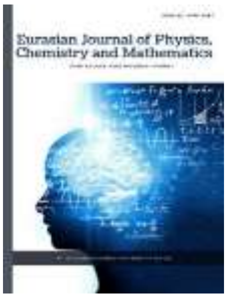

Restricted Maximum Likelihood Estimators for Unbalanced of One-way Repeated Measurements Model

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ABSTRACT

In this paper, we define and explain the one-way repeated measurements model for unbalanced data. Also next explore an alternative approaches to estimate the model's parameters and some of its properties.

Keywords: One-Way Repeated Measurements Model; Maximum Likelihood Method; Restricted Maximum Likelihood Method RMLM

1.Introduction

Several disciplines, including epidemiology, biomedical research, health and life sciences, and others frequently use repeated measurements analysis. The univariate repeated measurements analysis of variance has received a lot of literature. [6][12]. The term "repeated measurements" refers to data when each experimental unit's response variable is measured repeatedly, sometimes under various experimental conditions. [10]. Repeated measures in a number of research contexts, designs with two or more independent groups are among the most popular. When parametric model assumptions are broken, a number of statistical techniques have been offered for interpreting data from split-plot designs. [7]. Data comes in two varieties: balanced data and unbalanced data. In a balance RM design, all of the experimental units' measurement occasions are the same, however in an unbalance data design, some of the experimental units' measurement times are different. [11]. Several studies and research projects are interested in examining repeated measures models, particularly in those models' analysis of variance and model parameter estimation. Following are some recent research on estimators and analysis of variance for repeated measurement models that we will quickly cover in this historical review: Al-Mouel. in (2004)[2], compared estimators and analyzed the multivariate repeated measures models. Mohaisen and Swadi in (2014) , studied the one-way repeated measurements model by using the Bayesian approach based on Markov Chain Monte Carlo. The one-way repeated measurements model is studied using the Bayesian approach as a mixed model, and they investigate the consistency property of the Bayes factor for testing the fixed effects linear one-way repeated measurements model against the mixed one-way repeated measurements alternative model. They discovered the analytical form of the Bayes factor and demonstrated its consistency under some prior and design matrix constraints. [7],[8],[9],[10].AL-Mouel and Al-Isawi in (2019)[1], They computed the quadratic unbiased estimator that has the least variance (best quadratic unbiased estimate (BQUE)) by using the analysis of variance (ANOVA) method of estimating the variance components. They studied best quadratic unbiased estimator of variance components for balanced data for repeated measurement model (RMM).. AL-Mouel and AL-Hasan in(2021)[4], analyzed statistical inference in models with repeated measurements and variance components..AL-Mouel and Ali in(2021)[5], research on the repeated measurements model's random effects.AL-Mouel and Abd-Ali in(2021)[3], study on the estimate of variance components for the repeated measurements model.

In this paper, studied the maximum likelihood estimators and restricted maximum likelihood estimators of the parameters for our model.

2. One-Way Repeated Measurements Model

For unbalanced data, we take into account the one-way repeated measurements model (where the number of observations is unequal in each levels). The model is given a mathematical formulation. Then, we go over a few estimator properties and methods for estimating the model's parameters.

2.1 The Mathematical model

Consider the one-way repeated measurements model for unbalanced data,

$$\upsilon_{\ell\mathcal{L}\varkappa} = \vartheta + \xi_{\mathcal{L}} + \zeta_{\varkappa} + (\xi\zeta)_{\mathcal{L}\varkappa} + \varsigma_{\ell(\mathcal{L})} + e_{\ell\mathcal{L}\varkappa}, \quad (2.1.1)$$

where : $\upsilon_{\ell\mathcal{L}\varkappa}$ is the response measurement at time $\varkappa$ for unit ℓ within group $\mathcal{L}$,

$\ell = 1,2,\dots,n_{\mathcal{L}}$ is an index for experimental unit within group $\mathcal{L}$,

$\mathcal{L} = 1,2,\dots,q$ is an index for levels of the between-units factor (Group) ,

$\varkappa = 1,2,\dots,p$ is an index for levels of the within-units factor (Time) ,

ϑ is the overall mean ,and $\boldsymbol{\xi_{\mathcal{L}}}$ is the added effect for treatment group $\mathcal{L}$,

$\zeta_{\varkappa}$ is the added effect for time $\varkappa$,

$(\xi\zeta)_{\mathcal{L}\varkappa}$ is the added effect for the group $\mathcal{L}$ × time $\varkappa$ (Interaction) ,

$\varsigma_{\ell(\mathcal{L})}$ is the random effect due to experimental unit ℓ within treatment group $\mathcal{L}$,

$e_{\ell\mathcal{L}\varkappa}$ is the random error time $\varkappa$ for unit ℓ within group $\mathcal{L}$,

under the following considerations for the added parameter (added effect)

$$\left.\begin{aligned} &\sum_{\mathcal{L}=1}^{q}\xi_{\mathcal{L}} = 0 \ ,\sum_{\varkappa=1}^{p}\zeta_{\varkappa} = 0 \ ,\ \sum_{\mathcal{L}=1}^{q}(\xi\zeta)_{\mathcal{L}\varkappa} = 0 ,\forall\, \varkappa = 1,2,\dots,p \ , \\ &\sum_{\varkappa=1}^{p}(\xi\zeta)_{\mathcal{L}\varkappa} = 0 \ ,\ \forall\mathcal{L} = 1,2,\dots,q \ ,\ with\ N = \sum_{\mathcal{L}} n_{\mathcal{L}} \ , and\ \sum_{\mathcal{L}=1}^{q} n_{\mathcal{L}}\xi_{\mathcal{L}} = 0 \end{aligned}\right\} (2.1.2),$$

