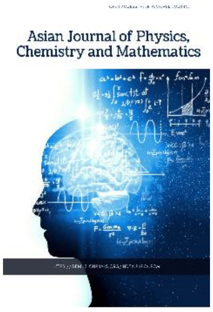
Volume 1| November 2021 ISSN: 2795-7667



Conditional Properties of Estimators of Repeated Measurements Model (Type I)

Hayder Abbood Kori

[a] Department of Economics, College of Administration and Economics, Thi-Qar University, Thi-Qar, Iraq
korihaydar@gmail.com

Abdulhussein Saber AL-Mouel

Department of mathematics, College of Education for Pure Sciences, University of Basrah, Basrah, Iraq
abdulhusseinsaber@yahoo.com

ABSTRACT

This paper studies the conditional biases and mean squared errors of the repeated measurements model for two types 1 and 2 estimators.

Keywords: Repeated Measurements Model; Type 1 Estimators; Type 2 Estimators

1. Introduction

Repeated measurements is a terminology used to describe data in which the response variable for each experimental units is observed on multiple times and possible under different experimental conditions. Repeated measurements analysis is used in several areas, for instance, in the health and life science, epidemiology, biomedical, agricultural, industrial, psychological, educational researches and so on,[13].

Many studies have explored the repeated measurement model, for example: Vonesh and Chinchilli (1997) discussed the univariate repeated measurements model , analysis of variance model,[13]. Al-Mouel and Wang in (2004) presented the sphericity test for the one-way multivariate repeated measurements analysis of variance model,[8]. Al-Mouel and Batto (2010) studied analysis of nested repeated measures model with applications,[3]. Naji and et al. (2011) studied analysis of variance and covariance repeated measurements model,[11] . Al-Mouel and Faker (2012) studied time-independent covariates in two-way multivariate repeated measurements model,[4]. Mohaisen and Swadi (2014) discussed a note on bayesian one-way repeated measurements model,[12]. Al-Mouel and Al-shmailawi (2016) studied a note of estimation of variance components by maximization,[7]. AL-Mouel, Mohaisen and Khawla (2017) studied bayesian one- way repeated measurements model based on bayes quadratic unbiased estimator,[6]. AL-Mouel and Al-Isawi (2018) computed the quadratic unbiased estimator, which has minimum variance (best quadratic unbiased estimate),[1]. Al-Isawi and Al-Mouel (2019) investigate the estimator of variance components of one-way repeated measurements model (RMM),[2]. This paper studies the conditional biases and mean squared errors of the one-way repeated measurements model for two types 1 and 2 estimators.

2.Setting Up The Model

The repeated measurement model can be summarized as following:

$h_{\text{abc}} = \theta + \text{A}_b + \pi_{a(b)} + \text{B}_c + (\text{AB})_{bc} + \epsilon_{abc}$ (1)

where

$a = 1, \dots, I$ "is an index for experimental unit within group (b)",

$b = 1, \dots, J$ "is an index for levels of the between-units factor (Group)",

$c = 1, \dots, K$ "is an index for levels of the within-units factor (Time)",

h_{abc} : "is the response measurement at time (c) for unit (a) within group (b)",

θ : "is the overall mean",

A_b : "is the added effect for treatment group (b)",

$\pi_{a(b)}$: "is the random effect for due to experimental unit (a) within treatment group (b)",

B_c : "is the added effect for time (c)",

$(\text{AB})_{bc}$: "is the added effect for the group $(b) \times$ time (c) interaction",

ϵ_{abc}: "is the random error on time (c) for unit (a) within group (b)".

For the parameterization to be of full rank, we imposed the following set of conditions:

$\sum_{b=1}^{J} \text{A}_b = 0;\ \sum_{c=1}^{K} \text{B}_c = 0;\ \sum_{b=1}^{J} (\text{AB})_{bc} = 0$ for each c = 1,…, K;

$\sum_{c=1}^{K} (\text{AB})_{bc} = 0$ for each b = 1,…, J.

and let, the ϵ_{abc} and $\pi_{a(b)}$are independent with

$\epsilon_{abc}\ i.i.d \sim N(0, \sigma_\epsilon^2)$ and $\pi_{a(b)}\ i.i.d \sim N(0, \sigma_\pi^2)$ (2)

Let

$\theta_{abc} = \theta + \text{A}_b + \pi_{a(b)} + \text{B}_c + (\text{AB})_{bc}$ (3)

represent the mean of time (c) for unit (a) within group (b).

and, let

$$H = \ell_0\theta + \sum_{b=1}^{J} \ell_b A_b + \sum_{a=1}^{I}\sum_{b=1}^{J} \ell_a \ell_b\, \pi_{a(b)} + \sum_{c=1}^{K} \ell_c B_c + \sum_{b=1}^{J}\sum_{c=1}^{k} \ell_b \ell_c (AB)_{bc} \qquad (4)$$

an arbitrary linear combination of parameters θ, $\text{A}_1, \dots, \text{A}_q$, $\pi_{1(1)}, \dots, \pi_{I(J)}$, $\text{B}_1, \dots, \text{B}_K$, $(\text{AB})_{11}, \dots, (\text{AB})_{JK}$.

the best linear unbiased estimators (BLUE's) of the estimable parameters $\theta, A_b, \pi_{a(b)}, B_c, (AB)_{bc}$and θ_{abc} are $\hat\theta = \bar h_{...}$, $\hat A_b = \bar h_{.b.} - \bar h_{...}$, $\hat\pi_{a(b)} = (1 - r)(\bar h_{ab.} - \bar h_{.b.})$, $\hat B_c = \bar h_{..c} - \bar h_{...}$, $\left(\widehat{AB}\right)_{bc} = \bar h_{.bc} + \bar h_{...} - \bar h_{.b.} - \bar h_{..c}$ and $\hat\theta_{abc} = (1 - r)\left(\bar h_{ab.} - \bar h_{.b.}\right) + \bar h_{.bc}$

where (r) is the rate of expected mean squares, note that $0 < r \le 1$ is known iff $\sigma_\epsilon^2/\sigma_\pi^2$ is known, [5].

by using maximum likelihood estimators and related estimators and Jeffreys' noninformative prior and proper bayes estimators, we have five groups as follows: [9] and [10]

Type 1: This type consists of estimators as follows:

$\hat\theta_1 = \bar h_{...}$,

$\hat\pi_{1;\, a(b)} = (1 - \hat r)(\bar h_{ab.} - \bar y_{.b.})$,

$\hat A_{1;\, b} = \hat A_{1,z\,;\, b} = \bar h_{.b.} - \bar h_{...}$,

$\hat B_{1;\, c} = \hat B_{1,z\,;\, c} = \bar h_{..c} - \bar h_{...}$,

$\left(\widehat{AB}\right)_{1;\, bc} = \left(\widehat{AB}\right)_{1,z;\, bc} = \bar h_{.bc} + \bar h_{...} - \bar h_{.b.} - \bar h_{..c}$,

$\hat\theta_{1;\, abc} = \hat\theta_{1,z;\, abc} = \bar h_{...} + \hat{\text{A}}_{1,z;\, b} + \hat\pi_{1,z;\, a(b)} + \hat{\text{B}}_{1,z;\, c} + \left(\widehat{\text{AB}}\right)_{1,z;bc}$,

$\hat\theta_{1,z;\, abc} = \bar h_{.bc} + \hat\pi_{1,z;\, a(b)}$,

and

$$\hat H_1 = \ell_0\theta + \sum_{b=1}^{J} \ell_b\left(\bar h_{.b.} - \bar h_{...}\right) + \sum_{a=1}^{I}\sum_{b=1}^{J} \ell_b \ell_c \left[(1 - \hat r)(\bar h_{ab.} - \bar y_{.b.})\right] + \sum_{c=1}^{K} \ell_c\left(\bar h_{..c} - \bar h_{...}\right) + \sum_{b=1}^{J}\sum_{c=1}^{K} \ell_b \ell_c\left(\bar h_{.bc} + \bar h_{...} - \bar h_{.b.} - \bar h_{..c}\right),$$

with

$\hat{r}_1 = \hat{r}_{1,z} = z\frac{SS_\epsilon}{SS_\pi}$.

where $(a = 1, \dots, I;\ b = 1, \dots, J;\ c = 1, \dots, K)$ and z is an arbitrary positive constant. [9] and [10]

Type 2: This type consists of estimators as follows:

$\hat{\theta}_2 = \bar{h}_{\dots}$,

$\hat{\pi}_{2;\,a(b)} = (1-\hat{r})(\bar{h}_{ab.} - \bar{y}_{.b.})$,

$\hat{A}_{2;\,b} = \hat{A}_{2,z\,;\,b} = \bar{h}_{.b.} - \bar{h}_{\dots}$,

$\hat{B}_{2;\,c} = \hat{B}_{2,z\,;\,c} = \bar{h}_{..c} - \bar{h}_{\dots}$,

$\left(\widehat{AB}\right)_{2;\,bc} = \left(\widehat{AB}\right)_{2,z;\,bc} = \bar{h}_{.bc} + \bar{h}_{\dots} - \bar{h}_{.b.} - \bar{h}_{..c}$,

$\hat{\theta}_{2;\,abc} = \hat{\theta}_{2,z;\,abc} = \bar{h}_{\dots} + \hat{\mathrm{A}}_{2,z;\,b} + \hat{\pi}_{2,z;\,a(b)} + \hat{\mathrm{B}}_{2,z;\,c} + \left(\widehat{\mathrm{AB}}\right)_{2,z;bc}$,

