On λ- group (special group)

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ABSTRACT

In this study , we define the concepts of inverse operation , identity element , inverse element , special identity element , λ-group (special group) , λ-ring (special ring) , λ-field (special field) and
show that ,

i) Let $(G ,*)$ is λ-group and $a*b=c$, then $e_a*b=a^{-1}*c$ for all $a,b,c\in G$ such that e_a is identity for a w.r.t $*$, a^{-1} is inverse for a w.r.t $*$.

ii) Let $(G,*)$ be a λ- group , if $a*b=c$ and $e_a=e_b$, then $b=a^{-1}*c$ for all $a,b,c\in G$ such that e_a is identity for a w.r.t $*$, e_b is identity for b w.r.t $*$ and a^{-1} is inverse for a w.r.t $*$.

iii) Let $(G,*)$ be a λ- group , if $a*b=c*b$ or $b*a=b*c$ and $e_a=e_b=e_c$, then $a=c$ for all $a,b,c\in G$ such that e_a is identity for a w.r.t $*$, e_b is identity for b w.r.t $*$ and e_c is identity for c w.r.t $*$.

iv) Let $(G,*)$ be a λ- group , e_a is identity for a w.r.t $*$ and e_b is identity for b w.r.t $*$, then e_a is left identity for $a*b$ w.r.t $*$ and e_b is right identity for $a*b$ w.r.t $*$.

Keywords: mathematical system , semi group , group , identity element , inverse element , special identity element , λ-group (special group) ,
λ-ging (special ring) and λ-field (special field)

1.Introduction .

Let G be a non empty set , then $(G,*)$ is called mathematical system if $*$ is binary operation defined on G . Let $(G,*)$ is mathematical system , then $(G,*)$ is called semi group if $*$ is associative i.e
$(a*b)*c=a*(b*c)$ for all $a,b,c\in G$.
Let $(G,*)$ is semi group , then $(G,*)$ is called group if there exist $e\in G$ such that
$a*e=e*a=a$ for all $a\in G$ and for all $a\in G$ there exist $a^{-1}\in G$ such that $a*a^{-1}=a^{-1}*a=e$.
The notion of group was introduced by Galuis in 1830 [1], [2].
Let $(G,*)$ be a commutative group and $(G,\#)$ be a semi group , then $(G,*,\#)$ is called a ring if
$a\#(b*c)=(a\#b)*(a\#c)$ and
$(a*b)\#c=(a\#c)*(b\#c)$for all $a,b,c\in G$
[3] . Let $(G,*)$,
$(G-\{e\},\#)$ be two commutative group , such that $a*e=e*a=a$
for all $a\in G$, then $(G,*,\#)$ is called a field if

$a\#(b*c)=(a\#b)*(a\#c)$ and $(a*b)\#c=(a\#c)*(b\#c)$ for all $a,b,c\in G$ [4] .

Example 1.1:

Let $A=\left\{\begin{bmatrix}a & 0\\ 0 & 0\end{bmatrix}: a\ is\ real\ number\ ,\ a\neq 0\right\}$, $\times$ is ordinary multiplication operation on matrices such that

$\begin{bmatrix}a & 0\\ 0 & 0\end{bmatrix}\times\begin{bmatrix}b & 0\\ 0 & 0\end{bmatrix}=\begin{bmatrix}ab & 0\\ 0 & 0\end{bmatrix}$ for all $\begin{bmatrix}a & 0\\ 0 & 0\end{bmatrix},\begin{bmatrix}b & 0\\ 0 & 0\end{bmatrix}\in A$, $(A,\times)$ is semi group , there exist $\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\in A$ such that $\begin{bmatrix}a & 0\\ 0 & 0\end{bmatrix}\times\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\times\begin{bmatrix}a & 0\\ 0 & 0\end{bmatrix}=\begin{bmatrix}a & 0\\ 0 & 0\end{bmatrix}$ for all $\begin{bmatrix}a & 0\\ 0 & 0\end{bmatrix}\in A$.

For all $\begin{bmatrix}a & 0\\ 0 & 0\end{bmatrix}\in A$ there exist $\begin{bmatrix}\frac{1}{a} & 0\\ 0 & 0\end{bmatrix}\in A$ such that

$\begin{bmatrix}a & 0\\ 0 & 0\end{bmatrix}\times\begin{bmatrix}\frac{1}{a} & 0\\ 0 & 0\end{bmatrix}=\begin{bmatrix}\frac{1}{a} & 0\\ 0 & 0\end{bmatrix}\times\begin{bmatrix}a & 0\\ 0 & 0\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$

We get $(A,\times)$ is group .