Considering that the $\varsigma_{\ell(\mathcal{L})}'s$ and $e_{\ell\mathcal{L}\varkappa}'s$ are independent and identically with

$$e_{\ell\mathcal{L}\varkappa} \sim i.i.d\ N(0,\sigma_e^2) \quad , \quad \varsigma_{\ell(\mathcal{L})} \sim i.i.d\ N\left(0,\sigma_{\varsigma}^2\right) \ , \quad (2.1.3),$$

2.2. Setting the One-way Repeated Measurements Model in Matric Form

We can rewrite the model (2.1.1) as

$$\upsilon_{\ell\mathcal{L}\varkappa} = \vartheta + \xi_{\mathcal{L}} + \zeta_{\varkappa} + (\xi\zeta)_{\mathcal{L}\varkappa} + \Psi_{\ell\mathcal{L}\varkappa} \quad (2.2.1)$$

where , $\Psi_{\ell\mathcal{L}\varkappa} = \varsigma_{\ell(\mathcal{L})} + e_{\ell\mathcal{L}\varkappa}$, $\Psi_{\ell\mathcal{L}\varkappa} \sim N(0,\sigma_{\Psi}^2)$, $\sigma_{\Psi}^2 = \sigma_{\varsigma}^2 + \sigma_e^2$, thus by taking matrix notation as

$$\Upsilon = \aleph\Phi + \Psi \quad (2.2.2)$$

where

$\Upsilon = \left(\Upsilon_{111},\dots,\Upsilon_{n_1 1p},\Upsilon_{121},\dots,\Upsilon_{n_2 2p},\dots,\Upsilon_{n_q q1},\dots,\Upsilon_{n_q qp}\right)^T$ is Np×1 -dimensional response vector, where $N = \sum_{\mathcal{L}=1}^{q} n_{\mathcal{L}}$,

$\aleph = [\aleph_1,\aleph_2,\dots,\aleph_q]^T$ is a $Np \times (q+1)(p+1)$ design matrix

Φ: is $(q+1)(p+1)$× 1 -dimensional vector of fixed effects parameters, and

$\Psi = \left(\Psi_{111},\dots,\Psi_{n_1 1p},\Psi_{121},\dots,\Psi_{n_2 2p},\dots,\Psi_{n_q q1},\dots,\Psi_{n_q qp}\right)^T$ is Np×1 -dimensional

The matrices of model in (2.2.1) are defined as :

$$\Phi_{(q+1)(p+1)\times 1} = \begin{bmatrix} \vartheta \\ \xi \\ \zeta \\ \rho \end{bmatrix}, \vartheta_{N\times 1} = \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \\ \vdots \\ \vartheta_N \end{bmatrix}, \xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_q \end{bmatrix}, \zeta = \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_q \end{bmatrix}, \varsigma_{\mathcal{L}_{n_{\mathcal{L}}\times 1}} = \begin{bmatrix} \varsigma_1 \\ \varsigma_2 \\ \vdots \\ \varsigma_{n_{\mathcal{L}}} \end{bmatrix},$$

$$\rho = \begin{bmatrix} (\xi\zeta)_{11} \\ (\xi\zeta)_{12} \\ \vdots \\ (\xi\zeta)_{qp} \end{bmatrix}, and\ e_{Np\times 1} = (e_{111}, \cdots, e_{n_1 1p}, e_{121}, \cdots, e_{n_2 2p}, \cdots, e_{n_q qp})', \quad (2.2.3)$$

And design matrices ℵ is

$$\aleph = \begin{bmatrix} \aleph_1 \\ \aleph_2 \\ \vdots \\ \aleph_q \end{bmatrix}_{Np\times(q+1)(p+1)}, \quad (2.2.4)$$

where $\aleph_{\mathcal{L}} = j_{n_{\mathcal{L}}} \otimes [j_p, e_{\mathcal{L}} \otimes j_p, I_n, e_{\mathcal{L}} \otimes I_n]_{p\times(q+1)(p+1)}$,

$$\aleph_{\mathcal{L}} = j_{n_{\mathcal{L}}} \otimes [\aleph]_{p\times(q+1)(p+1)}, \quad (2.2.5)$$

$e = I_q = [e_1, e_2, \cdots, e_q]'$ and $\otimes$ denotes the Kronecker product .From the model (2.2.2) we have $\Upsilon \sim N_{Np}(\aleph\Phi, \Theta)$, where

$\Theta = E(\Psi\Psi^T)$,

$\Theta = I_N \otimes (\sigma_e^2 I_p + \sigma_\varsigma^2 ee^T)$,

$\Theta = \sigma_e^2 (I_N \otimes I_p) + \sigma_\varsigma^2 (I_N \otimes ee^T)$,

replace I_p by $(E_p + J_p)$ and ee^T by pJ_p where $J_p = \frac{1}{p} ee^T$ and $E_p = I_p - J_p$,

then

$\Theta = \sigma_e^2 \left(I_N \otimes (E_p + J_p)\right) + \sigma_\varsigma^2 (I_N \otimes pJ_p)$,

$\Theta = \sigma_e^2 (I_N \otimes E_p) + \sigma_e^2 (I_N \otimes J_p) + p\sigma_\varsigma^2 (I_N \otimes J_p)$,

By assembling concepts that have the same matrices, we obtain

$\Theta = \sigma_e^2 (I_N \otimes E_p) + (\sigma_e^2 + p\sigma_\delta^2)(I_N \otimes J_p) = \sigma_e^2 A + \sigma_1^2 B$,

where, $\sigma_1^2 = \sigma_e^2 + p\sigma_\delta^2$, $A = I_N \otimes E_p$, $B = I_N \otimes J_p$,and $\Theta^{-1} = \frac{A}{\sigma_e^2} + \frac{B}{\sigma_1^2}$,

$|\Theta| = (\sigma_e^2)^{(N-q)(p-1)} (\sigma_1^2)^{(N-q)}$, where $|\Theta|$ =product of its characteristic roots .