$\hat{\theta}_{2,z;\,abc} = \bar{h}_{.bc} + \hat{\pi}_{2,z;\,a(b)}$,

and

$$\hat{H}_1 = \ell_0\theta + \sum_{b=1}^{J}\ell_b\left(\bar{h}_{.b.} - \bar{h}_{\dots}\right) + \sum_{a=1}^{I}\sum_{b=1}^{J}\ell_b\ell_c\left[(1-\hat{r})(\bar{h}_{ab.} - \bar{y}_{.b.})\right] + \sum_{c=1}^{K}\ell_c\left(\bar{h}_{..c} - \bar{h}_{\dots}\right) + \sum_{b=1}^{J}\sum_{c=1}^{K}\ell_b\ell_c\left(\bar{h}_{.bc} + \bar{h}_{\dots} - \bar{h}_{.b.} - \bar{h}_{..c}\right),$$

with

$\hat{r}_2 = \hat{r}_{2,z} = \min\{z\frac{SS_\epsilon}{SS_\pi}, 1\}$,

where $(a = 1, \dots, I;\ b = 1, \dots, J;\ c = 1, \dots, K)$ and z is an arbitrary positive constant. [9] and [10]

Type 3: This type consists of estimators as follows:

$\hat{\theta}_3 = \bar{h}_{\dots}$,

$\hat{\pi}_{3;\,a(b)} = (1-\hat{r})(\bar{h}_{ab.} - \bar{y}_{.b.})$,

$\hat{A}_{3;\,b} = \hat{A}_{3,z\,;\,b} = \bar{h}_{.b.} - \bar{h}_{\dots}$,

$\hat{B}_{3;\,c} = \hat{B}_{3,z\,;\,c} = \bar{h}_{..c} - \bar{h}_{\dots}$,

$\left(\widehat{AB}\right)_{3;\,bc} = \left(\widehat{AB}\right)_{3,z;\,bc} = \bar{h}_{.bc} + \bar{h}_{\dots} - \bar{h}_{.b.} - \bar{h}_{..c}$,

$\hat{\theta}_{3;\,abc} = \hat{\theta}_{3,z;\,abc} = \bar{h}_{\dots} + \hat{\mathrm{A}}_{3,z;\,b} + \hat{\pi}_{3,z;\,a(b)} + \hat{\mathrm{B}}_{3,z;\,c} + \left(\widehat{\mathrm{AB}}\right)_{3,z;bc}$,

$\hat{\theta}_{3,z;\,abc} = \bar{h}_{.bc} + \hat{\pi}_{3,z;\,a(b)}$,

and

$$\hat{H}_3 = \ell_0\theta + \sum_{b=1}^{J}\ell_b\left(\bar{h}_{.b.} - \bar{h}_{\dots}\right) + \sum_{a=1}^{I}\sum_{b=1}^{J}\ell_b\ell_c\left[(1-\hat{r})(\bar{h}_{ab.} - \bar{y}_{.b.})\right] + \sum_{c=1}^{K}\ell_c\left(\bar{h}_{..c} - \bar{h}_{\dots}\right) + \sum_{b=1}^{J}\sum_{c=1}^{K}\ell_b\ell_c\left(\bar{h}_{.bc} + \bar{h}_{\dots} - \bar{h}_{.b.} - \bar{h}_{..c}\right),$$

with

$\hat{r}_3 = \hat{r}_{3,z} = \mathrm{f}_3(SS_\pi, SS_\epsilon)$,

where $(a = 1, \dots, I;\ b = 1, \dots, J;\ c = 1, \dots, K)$ and $\mathrm{f}_3(x, y)$ is an arbitrary function of $x, y > 0$. [9] and [10]

Type 4: This type consists of the following estimators:

$\hat{\theta}_4 = \bar{h}_{\dots}$,

$\hat{\pi}_{4;\,a(b)} = (1-\hat{r})(\bar{h}_{ab.} - \bar{y}_{.b.})$,

$\hat{A}_{4;\,b} = \hat{A}_{4,z\,;\,b} = \bar{h}_{.b.} - \bar{h}_{\dots}$,

$\hat{B}_{4;\,c} = \hat{B}_{4,z\,;\,c} = \bar{h}_{..c} - \bar{h}_{\dots}$,

$\left(\widehat{AB}\right)_{4;\,bc} = \left(\widehat{AB}\right)_{4,z;\,bc} = \bar{h}_{.bc} + \bar{h}_{\dots} - \bar{h}_{.b.} - \bar{h}_{..c}$,

$\hat{\theta}_{4;\,abc} = \hat{\theta}_{4,z;\,abc} = \bar{h}_{\dots} + \hat{\mathrm{A}}_{4,z;\,b} + \hat{\pi}_{4,z;\,a(b)} + \hat{\mathrm{B}}_{4,z;\,c} + \left(\widehat{\mathrm{AB}}\right)_{4,z;bc}$,

$\hat{\theta}_{4,z;\,abc} = \bar{h}_{.bc} + \hat{\pi}_{4,z;\,a(b)}$,

and

$$\widehat{H}_4 = \ell_0\theta + \sum_{b=1}^{J}\ell_b(\bar{h}_{.b.} - \bar{h}_{...}) + \sum_{a=1}^{I}\sum_{b=1}^{J}\ell_b\ell_c\left[(1-\hat{r})(\bar{h}_{ab.} - \bar{y}_{.b.})\right]$$
$$+\sum_{c=1}^{K}\ell_c(\bar{h}_{..c} - \bar{h}_{...}) + \sum_{b=1}^{J}\sum_{c=1}^{K}\ell_b\ell_c(\bar{h}_{.bc} + \bar{h}_{...} - \bar{h}_{.b.} - \bar{h}_{..c}),$$

with
$\hat{r}_4 = \hat{r}_4, z = \mathrm{f}_4\left(\frac{SS_\epsilon}{SS_\pi}\right),$
where $(i = 1, \dots, n, j = 1, \dots, q, k = 1, \dots, q)$ and $\mathrm{f}_4(x_1, x_2)$ is an arbitrary positive function of $x_1 > 0$ and $x_2 > 0$. [9] and [10]
Type 5: this type consists of estimators as follows:
$\hat{\theta}_5 = \frac{L\bar{h}_{...}+L^*\bar{h}^*_{...}}{(L+L^*)},$
$\hat{\pi}_{5;\,a(b)} = \hat{\pi}_{5,z\,;a(b)} = (1-\hat{r}_{5,z})(\bar{h}_{ab.} - \bar{h}_{.b.}),\ \ (a = 1, \dots, I;\ b = 1, \dots, J)$
$\hat{A}_{5,z;b} = \frac{L(\bar{h}_{.b.}-\bar{h}_{...})+L^*(\bar{h}^*_{.b.}-\bar{h}^*_{...})}{(L+L^*)},$
$\hat{B}_{5,z;c} = \frac{L(\bar{h}_{..c}-\bar{h}_{...})+L^*(\bar{h}^*_{..c}-\bar{h}^*_{...})}{(L+L^*)},$
$(\widehat{AB})_{5,z;\,bc} = \frac{L(\bar{h}_{.bc}+\bar{h}_{...}-\bar{h}_{.b.}-\bar{h}_{..c})+L^*(\bar{h}^*_{.bc}+\bar{h}^*_{...}-\bar{h}^*_{.c.}-\bar{h}^*_{..c})}{(L+L^*)}, (L = IJK, L^* = l^*_\epsilon + l^*_\pi + 1)$
$\hat{\theta}_{5;abc} = \frac{L\bar{h}_{...}-L^*\bar{h}^*_{...}}{(L+L^*)} + \frac{L(\bar{h}_{.b.}-\bar{h}_{...})+L^*(\bar{h}^*_{.b.}-\bar{h}^*_{...})}{(L+L^*)} + \hat{\pi}_{5,z\,;\,a(b)} + \frac{L(\bar{h}_{..c}-\bar{h}_{...})+L^*(\bar{h}^*_{..c}-\bar{h}^*_{...})}{(L+L^*)} +$
$\frac{L(\bar{h}_{.bc}+\bar{h}_{...}-\bar{h}_{.b.}-\bar{h}_{..c})+L^*(\bar{h}^*_{.bc}+\bar{h}^*_{...}-\bar{h}^*_{.b.}-\bar{h}^*_{..c})}{(L+L^*)}$
$\hat{\theta}_{5;\,abc} = \frac{L\bar{h}_{bc}-2L^*\bar{h}^*_{...}+L^*\bar{h}^*_{.bc}}{(L+L^*)} + \hat{\pi}_{5,z\,;\,a(b)}$
and
$\widehat{H}_5 = \ell_0\hat{\theta}_{5;\,abc} + \sum_{b=1}^{J}\ell_b A_{5,z;\,b} + \sum_{a=1}^{I}\sum_{b=1}^{j}\ell_a\ell_b\,\hat{\delta}_{5,z\,;\,a(b)} + \sum_{c=1}^{K}\ell_c\gamma_{5,z;\,c} + \sum_{b=1}^{J}\sum_{c=1}^{K}\ell_b\ell_c(\tau\gamma)_{5,z;\,bc}$
where $(a = 1, \dots, I;\ b = 1, \dots, J;\ c = 1, \dots, K)$ and $\hat{r}_5$ depends (nontrivially) on G as well as SS_ϵ and SS_π.We will discussed types 1 , 2. [9] and [10]
Let $\boldsymbol{x} = (x_{11}, x_{12}, \dots, x_{J(I-1)})^T$ be a normally distributed random matrix with mean $\boldsymbol{\theta} = (\theta_{11}, \theta_{12}, \dots, \theta_{J(I-1)})^T$ and variance covariance matrix **I**. Then $\boldsymbol{x}^T\boldsymbol{x} \sim \chi^2(J(I-1), \varepsilon)$, with $\varepsilon = \frac{\boldsymbol{\lambda}^T\boldsymbol{\lambda}}{2}$.
Define N be a Poisson random variable with parameter $\varepsilon > 0$ (if $\varepsilon = 0 \rightarrow N = 0$), let v represent a random variable whose joint distribution, with N, which is conditional distribution of v given $N \sim \chi^2(J(I-1) + 2N)$. Then, the p.d.f of the marginal distribution of v is:

$$g(x) = \sum_{\phi=0}^{\infty}\frac{e^{-\varepsilon}\varepsilon^{\phi}}{\phi!}\chi^2(x; J(I-1) + 2\phi) \qquad (5)$$

where $\chi^2(.\,; v)$ denote the p.d.f. of a central chi-square distribution with v degree of freedom.
Let f(.) be a measurable positive function such that:
1- $E[|x_{ab}|f(\boldsymbol{x}^T\boldsymbol{x})] < \infty$.
2- $E[|x_{ab}x_{bc}|f(\boldsymbol{x}^T\boldsymbol{x})] < \infty$. $\quad (a = 1, \dots, I-1;\ b = 1, \dots, J; c = 1, \dots, K)$

let $g(N) = E[f(v)|N]$ (6)

$$E[f(\boldsymbol{x}^T\boldsymbol{x})] = E\left[\sum_{J(I-1)=0}^{\infty}\frac{e^{-\frac{\varepsilon}{2}}\left(\frac{\varepsilon}{2}\right)^{J(I-1)}}{J(I-1)!}f_{(2I+J(I-1))}(x)\right]$$

$$E[f(\boldsymbol{x}^T\boldsymbol{x})] = E\big[E[f(v)|N]\big] = E[g(N)] \qquad (7)$$

In particular,

$$E\left[\frac{1}{x^Tx}\right] = E\left[\frac{1}{2N+J(I-1)-2}\right] \qquad (8)$$

$$E\left[\frac{1}{(x^Tx)^2}\right] = E\left[\frac{1}{(2N+J(I-1)-2)(2N+J(I-1)-4)}\right] \qquad (9)$$

an important recursive property of the Poisson distribution is

$$E[Ng(N)] = \varepsilon E[g(N+1)] \qquad (10)$$

Take the model to be the repeated measurements model.

