omega-\omega_N\|^2=\sum_{k=N+1}^{\infty}\left(\omega_k^+(t)\right)^2=\sum_{k=N+1}^{\infty}\left\{\overline{\varphi}_{1k}^+\right\}^2\cosh^2\left(\sqrt{\left|\mu_k^++\lambda_1\right|}t\right), \qquad (19)$$

under the condition $\|\omega(x,T)\|\le m_1$, where $m_1=\left(|b_2|+|\lambda_1-a_1|\right)m$. It's easy to notice that

$$\sum_{k=1}^{\infty}\left\{\overline{\varphi}_{1k}^+\right\}^2\cosh^2\left(\sqrt{\left|\mu_k^++\lambda_1\right|}T\right)\le m_1^2 \qquad (20)$$

From here it can be seen that (19) reaches its maximum value under condition (20) in the case when the coefficients

$$\overline{\varphi}_{1k}^+=\begin{cases} m_1\cdot\cosh^{-1}\left(\sqrt{\left|\mu_k^++\lambda_1\right|}T\right), & k=N+1,\\ 0, & k\ne N+1.\end{cases}$$

So, we have

$$\|\omega-\omega_N\|\le m_1\cdot\frac{\cosh\left(\sqrt{\left|\mu_{N+1}^++\lambda_1\right|}t\right)}{\cosh\left(\sqrt{\left|\mu_{N+1}^++\lambda_1\right|}T\right)} \qquad (21)$$

Under the condition $\|\vartheta(x,T)\| \le m_2$, evaluating the expressions $\|\vartheta - \vartheta_N\|$ we get

$$\|\vartheta - \vartheta_N\| \le m_2 \cdot \frac{\sinh\left(\sqrt{\left|\mu_{N+1}^+ + \lambda_2\right|}t\right)}{\sinh\left(\sqrt{\left|\mu_{N+1}^+ + \lambda_2\right|}T\right)}, \qquad (22)$$

where $m_2 = \left(|b_2| + |\lambda_2 - a_1|\right)m$.

Combining estimates (21) and (22) we have

$$\|u - u_N\| \le |A_1| m_1 \cdot \frac{\cosh\left(\sqrt{\left|\mu_{N+1}^+ + \lambda_1\right|}t\right)}{\cosh\left(\sqrt{\left|\mu_{N+1}^+ + \lambda_1\right|}T\right)} + |A_2| m_2 \cdot \frac{\sinh\left(\sqrt{\left|\mu_{N+1}^+ + \lambda_2\right|}t\right)}{\sinh\left(\sqrt{\left|\mu_{N+1}^+ + \lambda_2\right|}T\right)}.. \qquad (23)$$

We substitute (23) and (18) into inequality (15), then we get that

$$\begin{aligned}\|u - u_{N\varepsilon}\| \le\, & |A_1| m_1 \cdot \frac{\cosh\left(\sqrt{\left|\mu_{N+1}^+ + \lambda_1\right|}t\right)}{\cosh\left(\sqrt{\left|\mu_{N+1}^+ + \lambda_1\right|}T\right)} + d\,|A_1| \cosh\left(\sqrt{\lambda_1 + \mu_N^+}\,t\right)\cdot\varepsilon + \\ & |A_2| m_2 \cdot \frac{\sinh\left(\sqrt{\left|\mu_{N+1}^+ + \lambda_2\right|}t\right)}{\sinh\left(\sqrt{\left|\mu_{N+1}^+ + \lambda_2\right|}T\right)} + d\cdot C_4 |A_2| \sinh\left(\sqrt{\lambda_2 + \mu_N^+}\,t\right)\cdot\varepsilon.\end{aligned} \qquad (24)$$

Let us estimate inequality (16). Note that for the expression $\|v_N - v_{N\varepsilon}\|$ correct estimate

$$\begin{aligned}\|v_N - v_{N\varepsilon}\| \le\, & |A_3|\left(\|\omega_N - \omega_{N\varepsilon}\| + \|\vartheta_N - \vartheta_{N\varepsilon}\|\right) \le \\ & d\,|A_3|\left(\cosh\left(\sqrt{\lambda_1 + \mu_N^+}\,t\right) + C_4 \sinh\left(\sqrt{\lambda_2 + \mu_N^+}\,t\right)\right)\varepsilon.\end{aligned}$$

For $\|v - v_N\|$ we have

$$\|v - v_N\| \le |A_3| m_1 \cdot \frac{\cosh\left(\sqrt{\left|\mu_{N+1}^+ + \lambda_1\right|}t\right)}{\cosh\left(\sqrt{\left|\mu_{N+1}^+ + \lambda_1\right|}T\right)} + |A_3| m_2 \cdot \frac{\sinh\left(\sqrt{\left|\mu_{N+1}^+ + \lambda_2\right|}t\right)}{\sinh\left(\sqrt{\left|\mu_{N+1}^+ + \lambda_2\right|}T\right)}.$$

Finally we get

$$\begin{aligned}\|v - v_{N\varepsilon}\| \le\, & |A_3| m_1 \cdot \frac{\cosh\left(\sqrt{\left|\mu_{N+1}^+ + \lambda_1\right|}t\right)}{\cosh\left(\sqrt{\left|\mu_{N+1}^+ + \lambda_1\right|}T\right)} + d\,|A_3| \cosh\left(\sqrt{\lambda_1 + \mu_N^+}\,t\right)\cdot\varepsilon + \\ & |A_3| m_2 \cdot \frac{\sinh\left(\sqrt{\left|\mu_{N+1}^+ + \lambda_2\right|}t\right)}{\sinh\left(\sqrt{\left|\mu_{N+1}^+ + \lambda_2\right|}T\right)} + d\cdot C_4 |A_3| \sinh\left(\sqrt{\lambda_2 + \mu_N^+}\,t\right)\varepsilon\end{aligned}. \qquad (25)$$

Minimizing the right side of inequalities (24) and (25) with respect to $m, \varepsilon, T$ we find the corresponding regularization parameter $N$.

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