2.3 Estimation

There are various methods available for estimating the model parameters from unbalanced data, we discuss some these methods and their properties**.**

2.3.1- Maximum Likelihood Method

The density function of $\Upsilon \sim N_{Np}(\aleph\Phi, \Theta)$, is

$$f(\Upsilon/\Phi, \Theta) = (2\pi)^{\frac{-Np}{2}} |\Theta|^{\frac{-1}{2}} \exp\left(-\frac{1}{2}(\Upsilon - \aleph\Phi)^T \Theta^{-1} (\Upsilon - \aleph\Phi)\right),$$

Then, the likelihood function is the joint density of the ϒ's that is

$$L(\Upsilon/\Phi, \Theta) = (2\pi)^{\frac{-Np}{2}} |\Theta|^{\frac{-1}{2}} exp\left(-\frac{1}{2}(\Upsilon - \aleph\Phi)^T \Theta^{-1} (\Upsilon - \aleph\Phi)\right), \text{ then}$$

$$L(\Upsilon/\Phi, \sigma_1^2, \sigma_e^2) = (2\pi)^{\frac{-Np}{2}} (\sigma_e^2)^{\frac{-(N-q)(p-1)}{2}} (\sigma_1^2)^{\frac{-(N-q)}{2}} \exp\left(-\frac{1}{2}(\Upsilon - \aleph\Phi)^T (\frac{A}{\sigma_e^2} + \frac{B}{\sigma_1^2})(\Upsilon - \aleph\Phi)\right), (2.3.1)$$

$$\ln(L) = \frac{-Np}{2}\ln(2\pi) - \frac{(N-q)(p-1)}{2}\ln(\sigma_e^2) - \frac{(N-q)}{2}\ln(\sigma_1^2) - \frac{1}{2}(\Upsilon - \aleph\Phi)^T \left(\frac{A}{\sigma_e^2} + \frac{B}{\sigma_1^2}\right)(\Upsilon - \aleph\Phi) .$$

Since, $(\Upsilon - \aleph\Phi)^T \Theta^{-1} (\Upsilon - \aleph\Phi) = \Upsilon^T \Theta^{-1} \Upsilon - 2\Phi^T \aleph^T \Theta^{-1} \Upsilon + \Phi^T \aleph^T \Theta^{-1} \aleph\Phi$

the result is obtained by differentiating ln(L) with respect to ϕ and equalizing to zero.

$$\frac{\partial \ln(L)}{\partial \Phi} = \frac{-1}{2}(2\aleph^T \Theta^{-1} \aleph\Phi - 2\aleph^T \Theta^{-1} \Upsilon) = 0 , \Rightarrow \aleph^T \Theta^{-1} \aleph\widehat{\Phi} = \aleph^T \Theta^{-1} \Upsilon ,$$

$$\therefore \widehat{\Phi} = (\aleph^T \Theta^{-1} \aleph)^{-1} (\aleph^T \Theta^{-1} \Upsilon). \quad (2.3.2)$$

We can write ln(L) as

$$\ln(L) = \frac{-Np}{2}\ln(2\pi) - \frac{(N-q)(p-1)}{2}\ln(\sigma_e^2) - \frac{(N-q)}{2}\ln\left(\sigma_e^2 + p\sigma_\varsigma^2\right) - \frac{1}{2}(\Upsilon - \aleph\Phi)^T\left(\frac{A}{\sigma_e^2} + \frac{B}{\sigma_e^2 + p\sigma_\varsigma^2}\right)(\Upsilon - \aleph\Phi).$$

Now, differentiate ln(L) with respect to σ_ς^2 *and* σ_e^2 and equate to zero, we get

$$\frac{\partial\, ln(L)}{\partial\sigma_e^2} = -\frac{(N-q)(p-1)}{2\sigma_e^2} - \frac{(N-q)}{2(\sigma_e^2 + p\sigma_\varsigma^2)} + \frac{1}{2}\left(\Upsilon - \aleph\widehat{\Phi}\right)^T\left(\frac{A}{\sigma_e^4} + \frac{B}{\left(\sigma_e^2 + p\sigma_\varsigma^2\right)^2}\right)\left(\Upsilon - \aleph\widehat{\Phi}\right) = 0\,, \quad (2.3.3)$$

and $\frac{\partial\, ln(L)}{\partial\sigma_\delta^2} = -\frac{(N-q)p}{2(\sigma_e^2 + p\sigma_\varsigma^2)} + \frac{1}{2}\left(\Upsilon - \aleph\widehat{\Phi}\right)^T\left(\frac{pB}{\left(\sigma_e^2 + p\sigma_\varsigma^2\right)^2}\right)\left(\Upsilon - \aleph\widehat{\Phi}\right) = 0\,,$ (2.3.4)

From (2.3.3), we get $\rightarrow \frac{\partial\, ln(L)}{\partial\sigma_e^2} = \frac{-(N-q)(p-1)\hat{\sigma}_e^2 + (\Upsilon - \aleph\widehat{\Phi})^T A(\Upsilon - \aleph\widehat{\Phi})}{2\hat{\sigma}_e^4} = 0$,

$\rightarrow \left(\Upsilon - \aleph\widehat{\Phi}\right)^T A\left(\Upsilon - \aleph\widehat{\Phi}\right) = (N-q)(p-1)\hat{\sigma}_e^2$,

$\therefore \hat{\sigma}_e^2 = \frac{1}{(N-q)(p-1)}\left(\Upsilon - \aleph\widehat{\Phi}\right)^T A\left(\Upsilon - \aleph\widehat{\Phi}\right).$ (2.3.5)

Now, from (2.3.4)

$$\frac{\partial\, ln(L)}{\partial\sigma_\varsigma^2} = \frac{-(N-q)p\left(\hat{\sigma}_e^2 + p\hat{\sigma}_\varsigma^2\right) + p\left(\Upsilon - \aleph\widehat{\Phi}\right)^T B\left(\Upsilon - \aleph\widehat{\Phi}\right)}{2\left(\hat{\sigma}_e^2 + p\hat{\sigma}_\varsigma^2\right)^2} = 0\,,$$