Let $\bar{\boldsymbol{h}} = (\bar{h}_{11.}, \dots, \bar{h}_{IJ.})^T$ and $\boldsymbol{\theta} = (\theta_{111}, \dots, \theta_{IJK})^T$. Then conditional at π, $\bar{\boldsymbol{h}} \sim N\left(\boldsymbol{\theta}, \frac{\sigma_\epsilon^2}{K}\boldsymbol{I}\right)$. Define $\mathbf{1}$ to be an $IJ \times 1$ vector of 1's, and take

$$\boldsymbol{C} = \boldsymbol{I} - \frac{1}{I}\mathbf{1}^T\mathbf{1} \qquad (11)$$

Where $\boldsymbol{C}$ is a symmetric idempotent matrix of rank $J(I-1)$ and that

$$SS_\pi = K\sum_{a=1}^{I}\sum_{b=1}^{J}(\bar{h}_{ab.} - \bar{h}_{.b.})^2 = K\bar{\boldsymbol{h}}^T(\boldsymbol{C}\bar{\boldsymbol{h}}) \qquad (12)$$

let **P** be an orthogonal $IJ \times IJ$ matrix such that

$$\boldsymbol{P}^T(\boldsymbol{CP}) = \begin{bmatrix} \boldsymbol{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}. \qquad (13)$$

define

$$\boldsymbol{x} = (x_{11}, \dots, x_{\boldsymbol{IJ}})^{\boldsymbol{T}} = \frac{\sqrt{K}}{\sigma_\epsilon}\boldsymbol{P}^T\bar{\boldsymbol{h}} \qquad (14)$$

and

$$\boldsymbol{\lambda} = (\lambda_{11}, \dots, \lambda_{\boldsymbol{IJ}})^{\boldsymbol{T}} = \frac{\sqrt{K}}{\sigma_\epsilon}\boldsymbol{P}^T\boldsymbol{\theta} \qquad (15)$$

note that, conditional at $\boldsymbol{\pi}$, $\boldsymbol{x} \sim N(\boldsymbol{\lambda}, \boldsymbol{I})$. We also have that

$$SS_\pi = K\bar{\boldsymbol{h}}^T(\boldsymbol{C}h) = K\frac{\sigma_\epsilon}{\sqrt{K}}\boldsymbol{x}^{\boldsymbol{T}}(\boldsymbol{P}^{\boldsymbol{T}})^{-1}\frac{\sigma_\epsilon}{\sqrt{K}}(\boldsymbol{CP}^{-1}\boldsymbol{x}),\ \boldsymbol{P}^{-1} = \boldsymbol{P}^{\boldsymbol{T}}$$
$$= \sigma_\epsilon^2\boldsymbol{x}^{\boldsymbol{T}}\boldsymbol{P}^{\boldsymbol{T}}(\boldsymbol{CPx})$$
$$= \sigma_\epsilon^2\sum_{a=1}^{I-1}\sum_{b=1}^{J}x_{ab}^2 \qquad (16)$$

define

$$\boldsymbol{\Upsilon} = \begin{bmatrix} \varepsilon_{11} & \cdots & \varepsilon_{1J} \\ \vdots & \ddots & \vdots \\ \varepsilon_{I1} & \cdots & \varepsilon_{IJ} \end{bmatrix} = \frac{\sigma_\epsilon}{\sqrt{K}}(\boldsymbol{CP}) \qquad (17)$$

note that $\varepsilon_{aJ} = 0$ for $a = 1, \dots, I$, this mean that the last column of $\boldsymbol{\Upsilon}$ is null. We have that

$$\boldsymbol{C}\bar{\boldsymbol{h}} = \frac{\sigma_\epsilon}{\sqrt{K}}(\boldsymbol{CPx}) = \boldsymbol{\Upsilon x} \qquad (18)$$

and hence,

$$\bar{h}_{ab.} - \bar{h}_{.b.} = ab\ th \text{ components of } \boldsymbol{C}\bar{\boldsymbol{h}} = \sum_{c=1}^{J(I-1)}\varepsilon_{abc}x_c$$
$$(a = 1, \dots, I; b = 1, \dots, J) \qquad (19)$$

we have that

$$\boldsymbol{\Upsilon\Upsilon}^T = \left(\frac{\sigma_\epsilon}{\sqrt{K}}(\boldsymbol{CP})\right)\left(\frac{\sigma_\epsilon}{\sqrt{K}}(\boldsymbol{CP})\right)^T = \frac{\sigma_\epsilon^2}{K}(\boldsymbol{CPP}^{\boldsymbol{T}})\boldsymbol{C} = \frac{\sigma_\epsilon^2}{K}\boldsymbol{P} \qquad (20)$$

we implying that

$$\sum_{c=1}^{J(I-1)}\theta_{abc}^2 = \sum_{c=1}^{IJ}\theta_{abc}^2 = \frac{\sigma_\epsilon^2}{K}\ (ab\ th \text{ diagonal element of } \mathbf{C})$$
$$= \frac{\sigma_\epsilon^2(I-1)}{IJK}\ \ (a = 1, \dots, I-1; b = 1, \dots, J) \qquad (21)$$

and,

$$\boldsymbol{\Upsilon\lambda} = \frac{\sigma_\epsilon}{\sqrt{K}}(\boldsymbol{CP})\frac{\sqrt{K}}{\sigma_\epsilon}\boldsymbol{P}^T\boldsymbol{\theta} = \boldsymbol{CPP}^T\boldsymbol{\theta} = \boldsymbol{C\theta}, \qquad (22)$$

so that

$$\sum_{c=1}^{IJ}\varepsilon_{abc}\lambda_c = ab\ th \text{ element of } \boldsymbol{C\theta} = \pi_{a(b)} - \bar{\pi}_{.(b)} = \pi_{a(b)}^*$$
$$(a = 1, \dots, I-1; b = 1, \dots, J). \qquad (23)$$

Recall that $\boldsymbol{\pi}^* = (\pi_{1(b)}^*, \dots, \pi_{a(b)}^*)^{\boldsymbol{T}}$, $(b = 1, \dots, J)$. Using result (13), we have

$$\sum_{t=1}^{J(I-1)}\lambda_t^2 = \boldsymbol{\lambda}^T\boldsymbol{P}^T(\boldsymbol{CP\lambda}) = \frac{\sqrt{K}}{\sigma_\epsilon}\boldsymbol{P\theta}^{\boldsymbol{T}}\boldsymbol{P}^T\left(\boldsymbol{CP}\frac{\sqrt{K}}{\sigma_\epsilon}\boldsymbol{P}^T\boldsymbol{\theta}\right) = \frac{K}{\sigma_\epsilon^2}\boldsymbol{P\theta}^{\boldsymbol{T}}\boldsymbol{P}^T\boldsymbol{CPP}^T\boldsymbol{\theta}$$
$$= \frac{K}{\sigma_\epsilon^2}\boldsymbol{\mu}^{\boldsymbol{T}}\boldsymbol{P}^{\boldsymbol{T}}(\boldsymbol{P\theta}) = \frac{K}{\sigma_\epsilon^2}\sum_{a=1}^{I}\sum_{b=1}^{J}\pi_{a(b)}^{*2} = \frac{K}{\sigma_\epsilon^2}\boldsymbol{\pi}^{*T}\boldsymbol{\pi}^*. \qquad (24)$$

From (24) we implies that

$\varepsilon = \frac{K}{2\sigma_\epsilon^2}\boldsymbol{\pi}^{*T}\boldsymbol{\pi}^* = \frac{1}{2}\sum_{t=1}^{J(I-1)} \lambda_t^2$ (25)

Take N to be a random variable whose distribution is Poisson with parameter ε. If N = 0 which implies that $\varepsilon = 0$. And also $\varepsilon = 0$ iff $\boldsymbol{\pi}^* = \boldsymbol{0}$. Let t_1, t_2, t_3, t_4 and t_5 be a random variables such that, conditional on N, $t_1 \sim \chi^2(J(pK-1)(I-1))$, $t_2 \sim \chi^2(J(I-1)+2N), t_3 \sim \chi^2(J(I+1)+2N), t_4 \sim \chi^2(J(I-1))$ and $t_5 \sim \chi^2(J(I-1))$ with t_1 is distributed independently of t_2, t_3, t_4 and t_5. Define s_π, s_b, s_c and s_d to be random variables such that, conditional on N, $s_a \sim beta(v_\epsilon, v_\pi)$, $s_b \sim beta(v_\epsilon, v_\pi), s_c \sim beta(v_\epsilon, v_\pi)$ and $s_d \sim beta(v_\epsilon, v_\pi)$ with $v_\epsilon = \frac{J(K-1)(I-1)}{2}, v_\pi = \frac{J(I-1)+2N}{2}, v_b = \frac{J(I+1)+2N}{2}, v_c = \frac{J(I+1)}{2}$ and $v_d = \frac{J(I-1)}{2}$.

