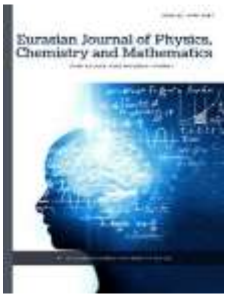

Some Cardinal Properties of Weakly Separable Spaces

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ABSTRACT

This article proves theorems on cardinal properties of weakly separable spaces, which are one of the important spaces of topology. In particular, the images of weakly separable spaces in different mapping are presented with proofs of theorems about weakly separable spaces, and good results were obtained.

Keywords: topological space, hypersymmetric, homeomorphic, weakly separable, caliber, precaliber.

For a topological space X, we denote the family of nonempty closed subsets of X by $expX$. A finite family of open subsets of X is let it be $U_1, . . . , U_k$.

Let's set the relation $O < U_1, . . . , U_k>= \{F \in \exp X : F \subset \cup^k_1 U_i, F \cap U_i \neq \emptyset, i = 1, . . . , k\}$. According to the hyperspace of closed part sets of X, the set $expX$ is called equipped (filled, saturated) with Vietors topology. Its open base consists of the set U_i, $O <U_1, . . . , U_k>$ that open in X.

Now we show the subsets of $expX$. The first of them is the n[th] degree hypersymmetric subspace exp_nX of the space X, n being a positive integer. It consists of closed subsets of X with no more than n points. The second one is as follows:

$\exp_\omega X = \cup\{\exp_n X : n = 1, 2, . . .\}$.

Finally, $\exp_c X$ is the set of all nonempty compact closed subsets of X. For an arbitrary space X, the following results arise:

(1.1) $\exp_n X \subset \exp_\omega X \subset \exp_c X \subset \exp X.$

If a space X is a T_1-space, it is itself homeomorphic to exp_1X.

Proposition 1. if the space X is a T_1-space, then:

1) $expX$ is T_1-space;

2) $exp_\omega X$ is dense in – $exp\ X$;

Proposition 2. If the space X is a T_1-space, then there is a continuous mapping $\pi_{n,X} \equiv \pi_n: X^n \to \exp_n$ which puts the point $\{x_0, . . . , x_{n-1}\} \in \exp_n X$ respectively to the point $(x_0, . . . , x_{n-1}) \in X^n$.

Let it be the G as a subgroup of the S_n symmetric group of all permutations group of the set $n = \{0, . . . , n-1\}$. For an arbitrary space X, the group G moves through every $g \in G$ in X^n such that $g(x_0, . . . , n - 1) = (x_{g(0)}, . . . , x_{g(n-1)})$.

Let SP_G^nX be the factor space of X^n/G and let $\pi_{G,X^n} \equiv \pi_G{}^n : X^n \to SP_G^nX$ be the factor mapping. In that case, the space SP_G^nX is called n^{th} - degree G-symmetric space of X.

Proposition 3. If the space X is a T_1-space, then SP_G^nX is also a T_1-space.

Obviously, the X^n space is T_1-space, the image of the X^n space in $\pi_G{}^n$ factor mapping is SP_G^nX. It can be seen that the unique mapping takes the following form: $\pi_n{}^G : SP_G^nX \to exp_nX$ follows from:

(1.2) $\pi_n = \pi_n{}^G \circ \pi_G{}^n.$

Proposition 4. If the space X is T_1-space, the mapping will be continuous:

$\pi_n^G: SP_G^nX \to \exp_n X$

Proof: It follows from **Proposition 2** and the equality in it that the continuous mapping π_n is the composition of the π_n^G mapping and the π_G^n factor mapping. In this situation, π_n^G is continious.

Theorem 5. For an arbitrary X - T_1-space, an n-natural number and a group G ⊂ S_n, the following conditions are equivalent:

1) X is weakly separable;
2) X^n is weakly separable;
3) SP_G^nX is a weakly separable;
4) $SP^{\,n}X$ *is a* weakly separable;
5) exp_nX is a weakly separable;
6) $exp_\omega X$ is a weakly separable;
7) exp_cX is a weakly separable;
8) $exp\ X$ is a weakly separable.