$\rightarrow p\left(\Upsilon - \aleph\widehat{\Phi}\right)^T B\left(\Upsilon - \aleph\widehat{\Phi}\right) = (N-q)p\left(\hat{\sigma}_e^2 + p\hat{\sigma}_\varsigma^2\right),$

$\therefore \hat{\sigma}_\varsigma^2 = \frac{1}{p(N-q)}\left(\Upsilon - \aleph\widehat{\Phi}\right)^T B\left(\Upsilon - \aleph\widehat{\Phi}\right) - \frac{1}{p}\hat{\sigma}_e^2\,.$ (2.3.6)

Thus, it is evident that

$$\hat{\sigma}_1^2 = \hat{\sigma}_e^2 + p\hat{\sigma}_\varsigma^2 = \hat{\sigma}_e^2 + \frac{p}{p(N-q)}\left(\Upsilon - \aleph\widehat{\Phi}\right)^T B\left(\Upsilon - \aleph\widehat{\Phi}\right) - \frac{p}{p}\hat{\sigma}_e^2\,,$$

$\therefore \hat{\sigma}_1^2 = \frac{1}{(N-q)}\left(\Upsilon - \aleph\widehat{\Phi}\right)^T B\left(\Upsilon - \aleph\widehat{\Phi}\right)$. ☐ (2.3.7)

2.3.1.1 Important Estimators' Characteristics

We present certain estimator qualities as theorems in this section.

Theorem 2.3.1

The maximum likelihood estimator of ϕ is the best linear unbiased estimator .

Proof:

Since $\hat{\phi} = (\aleph^T\Theta^{-1}\aleph)^{-1}(\aleph^T\Theta^{-1}\Upsilon)$,

$E(\hat{\phi}) = E[(\aleph^T\Theta^{-1}\aleph)^{-1}(\aleph^T\Theta^{-1}\Upsilon)],$

$= (\aleph^T\Theta^{-1}\aleph)^{-1}\aleph^T\Theta^{-1}E(\Upsilon),$

$= [(\aleph^T\Theta^{-1}\aleph)^{-1}(\aleph^T\Theta^{-1}\aleph)]\phi = \phi\,.$

$var(\hat{\phi}) = var[(\aleph^T\Theta^{-1}\aleph)^{-1}(\aleph^T\Theta^{-1}\Upsilon)],$

$= (\aleph^T\Theta^{-1}\aleph)^{-1}\aleph^T\Theta^{-1}\left(var(\Upsilon)\right)\Theta^{-1}\aleph(\aleph^T\Theta^{-1}\aleph)^{-1},$

$= (\aleph^T\Theta^{-1}\aleph)^{-1}(\aleph^T\Theta^{-1}\aleph)(\aleph^T\Theta^{-1}\aleph)^{-1} = (\aleph^T\Theta^{-1}\aleph)^{-1}\,.$

Now, let $\hat{\phi}^* = D\Upsilon$ is another unbiased estimator for ϕ , where

$D = (\aleph^T\Theta^{-1}\aleph)^{-1}\aleph^T\Theta^{-1}\aleph + G$, where G is (q+1)×Np matrix,

Since, $E(\hat{\phi}^*) = \phi$

$E(D\Upsilon) = E([(\aleph^T\Theta^{-1}\aleph)^{-1}\aleph^T\Theta^{-1} + G\,]\Upsilon),$

$= [(\aleph^T\Theta^{-1}\aleph)^{-1}\aleph^T\Theta^{-1} + G\,]E(\Upsilon)$

$= [(\aleph^T\Theta^{-1}\aleph)^{-1}\aleph^T\Theta^{-1} + G\,]\aleph\phi$,

E(D Υ)= $(\aleph^T\Theta^{-1}\aleph)^{-1}\aleph^T\Theta^{-1}\aleph\phi + G\aleph\phi = I.\phi + 0 = \phi$,

that is $\aleph\phi = 0$.

$var(\hat{\phi}^*) = var(D\Upsilon) = D\,var(\Upsilon)\,D^T$,

$= [(\aleph^T\Theta^{-1}\aleph)^{-1}\aleph^T\Theta^{-1} + G\,]\,\Theta\,[(\aleph^T\Theta^{-1}\aleph)^{-1}\aleph^T\Theta^{-1} + G\,]^T,$

$= (\aleph^T\Theta^{-1}\aleph)^{-1}(\aleph^T\Theta^{-1}\aleph)(\aleph^T\Theta^{-1}\aleph)^{-1} + G\,\Theta\,G^T$,

$=(\aleph^T\Theta^{-1}\aleph)^{-1} + G\,\Theta\,G^T$,

$= var(\hat{\phi}) + G\ \Theta\ G^T$, *thus* $var(\hat{\phi}) < var(\hat{\phi}^*)$. ▭

Theorem 2.3.2

The maximum likelihood estimators $\hat{\phi}$, $\hat{\sigma}_1^2$, *and* $\hat{\sigma}_e^2$ in our model are jointly sufficient for ϕ, σ_1^2, *and* σ_e^2 .