<u>Notice:</u> the symbol $CE[.]$ refer to the conditional expectations, $TCMSE[.]$ refer to total conditional mean square error and $TCB[.]$ denote the total conditional bias.

<u>Theorem 1</u>: Under the model (1), we have that,

$$CE\left[\frac{(\bar{h}_{ab.}-\bar{h}_{.b.})}{(SS_\pi)}\right]^2 = \begin{cases} \frac{1}{K\sigma_\epsilon^2}\left\{\begin{matrix} \frac{(I-1)}{I}E[(J(I-1)+2N))^{-1}(J(I-1)+2N-2)^{-1}] + \\ 2\pi_{a(b)}^*(\boldsymbol{\pi}^{*T}\boldsymbol{\pi}^*)^{-1}E[N(J(I-1)+2N))^{-1}(J(I-1)+2N-2)^{-1}] \end{matrix}\right\}, & \text{if } \boldsymbol{\pi}^* \neq \boldsymbol{0} \\ \frac{1}{IJK}(J(I-1)-2)^{-1} & \text{if } \boldsymbol{\pi}^* = \boldsymbol{0} \end{cases}$$

$$(I > 1; a = 1, \dots, I;\ b = 1, \dots, J) \quad (26)$$

<u>Corollary 1:</u> Let $\hat{r}_3 = f_3(\frac{SS_\epsilon}{SS_\pi})$ for some positive function f_3, and $\hat{\theta}_{3;abc}$ be the corresponding type 3 estimator of θ_{abc} $(a = 1, \dots, I;\ b = 1, \dots, J; c = 1, \dots, K)$. Then, under the model (1), we have that,

$$CMSE[(\hat{\theta}_{3;abc}, \theta_{abc})] =$$

$$\begin{cases} CE\left[(\hat{\theta}_{3;abc} - \theta_{abc})^2\right] = \frac{\sigma_\epsilon^2}{K}\{\frac{(K+I-1)}{I} + E\left[(J(I-1)(K-2)+2N)E\left[g_3\left(\frac{s_b}{1-s_b}\right)\right]\right] + \\ 2\pi_{a(b)}^*(\boldsymbol{\pi}^{*T}\boldsymbol{\pi}^*)^{-1}\left(E\left[Ng_3\left(\frac{s_b}{1-s_b}\right)\right] + E\left[2Nf_3\left(\frac{s_a}{1-s_a}\right)\right]\right)\}, & \text{if } \boldsymbol{\pi}^* \neq \boldsymbol{0} \\ \frac{\sigma_\epsilon^2}{K}\left\{\frac{(K+I-1)}{I} + E\left[(J(I-1)(K-2)+2N)g_3\left(\frac{s_c}{1-s_c}\right)\right]\right\} & \text{if } \boldsymbol{\pi}^* = \boldsymbol{0} \end{cases}$$

$$(a = 1, \dots, I;\ b = 1, \dots, J; c = 1, \dots, K) \quad (27)$$

<u>Corollary 2:</u> Let $\hat{r}_3 = f_3(\frac{SS_\epsilon}{SS_\pi})$ and $\widehat{\boldsymbol{\theta}}_{\boldsymbol{3}}$ be the corresponding type 3 estimator of $\boldsymbol{\theta}$. Then, under the model (1), we have that,

$$TCMSE(\widehat{\boldsymbol{\theta}}_3, \boldsymbol{\theta}) = \sum_{a=1}^{I}\sum_{b=1}^{J}\sum_{c=1}^{K} CE\left[(\hat{\theta}_{3;abc} - \theta_{abc})^2\right] = \sum_{a=1}^{I}\sum_{b=1}^{J}\sum_{c=1}^{K}\frac{\sigma_\epsilon^2}{K}E\left\{\frac{(K+I-1)}{I} + (J(I-1)(K-2) + 2N)E\left[g_3\left(\frac{s_\pi}{1-s_\pi}\right)(1-s_\pi)\right] + 2\pi_{a(b)}^*(\boldsymbol{\pi}^{*T}\boldsymbol{\pi}^*)^{-1}\left(Ng_3\left(\frac{s_\pi}{1-s_\pi}\right)(1-s_\pi) + 2Nf_3\left(\frac{s_\pi}{1-s_\pi}\right)\right)\right\}$$

$$\therefore TCMSE(\widehat{\boldsymbol{\theta}}_3, \boldsymbol{\theta}) = J\sigma_\epsilon^2 E\{(K+I-1) + IK(J(I-1)(K-2)+2N) \times E\left[g_3\left(\frac{s_\pi}{1-s_\pi}\right)(1-s_\pi)\right] + 2\left(\left[g_3\left(\frac{s_\pi}{1-s_\pi}\right)(1-s_\pi)\right] + 2\left[2Nf_3\left(\frac{s_\pi}{1-s_\pi}\right)\right]\right)\} \quad (28)$$

<u>Corollary 3</u>: Let $\hat{r}_3 = f_3(\frac{SS_\epsilon}{SS_\pi})$ for some positive function f_3 such that $CE[\hat{r}_3(\bar{h}_{ab.} - \bar{h}_{.b.})], (a = 1, \dots, I;\ b = 1, \dots, J)$ exsists. Then, under the one-way repeated measurement random model, we have that,

$$CE\left[f_3(\frac{SS_\epsilon}{SS_\pi})(\bar{h}_{ab.} - \bar{h}_{.b.})\right] = \begin{cases} \frac{2\sigma_\epsilon^2}{K}2\pi_{a(b)}^*(\boldsymbol{\pi}^{*T}\boldsymbol{\pi}^*)^{-1}E\left[Nf_3\left(\frac{s_\pi}{1-s_\pi}\right)\right] & \text{if } \boldsymbol{\pi}^* \neq \boldsymbol{0} \\ 0 & \text{if } \boldsymbol{\pi}^* = \boldsymbol{0} \end{cases}$$

$$(a = 1, \dots, I;\ b = 1, \dots, J) \quad (29)$$

<u>Lemma 1</u>: Let $\hat{r}_3 = f_3(\frac{SS_\epsilon}{SS_\pi})$, under the model (1), if $CE\left[f_3^2(\frac{SS_\epsilon}{SS_\pi})(\bar{h}_{ab.} - \bar{h}_{.b.})(\bar{h}_{a'b'.} - \bar{h}_{.b'.})\right]$ exists, then

$$CE\left[f_3^2(\tfrac{SS_\epsilon}{SS_\pi})\left(\bar{h}_{ab.}-\bar{h}_{.b.}\right)\left(\bar{h}_{a'b'.}-\bar{h}_{.b'.}\right)\right]=\pi^*_{a(b)}\pi^*_{a'(b')}CE\left[(1-s_a)f_3^2(\tfrac{s_\pi}{1-s_\pi})\right]$$
$$(a>a'=1,\ldots,I;\ b>b'=1,..,J) \qquad (30)$$

Theorem 2: Let $\hat{\theta}_{1,z;\,abc}$ be the corresponding type 1 estimator of θ_{abc}, and $z>0$. Then, under model (1), we have

$$CE\left[\hat{\theta}_{1,z;\,abc}-\theta_{abc}\right]=\begin{cases}-\dfrac{2z\sigma_\epsilon^2}{K}\pi^*_{a(b)}(\boldsymbol{\pi}^{*T}\boldsymbol{\pi}^*)^{-1}E\left[N\dfrac{J(K-1)(I-1)}{J(I-1)+2(N-1)}\right], if\boldsymbol{\pi}^*\neq 0\\ 0 \qquad if\boldsymbol{\pi}^*=0\end{cases}$$

0

$$(I>1, a=1,\ldots,I; b=1,\ldots,J; c=1,\ldots,K) \qquad (31)$$

Proof:

$CE\left[\left(\hat{\boldsymbol{\theta}}_{1,z;\,abc}-\boldsymbol{\theta}_{abc}\right)\right]=CE\left[\left(\bar{h}_{.bc}+(1-\hat{r}_1)\left(\bar{h}_{ab.}-\bar{h}_{.b.}\right)-\theta_{abc}\right)\right]$

$=CE\left[\bar{h}_{.bc}+\bar{h}_{ab.}-\bar{h}_{.b.}-\theta_{abc}-\hat{r}_1\left(\bar{h}_{ab.}-\bar{h}_{.c.}\right)\right]$

$=CE\left[(\bar{\epsilon}_{.bc}+\bar{\epsilon}_{ab.}-\bar{\epsilon}_{.b.})-\hat{r}_1\left(\bar{h}_{ab.}-\bar{h}_{.b.}\right)\right]$

$\therefore CE\left[\left(\hat{\theta}_{1,z;\,abc}-\theta_{abc}\right)\right]=-CE\left[\hat{r}_1\left(\bar{h}_{ab.}-\bar{h}_{.b.}\right)\right]$

using corollary (3),

$CE\left[f_3(\frac{SS_\epsilon}{SS_\pi})\left(\bar{h}_{ab.}-\bar{h}_{.b.}\right)\right]=\frac{2\sigma_\epsilon^2}{K}\pi^*_{a(b)}(\boldsymbol{\pi}^{*T}\boldsymbol{\pi}^*)^{-1}E\left[Nf_3\left(\frac{s_\pi}{1-s_\pi}\right)\right]$

and put $f_3(t)=zt, t>0$ (32)

we have,

$CE\left[\hat{\theta}_{1,z;\,abc}-\theta_{abc}\right]=-\frac{2z\sigma_\epsilon^2}{p}\pi^*_{a(b)}(\boldsymbol{\pi}^{*T}\boldsymbol{\pi}^*)^{-1}E[Ns_\pi(1-s_\pi)^{-1}]$

$s_\pi(1-s_\pi)^{-1}=\int_0^\infty s_\pi(1-s_\pi)^{-1}\frac{1}{\beta(v_\epsilon,v_\pi)}s_\pi^{v_\epsilon-1}(1-s_\pi)^{v_\pi-1}\,ds_\pi$