Theorem 1.2 :[2]

Let $(G,*)$ be a group , if $a*b=c$ then $b=a^{-1}*c$ for all $a,b,c\in G$.

3. λ- group :

In this section we present the concepts of identity element , inverse element , special identity element and λ-group .

Definition 3.1:

Let $(G,*)$ is mathematical system and $a,e_a\in G$, then e_a is called identity for a w.r.t $*$ (special identity for a) if

$a*e_a=e_a*a=a$.

e_a is called right identity for a w.r.t $*$ if $a*e_a=a$.

e_a is called left identity for a w.r.t $*$ if $e_a*a=a$.

Example 3.2 :

(p(x) , ∩) is mathematical system , let $A,B\in$ p(x) such that A is subset of B , then B is identity for A w.r.t ∩ since

$A\cap B=B\cap A=A$.

Example 3.3 :

(N ,×) is mathematical system such that N is the set natural nambers and × is ordinary multiplication operation , let $n\in N$, then n is identity for 0 w.r.t × since $n\times 0=0\times n=0$.

Remark 3.4 :

Let $(G,*)$ be a group , then there exist $e\in G$ such that e is identity for all element in G , i.e e is general identity element .

Definition 3.5 :

Let $(G,*)$ is mathematical system and $a,a^{-1}\in G$, then a^{-1} is called inverse for a w.r.t $*$ if $a*a^{-1}=a^{-1}*a=e_a$ such that e_a is identity for a w.r.t $*$.

a^{-1} is called right inverse for a w.r.t $*$ if $a*a^{-1}=e_a$ such that e_a is right identity or left identity for a w.r.t $*$.

a^{-1} is called left inverse for a w.r.t $*$ if $a^{-1}*a=e_a$ such that e_a is right identity or left identity for a w.r.t $*$.

Example 3.6 :

(p(x) , ∩) is mathematical system , let $A,C\in$ p(x) such that A is subset of C , then C is inverse for A w.r.t ∩ since

$A\cap C=C\cap A=B=A$, then B is identity for A w.r.t ∩ since

$A\cap B=B\cap A=A$.

Example 3.7 :

(N ,×) is mathematical system such that N is the set of natural nambers and × is ordinary multiplication operation , let $n\in N$, then n is inverse for 0 w.r.t × since $n\times 0=0\times n=0$, then 0 is identity for 0 since $0\times 0=0\times 0=0$.

Lemma 2.8 :

Let $(G,*)$ is mathematical system , $*^{-1}$ be an inverse operation for $*$, then $a *^{-1} a$ is left identity for a w.r.t $*$

Proof.

Let $b=a *^{-1} a$ by Definition 2.1 , we get $b*a=a$ by Definition 2.1 , b is left identity for a w.r.t $*$.

Lemma 2.9 :

Let $(G,*)$ be a semi group , if e_a is left identity for a w.r.t $*$, then $e_a * e_a = e_a$

Proof .

Let e_a is left identity for a w.r.t $*$, by Definition 2.1 , we get $e_a * a = a$

$e_a * (e_a * a) = e_a * a$, $(G,*)$ be a semi group i.e $*$ is associative

$(e_a * e_a) * a = e_a * a$ by Lemma 2.6 (i) , we get $e_a * e_a = e_a$ **Definition 3.10 :**

Let $(G,*)$ be a semi group , then $(G,*)$ is called a λ-group if

i) For all $a \in G$ there exist $e_a \in G$ such that $a * e_a = e_a * a = a$ i.e e_a is identity for a w.r.t $*$

ii) For all $a \in G$ there exist $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e_a$ such that e_a is identity for a w.r.t $*$ ($a * e_a = e_a * a = a$) i.e a^{-1} is inverse for a w.r.t $*$.

Definition 3.11 :

Let $(G,*)$ be a semi group , then $(G,*)$ is called a λ-group (Special Group) if for all $a \in G$, there exist $b, c \in G$ such that

$a * b = b * a = a$ and $a * c = c * a = b$.