Proof:

The density function of Υ is

$f(\Upsilon/\phi,\Theta) = (2\pi)^{\frac{-Np}{2}}|\Theta|^{\frac{-1}{2}} \exp\left(-\frac{1}{2}(\Upsilon - \aleph\phi)^T\Theta^{-1}(\Upsilon - \aleph\phi)\right)$, by adding and subtracting $\aleph\hat{\theta}$ in the exponent

$$f(\Upsilon/\phi,\Theta) = (2\pi)^{\frac{-Np}{2}}|\Theta|^{\frac{-1}{2}} \exp\left(-\frac{1}{2}\left(\Upsilon - \aleph\hat{\phi} + \aleph\hat{\phi} - \aleph\phi\right)^T\Theta^{-1}\left(\Upsilon - \aleph\hat{\phi} + \aleph\hat{\phi} - \aleph\phi\right)\right),$$

$$= (2\pi)^{\frac{-Np}{2}}|\Theta|^{\frac{-1}{2}} \exp\left(-\frac{1}{2}\left[(\Upsilon - \aleph\hat{\phi}) + \aleph(\hat{\phi} - \phi)\right]^T\Theta^{-1}\left[(\Upsilon - \aleph\hat{\phi}) + \aleph(\hat{\phi} - \phi)\right]\right),$$

$$= (2\pi)^{\frac{-Np}{2}}|\Theta|^{\frac{-1}{2}} \exp\left(-\frac{1}{2}\left[(\Upsilon - \aleph\hat{\phi})^T\Theta^{-1}(\Upsilon - \aleph\hat{\phi}) + (\hat{\phi} - \phi)^T\aleph^T\Theta^{-1}\aleph(\hat{\phi} - \phi)\right]\right),$$

$$= (2\pi)^{\frac{-Np}{2}}|\Theta|^{\frac{-1}{2}} \exp\left(-\frac{1}{2}\left[\begin{array}{c}(\Upsilon - \aleph\hat{\phi})^T\left(\frac{A}{\sigma_e^2} + \frac{B}{\sigma_1^2}\right)(\Upsilon - \aleph\hat{\phi}) + (\hat{\phi} - \phi)^T\aleph^T\Theta^{-1}\aleph \\ (\hat{\phi} - \phi)\end{array}\right]\right),$$

$$= (2\pi)^{\frac{-Np}{2}}|\Theta|^{\frac{-1}{2}} \exp\left(-\frac{1}{2}\left[(\Upsilon - \aleph\hat{\phi})^T\frac{A}{\sigma_e^2}(\Upsilon - \aleph\hat{\phi}) + (\Upsilon - \aleph\hat{\phi})^T\frac{B}{\sigma_1^2}(\Upsilon - \aleph\hat{\phi}) + (\hat{\phi} - \phi)^T\aleph^T\Theta^{-1}\aleph(\hat{\phi} - \phi)\right]\right),$$

From theorem (2.3.1) we get

$$f(\Upsilon/\phi,\Theta) = (2\pi)^{\frac{-Np}{2}}|\Theta|^{\frac{-1}{2}} \exp\left(-\frac{1}{2}\left(\frac{(N-q)(p-1)\hat{\sigma}_e^2}{\sigma_e^2} + \frac{(N-q)\hat{\sigma}_1^2}{\sigma_1^2}\right) + (\hat{\phi} - \phi)^T\aleph^T\Theta^{-1}\aleph(\hat{\phi} - \phi)\right),$$

Now, we can write the density as:

$f(\Upsilon/\phi,\Theta) = g(\hat{\phi}, \hat{\sigma}_e^2, \hat{\sigma}_1^2, \phi, \sigma_e^2, \sigma_1^2)h(\Upsilon)$, where $h(\Upsilon) = 1$, therefore by the Neyman factorization theorem $\hat{\phi}$, $\hat{\sigma}_e^2$, *and* $\hat{\sigma}_1^2$ are jointly sufficient for ϕ, σ_e^2, *and* σ_1^2 . ☐

Theorem 2.3.3

The maximum likelihood estimator of ϕ is an efficient statistic for ϕ , when $\Upsilon \sim N_{Np}(\aleph\phi, \Theta)$, where $\aleph$ is p×(q+1)(p+1) of rank (q+1)(p+1) < Np and $\phi = [\vartheta, \xi, \zeta, \rho]^T$.

Proof:

The density function of Υ is

$$f(\Upsilon/\phi,\Theta) = (2\pi)^{\frac{-Np}{2}}|\Theta|^{\frac{-1}{2}} \exp\left(-\frac{1}{2}(\Upsilon - \aleph\phi)^T\Theta^{-1}(\Upsilon - \aleph\phi)\right),$$

and the likelihood function is

$$L(\Upsilon/\phi,\Theta) = (2\pi)^{\frac{-Np}{2}}|\Theta|^{\frac{-1}{2}} \exp\left(-\frac{1}{2}(\Upsilon - \aleph\phi)^T\Theta^{-1}(\Upsilon - \aleph\phi)\right),$$

then

$$\ln(L) = \frac{-Np}{2}\ln(2\pi) - \frac{(N-q)(p-1)}{2}\ln(\sigma_e^2) - \frac{(N-q)}{2}\ln(\sigma_1^2) - \frac{1}{2}(\Upsilon - \aleph\phi)^T\left(\frac{A}{\sigma_e^2} + \frac{B}{\sigma_1^2}\right)(\Upsilon - \aleph\phi),$$

$$\to \frac{\partial \ln(L)}{\partial\phi} = \frac{-1}{2}(2\aleph^T\Theta^{-1}\aleph\phi - 2\aleph^T\Theta^{-1}\Upsilon) \to \frac{\partial^2 \ln(L)}{\partial\phi^2} = -(\aleph^T\Theta^{-1}\aleph).$$

Then, the Rao – Cramer lower bounded is

$C.R.L = -E\left[\frac{\partial^2 ln(L)}{\partial\phi^2}\right]^{-1} = (\aleph^T\Theta^{-1}\aleph)^{-1} = var(\hat{\phi})$. ☐

2.3.b- Restricted Maximum Likelihood Method

In this part, we take into account a set of m linear constraints on the model's coefficients (2.2.2). It additionally examines the inferences. The one-way repeated measurements model for unbalanced data is inferred using the constrained maximum likelihood method. Consider the model (2.2.2) , we presume that