$=\frac{v_\epsilon}{(v_\pi-1)}$, where $v_\epsilon=\frac{J(K-1)(I-1)}{2}, v_\pi=\frac{J(I-1)+2N}{2}$, we get

$CE\left[\hat{\theta}_{1,z;\,abc}-\theta_{abc}\right]=-\frac{2z\sigma_\epsilon^2}{K}\pi^*_{a(b)}(\boldsymbol{\pi}^{*T}\boldsymbol{\pi}^*)^{-1}E\left[N\frac{J(K-1)(I-1)}{J(I-1)+2(N-1)}\right]$

Corollary 4: Let $\hat{\theta}_{1,z;\,abc}$ be the corresponding type 1 Estimator of θ_{abc}, and $z>0$. Then, under the model (1), we have

$TCB\left[\hat{\boldsymbol{\theta}}_{1,z}-\boldsymbol{\theta}|\boldsymbol{\pi}\right]=0$ (33)

Proof:

$TCB\left[\hat{\boldsymbol{\theta}}_{1,z}-\boldsymbol{\theta}\right]=E\left[\sum_{a=1}^I\sum_{b=1}^J\sum_{c=1}^K\left(\hat{\theta}_{1,z;\,abc}-\theta_{abc}\right)|\boldsymbol{\pi}\right]$

$=E\left[\sum_{a=1}^I\sum_{b=1}^J\sum_{c=1}^K\bar{h}_{.bc}+\left(1-\hat{r}_{1,z}\right)\left(\bar{h}_{ab.}-\bar{h}_{.b.}\right)-\theta_{abc}|\boldsymbol{\pi}\right]$

$=E\left[\sum_{a=1}^I\sum_{b=1}^J\sum_{c=1}^K\left(\bar{h}_{.bc}+\bar{h}_{ab.}-\bar{h}_{.b.}-\theta_{abc}\right)-\sum_{a=1}^I\sum_{b=1}^J\sum_{c=1}^K\hat{r}_{1,z}\left(\bar{h}_{ab.}-\bar{h}_{.b.}\right)|\boldsymbol{\pi}\right]$

$=E\left[\sum_{a=1}^I\sum_{b=1}^J\sum_{c=1}^K(\bar{\epsilon}_{.bc}+\bar{\epsilon}_{ab.}-\bar{\epsilon}_{.b.})-\sum_{a=1}^I\sum_{b=1}^J\sum_{c=1}^K\hat{r}_{1,z}\left(\pi_{(a)b}-\bar{\pi}_{.(b)}+\bar{\epsilon}_{ab.}-\bar{\epsilon}_{.b.}\right)|\boldsymbol{\pi}\right]=0.$

Theorem 3: Let $\hat{\theta}_{1,z;\,abc}$ be the corresponding type 1 estimator of θ_{abc}, and $z>0$. Then, under the model (1), we have

$CE\left[\hat{\theta}_{1,z;\,abc}-\theta_{abc}\right]=-\frac{2z\sigma_\epsilon^2}{K}\pi^*_{a(b)}(\boldsymbol{\pi}^{*T}\boldsymbol{\pi}^*)^{-1}\left(\frac{J(K-1)(I-1)}{J(I-1)+2(N-1)}\right)E[N]$ (34)

Proof:

$CE\left[\left(\hat{\theta}_{1,z;\,abc}-\theta_{abc}\right)\right]=CE\left[\left(\bar{h}_{.bc}+(1-\hat{r}_1)\left(\bar{h}_{ab.}-\bar{h}_{.b.}\right)-\theta_{abc}\right)\right]$

$=CE\left[\bar{h}_{.bc}+\bar{h}_{ab.}-\bar{h}_{.b.}-\theta_{abc}-\hat{r}_1\left(\bar{h}_{ab.}-\bar{h}_{.b.}\right)\right]$

$=CE\left[(\bar{\epsilon}_{.bc}+\bar{\epsilon}_{ab.}-\bar{\epsilon}_{.b.})-\hat{r}_1\left(\bar{h}_{ab.}-\bar{h}_{.b.}\right)\right]$

$\therefore CE\left[\left(\hat{\theta}_{1,z;\,abc}-\theta_{abc}\right)\right]=-CE\left[\hat{r}_1\left(\bar{h}_{ab.}-\bar{h}_{.b.}\right)\right]$

using corollary (3),

$CE\left[f_3(\frac{SS_\epsilon}{SS_\pi})\left(\bar{h}_{ab.}-\bar{h}_{.b.}\right)\right]=\frac{2\sigma_\epsilon^2}{K}\pi^*_{a(b)}(\boldsymbol{\pi}^{*T}\boldsymbol{\pi}^*)^{-1}E\left[Nf_3\left(\frac{s_\pi}{1-s_\pi}\right)\right]$

using result (32), we have

$CE\left[\hat{\theta}_{1,z;\,abc}-\theta_{abc}\right]=-\frac{2\sigma_\epsilon^2}{K}\pi^*_{a(b)}(\boldsymbol{\pi}^{*T}\boldsymbol{\pi}^*)^{-1}E[Ns_\pi(1-s_\pi)^{-1}]$

$E[s_\pi(1-s_\pi)^{-1}]=\int_0^1 s_\pi(1-s_\pi)^{-1}\frac{1}{\beta(v_\epsilon,v_\pi)}s_\pi^{v_\epsilon-1}(1-s_\pi)^{v_\pi-1}\,ds_\pi=\frac{v_\epsilon}{(v_\pi-1)}$

where $v_\epsilon=\frac{J(K-1)(I-1)}{2}, v_\pi=\frac{J(I-1)+2N}{2}$, we get

$CE\left[\hat{\theta}_{1,z;\,abc}-\theta_{abc}\right]=-\frac{2\sigma_\epsilon^2}{K}\pi^*_{a(b)}(\boldsymbol{\pi}^{*T}\boldsymbol{\pi}^*)^{-1}\left(\frac{J(K-1)(I-1)}{J(I-1)+2(N-1)}\right)E[N]$

Theorem 4: Let $\widehat{\boldsymbol{\theta}}_{\mathbf{1},\boldsymbol{z}}$ be the corresponding type 1 estimator of $\boldsymbol{\theta}$ and z > 0. Then, under the model (1), we have that,

$$TCMSE\left(\widehat{\boldsymbol{\theta}}_{\mathbf{1},\boldsymbol{z}},\boldsymbol{\theta}\right)=J\sigma_\epsilon^2E\{(K+I-1)+IK(J(I-1)(K-2)+2N)\times\left[\frac{z^2J(I-1)(K-1)[J(I-1)(K-1)+2]}{[J(I-1)(K-1)+J(I-1)+2N][J(I-1)+2N-2]}-\frac{2zJ(I-1)(K-1)}{[J(I-1)(K-1)+K(I-1)+2N]}\right]+2\left(\left[\frac{z^2J(I-1)(K-1)[J(I-1)(K-1)+2]}{[J(I-1)(K-1)+J(I-1)+2N][J(I-1)+2N-2]}-\frac{2zJ(I-1)(K-1)}{[J(I-1)(K-1)+J(I-1)+2N]}\right]+2\left[2z\frac{J(I-1)(K-1)}{[J(I-1)+2N-2]}\right]\right)\}\quad(35)$$

Proof: using corollary (2), we have

$$\therefore TCMSE\left(\widehat{\boldsymbol{\theta}}_3,\boldsymbol{\theta}\right)=J\sigma_\epsilon^2E\left\{(K+I-1)+IK(J(I-1)(K-2)+2N)\left[g_3\left(\frac{s_\pi}{1-s_\pi}\right)(1-s_\pi)\right]+2\left(\left[g_3\left(\frac{s_\pi}{1-s_\pi}\right)(1-s_\pi)\right]+2\left[2Nf_3\left(\frac{s_\pi}{1-s_\pi}\right)\right]\right)\right\}$$

we define $g_3(.)$ and $f_3(.)$ as

$$g_3(u)=z^2u^2-2zu,\qquad (u>0).\qquad(36)$$

$$E\left[g_3\left(\frac{s_\pi}{1-s_\pi}\right)(1-s_\pi)\middle|N\right]=E\left[z^2\left(\frac{s_\pi}{1-s_\pi}\right)^2(1-s_\pi)-2z\left(\frac{s_\pi}{1-s_\pi}\right)(1-s_\pi)\middle|N\right]$$

$$=E[z^2s_\pi^2(1-s_\pi)^{-1}-2zs_\pi|N]$$, we have

$$TCMSE(\widehat{\boldsymbol{\mu}}_1,\boldsymbol{\mu})=q\sigma_\epsilon^2E\{(K+I-1)+IK(J(I-1)(K-2)+2N)[z^2s_\pi^2(1-s_\pi)^{-1}-2zs_\pi|N]+2([z^2s_\pi^2(1-s_\pi)^{-1}-2zs_\pi|N]+2[2zs_\pi(1-s_\pi)^{-1}])\}\quad(37)$$

$$E[s_\pi^2(1-s_\pi)^{-1}]=\int_0^1 s_\pi^2(1-s_\pi)^{-1}\frac{1}{\beta(v_\epsilon,v_\pi)}s_\pi^{v_\epsilon-1}(1-s_\pi)^{v_\pi-1}\,ds_\pi=\frac{v_\epsilon(v_\epsilon+1)}{(v_\epsilon+v_\pi)(v_\pi-1)}$$

$$E[s_\pi^2(1-s_\pi)^{-1}]=\frac{J(I-1)(K-1)[J(I-1)(K-1)+2]}{[J(I-1)(K-1)+J(I-1)+2N][J(I-1)+2N-2]}\qquad(38)$$

$$E[s_\pi]=\int_0^1 s_\pi\frac{1}{\beta(v_\epsilon,v_\pi)}s_\pi^{v_\epsilon-1}(1-s_\pi)^{v_\pi-1}\,ds_\pi=\frac{v_\epsilon}{(v_\epsilon,v_\pi)}$$