Example 6. There exists such a separable X space that taken as

T_0-space, but exp_1X is not weakly separable.

We take an arbitrary uncountable set X and define its point x_0. A topology on X is defined as follows: a nonempty set U⊂ X is open, if x_0∈ U. In other words, X contains a dense set consisting of a single point x_0. Or, $exp_1X = X\setminus\{x_0\}$ is a discrete uncountable set.

Question 7: Is $expX$ (weakly) separable for any finite space X ?

Recall that if each family u of cardinality τ consisting of nonempty open subsets of the space X contains a nonempty intersection with the subset family u_0, then τ is an uncountable cardinal number is called the caliber of the X space (u_0 -centered). The following statements are clear and obvious:

Confirmation 8. If X is a separable space, then ω_1 is called the caliber of X.

Proposition 9. If ω_1 is a precaliber of X, then X has the Suslin property, that is, every family of non-empty non-intersecting sets of X is countable.

Proposition 10. If Y⊂X is dense in the space X and τ is a precaliber of X, then τ is a precaliber of Y.

Theorem 11. [3]. If X_α is separable for every α∈A, then ω_1 is called the caliber of the product $\Pi\{X\alpha: \alpha \in A\}$.

This theorem is derived in the following way:

Theorem 12. If X_α is weakly separable for every α∈A, then ω_1 is called the precaliber of the product $\Pi\{X\alpha: \alpha \in A\}$.

Proof: According to the theorem **[3]** (Every weakly separable space X has a separable extension eX), for each α there is an eX_α - separable extension. Let it be $eX = \Pi\{eX_\alpha: \alpha \in A\}$. According to the theorem 11, ω_1 is the caliber of eX. So, according to proposition 10. ω_1 is the precaliber of the space X.

From the proposition 9 and the theorem 12. follows:

Result 13. [2]. If X_α is weakly separable for every $\alpha \in A$, then the product $\Pi\{X_\alpha: \alpha \in A\}$ has the Suslin property.

Proposition 14. Let X_α be composed of one point, $\alpha \in A$ va $X = \Pi\{X\alpha : \alpha \in A\}$. Then the following conditions are equivalent:

1) X is weakly separable;
2) $card(A) \le 2^\omega$;
3) X is separable.

Proof. Let X be weakly separable. Then $bX = \Pi\{\beta X\alpha : \alpha \in A\}$ is a compact separable according to the theorem 5. According to the Pondiseri-Marchevsky theorem (literature [5], example 2.3.G), card(A) ≤ 2^ω is understandable. Since 2^ω is a product of separable spaces, the space X is separable (literature[5], theorem 2.3.15). Proposition 14. proved.

Example. 15. Weak separability of $C_p(X)$.

All spaces in this section are Tikhonov spaces. For the space X, we denote by $C_p(X)$ the set of all continuous real-valued functions. In this topology, $C_p(X)$ is a dense subset of the Tikhonov product - R^X.

Let's also remember that the cardinal number τ is called the i-weight of the space X (it is written as $\tau = iw(X)$), if τ is the smallest cardinal number, then $f: X \to Y$ is a one-to-one continuous mapping of τ

into space Y. In particular, if X is a compact space, then $iw(X) = w(X)$.

Theorem 16. [2]. The following equality holds for an arbitrary X space:

$$d(C_p(X))=iw(X).$$

Result 17. For an arbitrary non-measurable compact space X, the space $C_p(X)$ is not separable.

Now we characterize the spaces X and weakly separable $C_p(X)$,

Theorem 18. The following terms are equivalent for space X:

1) $C_p(X)$ is weakly separable
2) R^x is weakly separable
3) $card(A) \leq 2^{\omega}$
4) R^x is separable

Proof: we construct the following scheme using the proposition 15:

1) → 2) → 3) → 4) → 1)

Example 19. There exists a compact space X such that $C_p(X)$ is weakly separable but not separable.

For a space X, we can obtain a non-dimensional compact of arbitrary cardinality $card(A) \leq 2^{\omega}$. In fact, in this case, the result 17. and Theorem 18 satisfy the required conditions of $C_p(X)$.

In this scientific article, cardinal properties of weakly separable spaces are discussed and some other important properties are studied.

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