$R\Phi = r$, (2.3.8)

where R is m×g , r is m×1 , Φ is g×1 and g=(q+1)(p+1)+1 .Then, the restricted likelihood is

$$L(\Upsilon/\Phi,\sigma_1^2,\sigma_e^2) = (2\pi)^{\frac{-Np}{2}}(\sigma_e^2)^{\frac{-(N-q)(p-1)}{2}}(\sigma_1^2)^{\frac{-(N-q)}{2}} exp\left(-\frac{1}{2}(\Upsilon - \aleph\Phi^c)^T(\frac{A}{\sigma_e^2}+\frac{B}{\sigma_1^2})(\Upsilon - \aleph\Phi^c)\right) \quad (2.3.9)$$

where Φ^c is (1×g) the restricted vector of parameters, then

$$\ln(L) = \frac{-Np}{2}\ln(2\pi) - \frac{(N-q)(p-1)}{2}\ln(\sigma_e^2) - \frac{(N-q)}{2}\ln(\sigma_1^2) - \frac{1}{2}(\Upsilon - \aleph\Phi^c)^T\left(\frac{A}{\sigma_e^2}+\frac{B}{\sigma_1^2}\right)(\Upsilon - \aleph\Phi^c) \quad (2.3.10)$$

From theorem(2.3.1) we get

$$\hat{\sigma}_e^2 = \frac{1}{(N-q)(p-1)}(\Upsilon - \aleph\Phi^c)^T A(\Upsilon - \aleph\Phi^c), \quad (2.3.11)$$

and

$$\hat{\sigma}_1^2 = \frac{1}{(N-q)}(\Upsilon - \aleph\Phi^c)^T B(\Upsilon - \aleph\Phi^c). \quad (2.3.12)$$

Now, by substitute (2.3.11),(2.3.12) in (2.3.10) get

$$\ln(L) = \frac{-Np}{2}\ln(2\pi) - \frac{(N-q)(p-1)}{2}\ln\left(\frac{1}{(N-q)(p-1)}(\Upsilon - \aleph\Phi^c)^T A(\Upsilon - \aleph\Phi^c)\right) - \frac{(N-q)}{2} ln\left(\frac{1}{(N-q)}(\Upsilon - \aleph\Phi^c)^T B(\Upsilon - \aleph\Phi^c)\right) - \frac{1}{2}\left[\frac{(\Upsilon-\aleph\Phi^c)^T A(\Upsilon-\aleph\Phi^c)}{\frac{1}{(N-q)(p-1)}(\Upsilon-\aleph\Phi^c)^T A(\Upsilon-\aleph\Phi^c)}\right] - \frac{1}{2}\left[\frac{(\Upsilon-\aleph\Phi^c)^T B(\Upsilon-\aleph\Phi^c)}{\frac{1}{(N-q)}(\Upsilon-\aleph\Phi^c)^T B(\Upsilon-\aleph\Phi^c)}\right], \quad (2.3.13)$$

Thus, the restricted likelihood function is given by

$$L^* = (2\pi)^{\frac{-np}{2}}\left[\frac{1}{(N-q)}(\Upsilon - \aleph\Phi^c)^T B(\Upsilon - \aleph\Phi^c)\right]^{\frac{-(N-q)}{2}} e^{\frac{-(N-q)}{2}}\left[\frac{1}{(N-q)(p-1)}(\Upsilon-\aleph\Phi^c)^T A(\Upsilon-\aleph\Phi^c)\right]^{\frac{-(N-q)(p-1)}{2}} e^{\frac{-(N-q)(p-1)}{2}},$$

(2.3.14)

Then, we can rewrite the description of the problem as the following

$$Max_{\phi^c} lnL^* = \frac{-Np}{2} ln(2\pi) - \frac{(N-q)}{2}\ln\left[\frac{1}{(N-q)}(\Upsilon - \aleph\Phi^c)^T B(\Upsilon - \aleph\Phi^c)\right] - \frac{(N-q)}{2} - \frac{(N-q)(p-1)}{2} ln\left[\frac{1}{(N-q)(p-1)}(\Upsilon - \aleph\Phi^c)^T A(\Upsilon - \aleph\Phi^c)\right] - \frac{(N-q)(p-1)}{2}, \quad (2.3.15)$$

such that

$$R\phi^c = r . \quad (2.3.16)$$

Then, we can maximize this likelihood function (2.3.15) by minimizing the terms $(\Upsilon - \aleph\Phi^c)^T B(\Upsilon - \aleph\Phi^c)$ and $(\Upsilon - \aleph\Phi^c)^T A(\Upsilon - \aleph\Phi^c)$ .To satisfy (2.3.15) subject to constraint (2.3.16), we use the form of Lagrangian function where λ is (m×1) vector of Lagrangian multipliers, thus, we can get

$$\alpha_1 = \Upsilon^T B\Upsilon - 2\phi^{c^T}\aleph^T B\Upsilon + \phi^{c^T}\aleph^T B\aleph\phi^c - \lambda^T(r - R\phi^c),$$

$$\alpha_2 = \Upsilon^T A\Upsilon - 2\phi^{c^T}\aleph^T A\Upsilon + \phi^{c^T}\aleph^T B\aleph\phi^c - \lambda^T(r - R\phi^c),$$

$$\rightarrow \alpha_1 + \alpha_2 = \Upsilon^T\Upsilon - 2\theta^{c^T}\aleph^T\Upsilon + \phi^{c^T}\aleph^T\aleph\phi^c - 2\lambda^T(r - R\phi^c) , \quad (2.3.17)$$

where $A + B = I_{Np}$, then, differentiate (2.3.17) above with respect to ϕ^c and λ and equate zero to obtain

$$-2\aleph^T\Upsilon + 2\aleph^T\aleph\phi^c + 2R^T\lambda = 0 , \quad (2.3.18)$$

$$2(R\phi^c - r) = 0 , \quad (2.3.19)$$

Then, by multiply the equation (2.3.18) by $R(\aleph^T\aleph)^{-1}$, get

$$(R(\aleph^T\aleph)^{-1})(-2\aleph^T\Upsilon + 2\aleph^T\aleph\phi^c + 2R^T\lambda) = 0,$$