$$E[s_\pi]=\frac{J(I-1)(K-1)}{[J(I-1)(K-1)+J(I-1)+2N]}\qquad(39)$$

and

$$E[s_\pi(1-s_\pi)^{-1}]=\int_0^1 s_\pi(1-s_\pi)^{-1}\frac{1}{\beta(v_\epsilon,v_\pi)}s_\pi^{v_\epsilon-1}(1-s_\pi)^{v_\pi-1}\,ds_\pi$$

$$E[s_\pi(1-s_\pi)^{-1}]=\frac{J(I-1)(K-1)}{[J(I-1)+2N-2]}\qquad(40)$$

substituting the results (38), (39) and (40) in (37), we have

$$TCMSE\left(\widehat{\boldsymbol{\theta}}_1,\boldsymbol{\theta}\right)=J\sigma_\epsilon^2E\{(K+I-1)+IK(J(I-1)(K-2)+2N)\times\left[\frac{z^2J(I-1)(K-1)[J(I-1)(K-1)+2]}{[J(I-1)(K-1)+J(I-1)+2N][J(I-1)+2N-2]}-\frac{2zJ(I-1)(K-1)}{[J(I-1)(K-1)+J(I-1)+2N]}\right]+2\left(\left[\frac{z^2J(I-1)(K-1)[J(I-1)(K-1)+2]}{[J(I-1)(K-1)+J(I-1)+2N][J(I-1)+2N-2]}-\frac{2zJ(I-1)(K-1)}{[J(I-1)(K-1)+J(I-1)+2N]}\right]+2\left[2z\frac{J(I-1)(K-1))}{[J(I-1)+2N-2]}\right]\right)\}$$

result (46) is minimized uniformly with respect to z by

$$z^*=\frac{J(I-3)}{(IJK-JK-IJ-J+2)},\quad (I>3).\qquad(41)$$

The value of z^* is minimizes of $MSE\left(\widehat{\boldsymbol{\theta}}_{1,z;\,abc}-\boldsymbol{\theta}_{abc}\right)$ uniformly. The estimator

Theorem 5: Let $\hat{\theta}_{1,z;abc}$ be the corresponding type 1 estimator of $\theta_{abc}\,(a=1,\dots,I; b=1,..,J; c=1,..,K)$ and z > 0. Then, under the model (1), we have that,

$$CMSE[(\hat{\boldsymbol{\theta}}_{1,z;\,abc}-\boldsymbol{\theta}_{abc})]=\frac{\sigma_\epsilon^2}{K}\{\frac{(K+I-1)}{I}+E[J(I-1)(K-2)+2N\,)$$
$$(\frac{z^2J(I-1)(K-1)[J(I-1)(K-1)+2]}{[J(I-1)(K-1)+J(I-1)+2N][J(I-1)+2N-2]}-\frac{2zJ(I-1)(K-1)}{[J(I-1)(K-1)+J(I-1)+2N]})]+$$
$$2\alpha_{i(j)}^*(\boldsymbol{\pi}^{*T}\boldsymbol{\pi}^*)^{-1}\left(\left[\frac{z^2J(I-1)(K-1)[J(I-1)(K-1)+2]}{[J(I-1)(K-1)+J(I-1)+2N][J(I-1)+2N-2]}-\frac{2zJ(I-1)(K-1)}{[J(I-1)(K-1)+J(I-1)+2N]}\right]+\left[2z\frac{J(I-1)(K-1)}{[J(I-1)+2N-2]}\right]\right)\}$$
$$if\ \boldsymbol{\pi}^*\neq 0 \qquad CMSE[(\hat{\theta}_{1,z;abc},\theta_{abc})]=\frac{\sigma_\epsilon^2}{K}\{\frac{(K+I-1)}{I}+$$
$$E[J(I-1)(K-2)$$
$$\times\left[\frac{z^2J(I-1)(K-1)[J(I-1)(K-1)+2]}{[J(I-1)(K-1)+J(I-1)][J(I-1)-2]}-\frac{2zJ(I-1)(K-1)}{[J(I-1)(K-1)+J(I-1)]}\right]]\} \qquad if\ \boldsymbol{\pi}^*=0$$
$$(a=1,\dots,I;b=1,\dots,J;c=1,\dots,K) \qquad (42)$$

Proof: Using theorem (1), we have

$$CMSE[(\hat{\theta}_{3;abc},\theta_{abc})]=CE\left[(\hat{\theta}_{3;abc}-\theta_{abc})^2\right]=\frac{\sigma_\epsilon^2}{K}\left\{\frac{(K+I-1)}{I}+E\left[(J(I-1)(K-2)+\right.\right.$$
$$\left.\left.2N\,)E\left[g_3\left(\frac{s_\pi}{1-s_\pi}\right)(1-s_\pi)\right]\right]+2\pi_{a(b)}^*(\boldsymbol{\pi}^{*T}\boldsymbol{\pi}^*)^{-1}\left(E\left[Ng_3\left(\frac{s_\pi}{1-s_\pi}\right)(1-s_\pi)\right]+E\left[2Nf_3\left(\frac{s_\pi}{1-s_\pi}\right)(1-s_\pi)\right]\right)\right\}$$

using results (32) and (36), we have

$$CMSE[(\hat{\theta}_{3;abc},\theta_{abc})]=\frac{\sigma_\epsilon^2}{K}\{\frac{(K+I-1)}{I}+E[(J(I-1)(K-2)+2N\,)E[z^2s_\pi^2(1-s_\pi)^{-1}-2zs_\pi|N]]+$$
$$2\pi_{a(b)}^*(\boldsymbol{\pi}^{*T}\boldsymbol{\pi}^*)^{-1}(E[z^2s_\pi^2(1-s_\pi)^{-1}-2zs_\pi|N]+E[2zs_\pi(1-s_\pi)^{-1}])\} \qquad (43)$$

substituting results (38), (39) and (40) in (43), we have

$$CMSE\left[\left((\hat{\theta}_{3;abc},\theta_{abc})\right)\right]=\frac{\sigma_\epsilon^2}{K}\left\{\frac{(K+I-1)}{I}+E\left[(J(I-1)(K-2)+\right.\right.$$
$$\left.2N\,)\left[(\frac{z^2J(I-1)(K-1)[J(I-1)(K-1)+2]}{[J(I-1)(K-1)+J(I-1)+2N][J(I-1)+2N-2]}-\frac{2zJ(I-1)(K-1)}{[J(I-1)(K-1)+J(I-1)+2N]}\right]\right]+$$
$$\left.2\pi_{a(b)}^*(\boldsymbol{\pi}^{*T}\boldsymbol{\pi}^*)^{-1}\left(\left[\frac{z^2J(I-1)(K-1)[J(I-1)(K-1)+2]}{[J(I-1)(K-1)+J(I-1)+2N][J(I-1)+2N-2]}-\frac{2zJ(I-1)(K-1)}{[J(I-1)(K-1)+J(I-1)+2N]}\right]+\left[2z\frac{J(I-1)(K-1)}{[J(I-1)+2N-2]}\right]\right)\right\}$$
$$if\ \boldsymbol{\pi}^*\neq 0$$
$$MSE\left[\left((\hat{\theta}_{3;abc},\theta_{abc})\right)\right]=\frac{\sigma_\epsilon^2}{K}\left\{\frac{(K+I-1)}{I}+E\left[(J(I-1)(K-2)\,)\left[(\frac{z^2J(I-1)(K-1)[J(I-1)(K-1)+2]}{[J(I-1)(K-1)+J(I-1)+2N][J(I-1)+2N-2]}-\right.\right.\right.$$
$$\left.\left.\left.\frac{2zJ(I-1)(K-1)}{[J(I-1)(K-1)+J(I-1)+2N]}\right]\right]\right\} \qquad if\ \boldsymbol{\pi}^*=0$$

Theorem 6: Let $\hat{\theta}_{1,z;abc}$ be the corresponding type 1 estimator of θ_{abc} $(a=1,\dots,I;b=1,..,J;c=1,..,K)$ and z > 0. Under the model (1), we have that,

$$CE[(\hat{\theta}_{1,z;abc}-\theta_{abc})(\hat{\theta}_{1,z;a'b'c'}-\theta_{a'b'c'})]=$$
$$\begin{cases}\pi_{a(b)}^*\pi_{a'(b')}^*\left(\frac{z^2J(I-1)(K-1)[J(I-1)(J-1)+2]}{[J(I-1)(K-1)+J(I-1)][J(I-1)-2]}-\frac{2zJ(I-1)(K-1)}{[J(I-1)(K-1)+J(I-1)+2N]}\right)\\ +\frac{2\sigma_\epsilon^2}{k}(\boldsymbol{\pi}^{*T}\boldsymbol{\pi}^*)^{-1}\left(\frac{J(I-1)(K-1)}{[J(I-1)+2N-2]}\right) & if\boldsymbol{\pi}^*\neq\mathbf{0}\\ 0 & if\boldsymbol{\pi}^*=\mathbf{0}\end{cases}$$
$$I,K>1 \quad (a=1,\dots,I;b=1,\dots,J;c=1,\dots,K) \qquad (44)$$

Proof: Since

$$CE[(\hat{\theta}_{1,z;abc}-\theta_{abc})(\hat{\theta}_{1,z;a'b'c'}-\theta_{a'b'c'})]=\frac{(K+I-1)}{IK}\sigma_\epsilon^2-2\pi_{a(b)}^*\pi_{a'(b')}^*E\left[g_3\left(\frac{s_b}{1-s_b}\right)+\right.$$
$$\left.\frac{2\sigma_\epsilon^2}{K}(\boldsymbol{\pi}^{*T}\boldsymbol{\pi}^*)^{-1}Nf_3\left(\frac{s_\pi}{1-s_\pi}\right)\right]$$
$$(a>a'=1,\dots,I;\ b>b'=1,..,J);\ c>c'=1,..,K)$$

using results (32) and (36). then, we have

$CE[(\hat{\theta}_{1,z;abc}-\theta_{abc})(\hat{\theta}_{1,z;a'b'c'}-\theta_{a'b'c'})]=\pi^*_{a(b)}\pi^*_{a'(b')}E[z^2s_\pi^2(1-s_\pi)^{-1}-2zs_\pi|N]+$
$\frac{2\sigma_\epsilon^2}{K}(\boldsymbol{\pi}^{*T}\boldsymbol{\pi}^*)^{-1}E[zs_\pi(1-s_\pi)^{-1}]$ (45)