Example 3.12 :

(p(Z),∩) is λ- group such that Z is the set of integer numbers , since (p(Z),∩) is semi group and for all $A \in P(Z)$, there exist A , $A \in P(Z)$ such that

$A \cap A = A \cap A = A$ and $A \cap A = A \cap A = A$

i.e A is identity for A w.r.t $\cap$ and A is inverse for A w.r.t $\cap$. **Example 3.13 :**

$(Z , +)$ is λ- group such that Z is the set of integer numbers , $+$ is ordinary addition operation , since $(Z , +)$ is semi group and for all $n \in Z$ there exist 0 , $-n \in Z$ such that $n + 0 = 0 + n = n$ and $n + (-n) = (-n) + n = 0$

i.e 0 is identity for n w.r.t $+$ and $-n$ is inverse for n w.r.t $+$ **Example 3.14 :**

Let $A = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a\ is\ real\ number\ ,\ a \neq 0 \right\}$, $\times$ is ordinary multiplication operation on matrices such that

$\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} ab & 0 \\ 0 & 0 \end{bmatrix}$ for all $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix} \in A$, $(A ,\times)$ is semi group , for all $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \in A$ there exist $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{a} & 0 \\ 0 & 0 \end{bmatrix} \in A$ such that $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$

And $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

i.e $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is identity for $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ w.r.t $\times$ and $\begin{bmatrix} \frac{1}{a} & 0 \\ 0 & 0 \end{bmatrix}$ is inverse for $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ w.r.t $\times$, we get $(A ,\times)$ is π - group , also $(A ,\times)$ is group by Example 1.1 .

It is clear that every group , is a λ- group , but the converse is not true in general , as shown by the following examples :

Example 3.15 :

Let G is the set of real numbers , we define operation $*$ on G by $a * b = a$ for all $a , b \in G$, $*$ is binary operation and $*$ is associative since $(a * b) * c = a * b = a$ and $a * (b * c) = a$ i.e $(a * b) * c = a * (b * c)$ for all $a, b, c \in G$, we get $(G,*)$ is semi group , for all $a \in G$ there exist a , $a \in G$ such that $a * a = a * a = a$ and $a * a = a * a = a$ i.e a is identity for a w.r.t $*$ and a is inverse for a w.r.t $*$, we get $(G,*)$ is λ- group .

Let $e \in G$ such that $a * e = e * a = a$ for all $a \in G$, we get $e = a$ for all $a \in G$, we obtain $(G,*)$ is not group .

Example 3.16 :

Let $A = \left\{ \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} : a, b\ are\ real\ numbers\ ,\ b \neq 0 \right\}$, $\times$ is ordinary multiplication operation on matrices such that

$\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} \times \begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix}$

$\begin{bmatrix} 0 & ad \\ 0 & bd \end{bmatrix}$ for all $\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}, \begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix} \in A$, $(A ,\times)$ is semi group , for all $\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} \in A$ there exist

$\begin{bmatrix} 0 & \frac{a}{b} \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & \frac{a}{b^2} \\ 0 & \frac{1}{b} \end{bmatrix} \in A$ such that

$\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} \times \begin{bmatrix} 0 & \frac{a}{b} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{a}{b} \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}$

And $\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} \times \begin{bmatrix} 0 & \frac{a}{b^2} \\ 0 & \frac{1}{b} \end{bmatrix} = \begin{bmatrix} 0 & \frac{a}{b^2} \\ 0 & \frac{1}{b} \end{bmatrix} \times \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} =$

$\begin{bmatrix} 0 & \frac{a}{b} \\ 0 & 1 \end{bmatrix}$

i.e $\begin{bmatrix} 0 & \frac{a}{b} \\ 0 & 1 \end{bmatrix}$ is identity for $\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}$ w.r.t $\times$ and

$\begin{bmatrix} 0 & \frac{a}{b^2} \\ 0 & \frac{1}{b} \end{bmatrix}$ is inverse

for $\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}$ w.r.t $\times$, we get $(A,\times)$ is λ-group .

Let $\begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix} \in G$ such that $\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} \times \begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix} = \begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix} \times \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}$ for all $\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} \in G$,

we get $\begin{bmatrix} 0 & ad \\ 0 & bd \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}$ and

$\begin{bmatrix} 0 & cb \\ 0 & db \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}$ for all $\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} \in G$, we get

$d = 1$ and $c = \frac{a}{b}$, if $\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$ then

$c = 2$ i.e $\begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$, if

$\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ then $c = 0$ i.e $\begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$,

$\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, we obtain $(G,*)$ is not group .