$$\rightarrow -2R(\aleph^T\aleph)^{-1}\aleph^T\Upsilon + 2R(\aleph^T\aleph)^{-1}\aleph^T\aleph\phi^c + 2R(\aleph^T\aleph)^{-1}R^T\lambda = 0,$$

$$\rightarrow -2R(\aleph^T\Theta^{-1}\aleph)^{-1}(\aleph^T\Theta^{-1}\Upsilon) + 2R\phi^c + 2R(\aleph^T\aleph)^{-1}R^T\lambda = 0, \quad (2.3.20)$$

since $\hat{\phi} = (\aleph^T\Theta^{-1}\aleph)^{-1}(\aleph^T\Theta^{-1}\Upsilon)$ .Then (2.3.20) will be

$$\rightarrow -2R\hat{\phi} + 2R\phi^c + 2R(\aleph^T\aleph)^{-1}R^T\lambda = 0,$$

$$\rightarrow [R(\aleph^T\aleph)^{-1}R^T]\lambda = R\hat{\phi} + R\phi^c,$$

$$\therefore \lambda = [R(\aleph^T\aleph)^{-1}R^T]^{-1}(r - R\hat{\phi}) \ , since \ \ R\phi^c = r. \quad (2.3.21)$$

Now, substitute (2.3.21) in the equation(2.3.18) ,we get

$$-2\aleph^T\Upsilon + 2\aleph^T\aleph\phi^c - 2R^T[R(\aleph^T\aleph)^{-1}R^T]^{-1}(r - R\hat{\phi}) = 0 ,$$