$(a>a'=1,\dots,I;\ b>b'=1,..,J);\ c>c'=1,..,K)$

substituting results (38), (39) and (50) in (45), we have

$CE[(\hat{\theta}_{1,z;abc}-\theta_{abc})(\hat{\theta}_{1,z;a'b'c'}-\theta_{a'b'c'})]=\pi^*_{a(b)}\pi^*_{a'(b')}(\frac{z^2J(I-1)(K-1)[J(I-1)(K-1)+2]}{[J(I-1)(K-1)+J(I-1)][J(I-1)-2]}-$
$\frac{2zJ(I-1)(K-1)}{[J(I-1)(K-1)+J(I-1)+2N]}+\frac{2\sigma_\epsilon^2}{K}(\boldsymbol{\pi}^{*T}\boldsymbol{\pi}^*)^{-1}\left(\frac{J(I-1)(K-1)}{[J(I-1)+2N-2]}\right)$

Lemma 2: Let $\hat{\theta}_{1,z;abc}$ be the corresponding type 1 estimator of $\theta_{abc}\,(a=1,\dots,I;b=1,..,J;c=1,..,K)$ and z > 0. Under the model (1), we have that,

$CE\left[z\frac{SS_\epsilon}{SS_\pi}(\bar{h}_{ab.}-\bar{h}_{.b.})(\bar{h}_{a'b'.}-\bar{h}_{.b'.})\right]=z\pi^*_{a(b)}\pi^*_{a'(b')}\frac{J(I-1)(K-1)}{[J(I-1)(K-1)+J(I-1)+2N]}$

$(a>a'=1,\dots,I;\ b>b'=1,..,J)$ (46)

Proof: Since

$CE\left[f_3(\frac{SS_\epsilon}{SS_\pi})(\bar{h}_{ab.}-\bar{h}_{.b.})(\bar{h}_{a'b'.}-\bar{h}_{.b'.})\right]=\pi^*_{a(b)}\pi^*_{a'(b')}CE\left[(1-s_\pi)f_3(\frac{s_\pi}{1-s_\pi})\right]$

using result (32), we have

$CE\left[f_3(\frac{SS_\epsilon}{SS_\pi})(\bar{h}_{ab.}-\bar{h}_{.b.})(\bar{h}_{a'b'.}-\bar{h}_{.b'.})\right]=z\pi^*_{a(b)}\pi^*_{a'(b')}CE[s_\pi]$

using result (41), we have

$CE\left[z\frac{SS_\epsilon}{SS_\pi}(\bar{h}_{ab.}-\bar{h}_{.b.})(\bar{h}_{a'b'.}-\bar{h}_{.b'.})\right]=z\pi^*_{a(b)}\pi^*_{a'(b')}\frac{J(I-1)(K-1)}{[J(I-1)(K-1)+J(I-1)+2N]}$

Lemma 3: Let $\hat{\theta}_{1,z;abc}$ be the corresponding type 1 estimator of $\theta_{abc}\,(a=1,\dots,I;b=1,..,J;c=1,..,K)$ and z > 0. Under the model (1), we have that,

$E\left[\left(z\frac{SS_\epsilon}{SS_\pi}\right)^2(\bar{h}_{ab.}-\bar{h}_{.b.})(\bar{h}_{a'b'.}-\bar{h}_{.b'.})\right]=\pi^*_{a(b)}\pi^*_{a'(b')}\frac{J(I-1)(K-1)[J(I-1)(K-1)+2]}{[J(I-1)(K-1)+J(I-1)+2N][J(I-1)+2N-2]}$

$(a>a'=1,\dots,I;\ b>b'=1,..,J)$ (47)

Proof: By using lemma (1),we have

$CE\left[f_3^2(\frac{SS_\epsilon}{SS_\pi})(\bar{h}_{ab.}-\bar{h}_{.b.})(\bar{h}_{a'b'.}-\bar{h}_{.b'.})\right]=CE\left[\left(z\frac{SS_\epsilon}{SS_\pi}\right)^2(\bar{h}_{ab.}-\bar{h}_{.b.})(\bar{h}_{a'b'.}-\bar{h}_{.b'.})\right]=$
$\pi^*_{a(b)}\pi^*_{a'(b')}CE\left[(1-s_\pi)f_3^2(\frac{s_\pi}{1-s_\pi})\right]$

using result (32), we have

$CE\left[f_3^2(\frac{SS_\epsilon}{SS_\pi})(\bar{h}_{ab.}-\bar{h}_{.b.})(\bar{h}_{a'b'.}-\bar{h}_{.b'.})\right]=\pi^*_{a(b)}\pi^*_{a'(b')}CE[z^2s_\pi^2(1-s_\pi)^{-1}]$

Using result (38), we have

$CE\left[\left(z\frac{SS_\epsilon}{SS_\pi}\right)^2(\bar{h}_{ab.}-\bar{h}_{.b.})(\bar{h}_{a'b'.}-\bar{h}_{.b'.})\right]=\pi^*_{a(b)}\pi^*_{a'(b')}\frac{J(I-1)(K-1)[J(I-1)(K-1)+2]}{[J(I-1)(K-1)+J(I-1)+2N][J(I-1)+2N-2]}$

$(a>a'=1,\dots,I;\ b>b'=1,..,J)$

Lemma 4 Let $\hat{\theta}_{1,z;abc}$ be the corresponding type 1 estimator of $\theta_{abc}\,(a=1,\dots,I;b=1,..,J;c=1,..,K)$ and z > 0, under the model (1), we have that,

$CE\left[z\frac{SS_\epsilon}{SS_\pi}(\bar{h}_{ab.}-\theta_{abc})(\bar{h}_{a'b'.}-\bar{h}_{.b'.})\right]=\pi^*_{a(b)}\pi^*_{a'(b')}\left(\frac{J(I-1)(K-1)}{[J(I-1)(K-1)+J(I-1)+2N]}\right)-\frac{2\sigma_\epsilon^2}{K}\pi^*_{ia(b)}\big(\pi^*_{a(b)}-$
$D\big)(\boldsymbol{\pi}^{*T}\boldsymbol{\pi}^*)^{-1}\left(\frac{J(I-1)(K-1)}{[J(I-1)+2N-2]}\right)$ $if\ \boldsymbol{\pi}^*\neq\mathbf{0}$

= 0 $if\ \boldsymbol{\pi}^*=\mathbf{0}$

$D=-(B_c+(AB)_{bc}),(a>a'=1,\dots,I;\ b>b'=1,..,J)$ (48)

Proof: Using lemma (1), we have

$CE\left[z\frac{SS_\epsilon}{SS_\pi}(\bar{h}_{ab.}-\theta_{abc})(\bar{h}_{a'b'.}-\bar{h}_{.b'.})\right]=\pi^*_{a(b)}\pi^*_{a'(b')}E\left[f_3\left(\frac{s_b}{1-s_b}\right)\right]-\frac{2\sigma_\epsilon^2}{K}\pi^*_{a(b)}\big(\pi^*_{a(b)}-$
$D\big)(\boldsymbol{\pi}^{*T}\boldsymbol{\pi}^*)^{-1}E\left[Nf_3\left(\frac{s_\pi}{1-s_\pi}\right)\right]$

$D=-(B_c+(AB)_{bc})$,$(a>a'=1,\dots,I;\ b>b'=1,..,J)$

using result (32), we have

$CE\left[z\frac{SS_\epsilon}{SS_\pi}(\bar{h}_{ab.}-\theta_{abc})(\bar{h}_{a'b'.}-\bar{h}_{.b'.})\right]=\pi^*_{a(b)}\pi^*_{a'(b')}E[s_\pi]-\frac{2\sigma_\epsilon^2}{K}\pi^*_{a(b)}(\pi^*_{a(b)}-D)(\boldsymbol{\pi}^{*T}\boldsymbol{\pi}^*)^{-1}E\left[Nz\frac{s_\pi}{1-s_\pi}\right]$

$D=-(B_c+(AB)_{bc})$,$(a>a'=1,\dots,I;\ b>b'=1,..,J)$

Using results (39) and (40). Then, we have that

$CE\left[z\frac{SS_\epsilon}{SS_\pi}(\bar{h}_{ab.}-\theta_{abc})(\bar{h}_{a'b'.}-\bar{h}_{.b'.})\right]=\pi^*_{a(b)}\pi^*_{a'(b')}\left(\frac{J(I-1)(K-1)}{[J(I-1)(K-1)+J(I-1)+2N]}\right)-\frac{2\sigma_\epsilon^2}{K}\pi^*_{a(b)}(\pi^*_{a(b)}-D)(\boldsymbol{\pi}^{*T}\boldsymbol{\pi}^*)^{-1}\left(\frac{J(I-1)(K-1)}{[J(I-1)+2N-2]}\right)$ $\quad if\ \boldsymbol{\pi}^*\neq\mathbf{0}$

$=0 \quad if\ \boldsymbol{\pi}^*=\mathbf{0}$

$D=-(B_c+(AB)_{bc})$,$(a>a'=1,\dots,I;\ b>b'=1,..,J)$

The following theorem gives an expression (in terms of incomplete beta function ratios) for the conditional bias of a type 2 estimators of θ_{abc}.