Example 3.17 :

By Example 3.12 , then $(P(X),\cap)$ is λ- group , but $(P(X),\cap)$ is not group since , Let $B \in P(X)$ such that $A \cap B = B \cap A = A$ for all $A \in P(X)$, we get A is subset of B for all $A \in P(X)$ i.e $B = X$, Let for all $A \in P(X)$ there exist $C \in P(X)$ such that $A \cap C = C \cap A = X$, we get $A = C = X$, i.e $(P(X),\cap)$ is not group .

Theroem 3.18 :

Let $(G,*)$ be a λ- group , if $a * b = c$ then $e_a * b = a^{-1} * c$ for all $a,b,c \in G$ such that e_a is identity for a w.r.t $*$ and a^{-1} is inverse for a w.r.t $*$.

Proof .

Let $a * b = c$, $a^{-1} * (a * b) = a^{-1} * c$ such that a^{-1} is inverse for a w.r.t $*$, $(a^{-1} * a) * b = a^{-1} * c$ since $*$ is associative , we get $e_a * b = a^{-1} * c$.

Example 3.19 :

In Examle 3.16 , $(A,\times)$ is λ-group , find value of y if , $0x + 4y = 8$ and $0x + 2y = 4$, use matrixes mothed ?

Solution :

$\begin{bmatrix} 0 & 4 \\ 0 & 2 \end{bmatrix} \times \begin{bmatrix} 0 & x \\ 0 & y \end{bmatrix} = \begin{bmatrix} 0 & 8 \\ 0 & 4 \end{bmatrix}$ by theorem 3.18 , we get

$\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & x \\ 0 & y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & \frac{1}{2} \end{bmatrix} \times \begin{bmatrix} 0 & 8 \\ 0 & 4 \end{bmatrix}$

such that $\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$ is identity for $\begin{bmatrix} 0 & 4 \\ 0 & 2 \end{bmatrix}$ w.r.t $\times$ and $\begin{bmatrix} 0 & 1 \\ 0 & \frac{1}{2} \end{bmatrix}$ is inverse for $\begin{bmatrix} 0 & 4 \\ 0 & 2 \end{bmatrix}$ w.r.t $\times$,

$\begin{bmatrix} 0 & 2y \\ 0 & y \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 0 & 2 \end{bmatrix}$, we get $y = 2$.

Example 3.20 :

Let $= \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a , b \ are\ real\ numbers ,\ a \neq 0 , b \neq 0 \right\}$, $\times$ is ordinary multiplication operation on matrices such that $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \times \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} ac & 0 \\ 0 & bd \end{bmatrix}$ for all $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} \in A$, $(A,\times)$ is semi group ,

there exist $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in A$ such that $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ for all $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in A$.

For all $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in A$ there exist $\begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix} \in A$ such that

$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \times \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix} \times \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

We get $(A ,\times)$ is group , also $(A ,\times)$ is λ-group .

Now , Find value of pair (x , y) if , $3x + 0y = 12$ and $0x + 5y = 10$, use matrixes mothed ?

Solution :

$\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \times \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} = \begin{bmatrix} 12 & 0 \\ 0 & 10 \end{bmatrix}$ by Theorem 1.2 , we get

$\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}^{-1} \times \begin{bmatrix} 12 & 0 \\ 0 & 10 \end{bmatrix}$

$\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{5} \end{bmatrix} \times \begin{bmatrix} 12 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$, we get $(x , y) = (4 , 2)$. Also

$\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \times \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} = \begin{bmatrix} 12 & 0 \\ 0 & 10 \end{bmatrix}$ by theorem 3.18 , we get

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 1 \\ 0 & \frac{1}{5} \end{bmatrix} \times \begin{bmatrix} 12 & 0 \\ 0 & 10 \end{bmatrix}$ such that $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is identity for $\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$ w.r.t $\times$ and $\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{5} \end{bmatrix}$ is inverse for $\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$ w.r.t $\times$,$\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$, we get $(x , y) = (4 , 2)$.