$\rightarrow -\aleph^T Y + \aleph^T \aleph \phi^c - R^T [R(\aleph^T \aleph)^{-1} R^T]^{-1} (r - R\hat{\phi}) = 0$,
$\rightarrow \aleph^T \aleph \phi^c = \aleph^T \Upsilon + R^T [R(\aleph^T \aleph)^{-1} R^T]^{-1} (r - R\hat{\phi})$,
$\rightarrow \theta^c = (\aleph^T \Theta^{-1} \aleph)^{-1} \aleph^T \Theta^{-1} \Upsilon + (\aleph^T \aleph)^{-1} R^T [R(\aleph^T \aleph)^{-1} R^T]^{-1} (r - R\hat{\phi})$,
$\therefore \phi^c = \hat{\phi} + (\aleph^T \aleph)^{-1} R^T [R(\aleph^T \aleph)^{-1} R^T]^{-1} (r - R\hat{\phi})$, (2.3.22)
Then
$\phi^c - \hat{\phi} = (\aleph^T \aleph)^{-1} R^T [R(\aleph^T \aleph)^{-1} R^T]^{-1} (r - R\hat{\phi})$,
$\rightarrow (\aleph^T \aleph)(\phi^c - \hat{\phi}) = R^T [R(\aleph^T \aleph)^{-1} R^T]^{-1} (r - R\hat{\phi}) [R(\aleph^T \aleph)^{-1} R^T]^{-1} R(\aleph^T \aleph)^{-1}$
$R^T [R(\aleph^T \aleph)^{-1} R^T]^{-1} (r - R\hat{\phi})$,
$\rightarrow (\phi^c - \hat{\phi})^T (\aleph^T \aleph)(\phi^c - \hat{\phi}) = (r - R\hat{\phi})^T [R(\aleph^T \aleph)^{-1} R^T]^{-1} R(\aleph^T \aleph)^{-1}$
$R^T [R(\aleph^T \aleph)^{-1} R^T]^{-1} (r - R\hat{\phi})$,
We can rewrite (2.3.22) by multiply by $\aleph$, we get
$\aleph \phi^c = \aleph \hat{\phi} + \aleph(\aleph^T \aleph)^{-1} R^T [R(\aleph^T \aleph)^{-1} R^T]^{-1} (r - R\hat{\phi})$,
$\rightarrow \Upsilon - \aleph \phi^c = \Upsilon - \aleph \hat{\phi} - \aleph(\aleph^T \aleph)^{-1} R^T [R(\aleph^T \aleph)^{-1} R^T]^{-1} (r - R\hat{\phi})$,
$\rightarrow \Psi^c = \hat{\Psi} - \aleph(\aleph^T \aleph)^{-1} R^T [R(\aleph^T \aleph)^{-1} R^T]^{-1} (r - R\hat{\phi})$.
Where Ψ^c is the estimator residual from the constrained model.
$\rightarrow \Psi^{c^T} \Psi^c = \left[\hat{\Psi}^T - (r - R\hat{\phi})^T [R(\aleph^T \aleph)^{-1} R^T]^{-1} R(\aleph^T \aleph)^{-1} \aleph^T\right]$
$\left[\hat{\Psi} - \aleph(\aleph^T \aleph)^{-1} R^T [R(\aleph^T \aleph)^{-1} R^T]^{-1} (r - R\hat{\phi})\right]$,
Where $\Psi^{c^T} \Psi^c$ sum of squared errors from the constrained model.
$= \hat{\Psi}^T \hat{\Psi} - \hat{\Psi} \aleph(\aleph^T \aleph)^{-1} R^T [R(\aleph^T \aleph)^{-1} R^T]^{-1} (r - R\hat{\phi}) \quad -(r - R\hat{\phi})^T [R(\aleph^T \aleph)^{-1} R^T]^{-1} R(\aleph^T \aleph)^{-1} \aleph^T \hat{\Psi}$
$+(r - R\hat{\phi})^T [R(\aleph^T \aleph)^{-1} R^T]^{-1} R(\aleph^T \aleph)^{-1} \aleph^T \aleph(\aleph^T \aleph)^{-1} R^T [R(\aleph^T \aleph)^{-1} R^T]^{-1} (r - R\hat{\phi})$, (2.3.23)
where, $\hat{\Psi}$ is the estimated residual from unconstrained model
$\hat{\Psi}^T \hat{\Psi} = (\Upsilon - \aleph \hat{\phi})^T (Y - \aleph \hat{\phi}) = \Upsilon^T \Upsilon - 2\hat{\phi}^T \aleph^T \Upsilon + \hat{\phi}^T \aleph^T \aleph \hat{\phi}$, (2.3.24)
where
$\aleph^T \hat{\Psi} = \aleph^T (Y - \aleph \hat{\phi}) = \aleph^T (\Upsilon - \aleph[(\aleph^T \Theta^{-1} \aleph)^{-1} (\aleph^T \Theta^{-1} \Upsilon)])$,
$= \aleph^T \Upsilon - (\aleph^T \aleph)(\aleph^T \aleph)^{-1} (\aleph^T \Upsilon) = \aleph^T Y - (\aleph^T Y) = 0$, (2.3.25)
By taking information in equation(2.3.23)and (2.3.25),we get
$\Psi^{c^T} \Psi^c = \hat{\Psi}^T \hat{\Psi} - \hat{\Psi}^T \aleph(\aleph^T \aleph)^{-1} R^T [R(\aleph^T \aleph)^{-1} R^T]^{-1} (r - R\hat{\phi}) - (r -$
$R\hat{\phi})^T [R(\aleph^T \aleph)^{-1} R^T]^{-1} R(\aleph^T \aleph)^{-1} \aleph^T \hat{\Psi} + (r - R\hat{\phi})^T [R(\aleph^T \aleph)^{-1} R^T]^{-1} (r - R\hat{\phi})$
$= \hat{\Psi}^T \hat{\Psi} + (r - R\hat{\phi})^T [R(\aleph^T \aleph)^{-1} R^T]^{-1} (r - R\hat{\phi})$,
$\Psi^{c^T} \Psi^c - \hat{\Psi}^T \hat{\Psi} = (r - R\hat{\phi})^T [R(\aleph^T \aleph)^{-1} R^T]^{-1} (r - R\hat{\phi})$. (2.3.26)
We can write (2.3.22) as following
$\hat{\phi} = (\aleph^T \Theta^{-1} \aleph)^{-1} (\aleph^T \Theta^{-1} \Upsilon) = (\aleph^T \Theta^{-1} \aleph)^{-1} \aleph^T \Theta^{-1} (\aleph \phi + \Psi)$,
$\therefore \hat{\phi} = \phi + (\aleph^T \Theta^{-1} \aleph)^{-1} (\aleph^T \Theta^{-1} \Psi)$, (2.3.27)
then, substitute (2.3.27) in (2.3.22) to obtain
$\phi^c = \hat{\phi} + (\aleph^T \aleph)^{-1} R^T [R(\aleph^T \aleph)^{-1} R^T]^{-1} (r - R\hat{\phi})$,
$= \phi + (\aleph^T \Theta^{-1} \aleph)^{-1} (\aleph^T \Theta^{-1} \Psi) + (\aleph^T \aleph)^{-1} R^T [R(\aleph^T \aleph)^{-1} R^T]^{-1}$
$(r - R[\phi + (\aleph^T \Theta^{-1} \aleph)^{-1} (\aleph^T \Theta^{-1} \Psi)])$, $r = R\phi$,
$\rightarrow \phi^c - \phi = (\aleph^T \Theta^{-1} \aleph)^{-1} (\aleph^T \Theta^{-1} \Psi) - (\aleph^T \aleph)^{-1} R^T [R(\aleph^T \aleph)^{-1} R^T]^{-1} R(\aleph^T \Theta^{-1} \aleph)^{-1} (\aleph^T \Theta^{-1} \Psi)$,
$= [I - (\aleph^T \aleph)^{-1} R^T [R(\aleph^T \aleph)^{-1} R^T]^{-1} R](\aleph^T \Theta^{-1} \aleph)^{-1} (\aleph^T \Theta^{-1} \Psi)$,
$= M^c (\aleph^T \Theta^{-1} \aleph)^{-1} (\aleph^T \Theta^{-1} \Psi)$,
where, $M^c = [I - (\aleph^T \aleph)^{-1} R^T [R(\aleph^T \aleph)^{-1} R^T]^{-1} R]$ and $\Psi = \Upsilon - \aleph \phi$,
$\therefore \phi^c - \phi = M^c (\hat{\phi} - \phi)$. (2.3.28)

3.Conclusions

In this paper, we reached the following conclusions.:

1. The maximum likelihood estimators of parameters of repeated measurements model are $\hat{\Phi} = (\aleph^T \Theta^{-1} \aleph)^{-1} (\aleph^T \Theta^{-1} \Upsilon)$, $\hat{\sigma}_e^2 = \frac{1}{(N-q)(p-1)} \left(\Upsilon - \aleph\hat{\Phi}\right)^T A\left(\Upsilon - \aleph\hat{\Phi}\right)$,and $\hat{\sigma}_\varsigma^2 = \frac{1}{p(N-q)} \left(\Upsilon - \aleph\hat{\Phi}\right)^T B\left(\Upsilon - \aleph\hat{\Phi}\right) - \frac{1}{p}\hat{\sigma}_e^2$,
2. The maximum likelihood estimator of ϕ is the best linear unbiased estimator.
3. The maximum likelihood estimators $\hat{\phi}$, $\hat{\sigma}_1^2$, and $\hat{\sigma}_e^2$ in our model are jointly sufficient for ϕ , σ_1^2 , and σ_e^2 .
4. The maximum likelihood estimator of ϕ is an efficient statistic for ϕ.
5. The restricted maximum likelihood estimators of parameterϕ^c of repeated measurements model is $\phi^c = \phi + M^c(\hat{\phi} - \phi)$,

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