Theroem 3.21 :

Let $(G,*)$ be a λ- group , if $a * b = c$ and $e_a = e_b$, then $b = a^{-1} * c$ for all $a, b, c \in G$ such that e_a is identity for a w.r.t $*$, e_b is identity for b w.r.t $*$ and a^{-1} is inverse for a w.r.t $*$.

Proof .

Let $a * b = c$, $a^{-1} * (a * b) = a^{-1} * c$ such that a^{-1} is inverse for a w.r.t $*$, $(a^{-1} * a) * b = a^{-1} * c$, since $*$ is associative , we get : $e_a * b = a^{-1} * c$, since $e_a = e_b$, we get : $e_b * b = a^{-1} * c$ and $b = a^{-1} * c$.

Theroem 3.22 :

Let $(G,*)$ be a λ- group , if $a * b = c * b$ or $b * a = b * c$ and $e_a = e_b = e_c$, then $a = c$ for all $a, b, c \in G$ such that e_a is identity for a w.r.t $*$, e_b is identity for b w.r.t $*$ and e_c is identity for c w.r.t $*$.

Proof .

Let $a * b = c * b$, $(a * b) * b^{-1} = (c * b) * b^{-1}$, such that b^{-1} is inverse for b w.r.t $*$, $a * (b * b^{-1}) = c * (b * b^{-1})$, since $*$ is associative , we get $a * e_b = c * e_b$, since $e_a = e_b = e_c$, we get : $a * e_a = c * e_c$ and $a = c$.

Let $b * a = b * c$, $b^{-1} * (b * a) = b^{-1} * (b * c)$, such that b^{-1} is inverse for b w.r.t $*$, $(b^{-1} * b) * a = (b^{-1} * b) * c$, since $*$ is associative , we get $e_b * a = e_b * c$, since $e_a = e_b = e_c$, we get : $e_a * a = e_c * \mathrm{c}$ and $a = c$.

Theroem 3.23 :

Let $(G,*)$ be a λ- group , e_a is identity for a w.r.t $*$ and e_b is identity for b w.r.t $*$, then e_a is left identity for $a * b$ w.r.t $*$ and e_b is right identity for $a * b$ w.r.t $*$.

Proof .

since $*$ is associative , then $e_a * (a * b) = (e_a * a) * b = (a * b)$

and $(a * b) * e_b = a * (b * e_b) = (a * b)$

i.e e_a is left identity for $a * b$ w.r.t $*$ and e_b is right identity for $a * b$ w.r.t $*$.

4. λ- ring (special ring) and λ- field (special field) :

Definition 4.1 :

Let $(G,*)$ be a λ- group , then $(G,*)$ is called a commutative λ- group , if $a * b = b * a$ for all $a \in G$.

Example 4.2 :

(p(Z),∩) is a commutative λ- ring , such that Z is the set of integer . **Definition 4.3 :**

Anon empty set G with two binary operation $*$ and $\#$ is said to be λ-ring (special ring) if :

(i) $(G,*)$ is a commutative λ- group .

(ii) $(G,\#)$ is a semi group .

(iii) $a\#(b*c)=(a\#b)*(a\#c)$ (left distribution law) and

$(a*b)\#c=(a\#c)*(b\#c)$ (right distribution law) , for all

$a,b,c\in G$.

And it will be denoted by $(G,*,\#)$.

Remark 4.4 :

Every ring is λ-ring since every group is λ-group , but the converse is not true in general , as shown by the following examples :

Example 4.5 :

(p(Z),∩,∪) is λ- ring , such that Z is the set of integer ,

(p(Z),∩,∪) is not ring .

Definition 4.6 :

Anon empty set G with two binary operation $*$ and $\#$ is said to be λ-field (special field) if :

(i) $(G,*)$ is a commutative λ-group .

(ii) $(G-A,\#)$ is a commutative λ- group , such that

$$A=\{e\in G : e*a=a*e=a\,, for\ some\ a\in G\}$$

(iii) $a\#(b*c)=(a\#b)*(a\#c)$ (left distribution law) and

$(a*b)\#c=(a\#c)*(b\#c)$ (right distribution law)

for all $a,b,c\in G$.

And it will be denoted by $(G,*,\#)$.

Remark 4.7 :

Every field is λ-field since every group is λ-group , but the converse is not true in general , as shown by the following example :

Example 4.8 :

(p(Z),∩,∪) is λ- field , such that Z is the set of integer , but

(p(Z),∩,∪) is not field .

Reference:

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