Theorem 7: Let $\hat{\theta}_{2,z;abc}$ be the corresponding type 2 estimator of θ_{abc} $(a=1,\dots,I;b=1,..,J;c=1,..,K)$ and z > 0, under the model (1), we have that,

$$CE[(\hat{\theta}_{2,z;\,abc}-\theta_{abc})]=-\frac{2\sigma_\epsilon^2}{K}\pi^*_{a(b)}(\boldsymbol{\pi}^{*T}\boldsymbol{\pi}^*)^{-1}E\left[N\left(\frac{v_\epsilon}{v_\pi-1}I_x(v_{\epsilon+1},\ v_{\pi-1})+1-I_x(v_\epsilon,\ v_\pi)\right)\right] \quad if\ \boldsymbol{\pi}^*\neq\mathbf{0}$$

$$=0 \quad if\ \boldsymbol{\pi}^*=\mathbf{0}$$

$$I,K>1 \quad (a=1,\dots,I;b=1,\dots,J;c=1,\dots,K) \qquad (49)$$

Proof:

$CE[(\hat{\theta}_{2,z;\,abc}-\theta_{abc})]=CE[(\bar{h}_{.bc}+(1-\hat{r}_2)(\bar{h}_{ab.}-\bar{h}_{.b.})-\theta_{abc})]$

$=CE[\bar{h}_{.bc}+\bar{h}_{ab.}-\bar{h}_{.b.}-\theta_{abc}-\hat{r}_1(\bar{h}_{ab.}-\bar{h}_{.b.})]$

$\therefore CE[(\hat{\theta}_{2,z;\,abc}-\theta_{abc})]=-CE[\hat{r}_2(\bar{h}_{ab.}-\bar{h}_{.b.})]$

using corollary (3),

$CE\left[f_3(\frac{SS_\epsilon}{SS_\pi})(\bar{h}_{ab.}-\bar{h}_{.b.})\right]=\frac{2\sigma_\epsilon^2}{K}\pi^*_{a(b)}(\boldsymbol{\pi}^{*T}\boldsymbol{\pi}^*)^{-1}E\left[Nf_3\left(\frac{s_\pi}{1-s_\pi}\right)\right]$

Let $f_3(x)=min\{zx,1\}$ $\quad (x>0)$

$CE\left[f_3(\frac{SS_\epsilon}{SS_\pi})(\bar{h}_{ab.}-\bar{h}_{.b.})\right]=CE\left[min\left\{\frac{zs_\pi}{1-s_\pi},1\right\}(\bar{h}_{ab.}-\bar{h}_{.b.})\right]=\frac{2\sigma_\epsilon^2}{K}\pi^*_{a(b)}(\boldsymbol{\pi}^{*T}\boldsymbol{\pi}^*)^{-1}E\left[Nmin\left\{\frac{zs_\pi}{1-s_\pi},1\right\}\right]$

(50)

where

$E\left[\min\left\{\frac{zs_\pi}{1-s_\pi},1\right\}\right]=E\left[\frac{zs_\pi}{1-s_\pi}\right]+1-I_x(v_\epsilon,\ v_\pi)$

$E[s_\pi(1-s_\pi)^{-1}]=\int_0^x s_\pi(1-s_\pi)^{-1}\frac{1}{\beta(v_\epsilon,v_\pi)}s_\pi^{v_\epsilon-1}(1-s_\pi)^{v_\pi-1}\,ds_\pi+1-I_x(v_\epsilon,\ v_\pi)$

$=\frac{v_\epsilon}{v_\pi-1}I_x(v_{\epsilon+1},\ v_{\pi-1})+1-I_x(v_\epsilon,\ v_\pi) \qquad (51)$

substituting (51) in (50), we have (49).

Theorem 8: Let $\hat{\theta}_{2,z;abc}$ be the corresponding type 2 estimator of θ_{abc} $(a=1,\dots,I;b=1,..,J;c=1,..,K)$ and z > 0, under the model (1), we have that,

$$TCMSE(\hat{\boldsymbol{\theta}}_2,\boldsymbol{\theta})=J\sigma_\epsilon^2E\Bigg\{(K+I-1)+IK(J(I-1)(K-2)+2N)\left(\frac{z^2v_\epsilon(v_\epsilon+1)}{(v_\epsilon+v_\pi)(v_\pi-1)}I_x(v_\epsilon+2,v_\pi-1)-2z\frac{v_\epsilon}{(v_\epsilon+v_\pi)}I_x(v_\epsilon+1,v_\pi)-\frac{v_\pi}{(v_\epsilon+v_\pi)}I_x(v_\epsilon+1,v_\pi)\right)+2\frac{z^2v_\epsilon(v_\epsilon+1)}{(v_\epsilon+v_\pi)(v_\pi-1)}I_x(v_\epsilon+2,v_\pi-1)-4z\frac{v_\epsilon}{(v_\epsilon+v_\pi)}I_x(v_\epsilon+1,v_\pi)-2\frac{v_\pi}{(v_\epsilon+v_\pi)}I_x(v_\epsilon+1,v_\pi)+4N\left(\frac{zv_\epsilon}{v_\pi-1}I_x(v_{\epsilon+1},\ v_{\pi-1})+1-I_x(v_\epsilon,\ v_\pi)\right)\Bigg\}$$

(52)

Proof: using corollary (2), we have

$$TCMSE(\widehat{\boldsymbol{\theta}}_2,\boldsymbol{\theta}) = J\sigma_\epsilon^2 E\left\{(K+I-1)+IK(J(I-1)(p-2)+2N)E\left[g_3\left(\frac{s_\pi}{1-s_\pi}\right)(1-s_\pi)\right]+ 2\left(\left[g_3\left(\frac{s_\pi}{1-s_\pi}\right)(1-s_\pi)\right]+2\left[2Nf_3\left(\frac{s_\pi}{1-s_\pi}\right)\right]\right)\right\}$$

Let $f_3(x) = min\{zx, 1\}$ $and\ g_3(x) = {f_3}^2(x) - 2f_3(x)$ $, (x > 0)$

$$TCMSE(\widehat{\boldsymbol{\theta}}_2,\boldsymbol{\theta}) = J\sigma_\epsilon^2 E\left\{(K+I-1)+IK(J(I-1)(K-2)+2N)E\left[(1-s_\pi)min\left\{\frac{z^2s_\pi^2}{(1-s_\pi)^2},1\right\}- 2(1-s_\pi)min\left\{\frac{zs_\pi}{1-s_\pi},1\right\}\right]+2\left(E\left[(1-s_\pi)min\left\{\frac{z^2s_\pi^2}{(1-s_\pi)^2},1\right\}-2(1-s_\pi)min\left\{\frac{zs_\pi}{1-s_\pi},1\right\}\right]+ 2E\left[2Nmin\left\{\frac{zs_\pi}{1-s_\pi},1\right\}\right]\right)\right\} \quad (53)$$

$$E[s_\pi^2(1-s_\pi)^{-1}] = \int_0^x s_\pi^2(1-s_\pi)^{-1}\frac{1}{\beta(v_\epsilon, v_\pi)}s_\pi^{v_\epsilon-1}(1-s_\pi)^{v_\pi-1}\,ds_\pi$$

$$= \frac{v_\epsilon(v_\epsilon+1)}{(v_\epsilon+v_\pi)(v_\pi-1)}I_x(v_\epsilon+2, v_\pi-1) \quad (54)$$

$$E[s_\pi] = \int_0^x s_\pi\frac{1}{\beta(v_\epsilon,v_\pi)}s_\pi^{v_\epsilon-1}(1-s_\pi)^{v_\pi-1}\,ds_\pi = \frac{v_\epsilon}{(v_\epsilon+v_\pi)}I_x(v_\epsilon+1, v_\pi) \quad (55)$$

and

$$E[(1-s_\pi)] = \int_0^x(1-s_\pi)\frac{1}{\beta(v_\epsilon, v_\pi)}s_\pi^{v_\epsilon-1}(1-s_\pi)^{v_\pi-1}\,ds_\pi$$

$$= \frac{\beta_x(v_\epsilon,v_\pi+1)}{\beta(v_\epsilon,v_\pi)} = \frac{v_\epsilon}{(v_\pi+v_\epsilon)}I_x(v_\epsilon, v_\pi+1) \quad (56)$$

$$E\left[(1-s_\pi)min\left\{\frac{z^2s_\pi^2}{(1-s_\pi)^2},1\right\}-2(1-s_\pi)min\left\{\frac{zs_\pi}{1-s_\pi},1\right\}\right] = E\left[\frac{z^2s_\pi^2}{1-s_\pi}-2zs_\pi-(1-s_\pi)(1-I_x(v_\epsilon,v_{\pi+1})\right] \quad (57)$$

Substituting (54), (55) and (56), we have

$$E\left[(1-s_\pi)min\left\{\frac{z^2s_\pi^2}{(1-s_\pi)^2},1\right\}-2(1-s_\pi)min\left\{\frac{zs_\pi}{1-s_\pi},1\right\}\right] = \frac{z^2v_\epsilon(v_\epsilon+1)}{(v_\epsilon+v_\pi)(v_\pi-1)}I_x(v_\epsilon+2, v_\pi-1) - 2z\frac{v_\epsilon}{(v_\epsilon+v_\pi)}I_x(v_\epsilon+1, v_\pi) - \frac{v_\pi}{(v_\epsilon+v_\pi)}I_x(v_\epsilon+1, v_\pi) \quad (58)$$

substituting results (51) and (58) in (53). we have that`

$$TCMSE(\widehat{\boldsymbol{\theta}}_2,\boldsymbol{\theta}) = J\sigma_\epsilon^2 E\left\{(K+I-1)+IK(J(I-1)(p-2)+2N)\left(\frac{z^2v_\epsilon(v_\epsilon+1)}{(v_\epsilon+v_\pi)(v_\pi-1)}I_x(v_\epsilon+2, v_\pi-1) - 2z\frac{v_\epsilon}{(v_\epsilon+v_\pi)}I_x(v_\epsilon+1, v_\pi) - \frac{v_\pi}{(v_\epsilon+v_\pi)}I_x(v_\epsilon+1, v_\pi)\right) + 2\frac{z^2v_\epsilon(v_\epsilon+1)}{(v_\epsilon+v_\pi)(v_\pi-1)}I_x(v_\epsilon+2, v_\pi-1) - 4z\frac{v_\epsilon}{(v_\epsilon+v_\pi)}I_x(v_\epsilon+1, v_\pi) - 2\frac{v_\pi}{(v_\epsilon+v_\pi)}I_x(v_\epsilon+1, v_\pi) + 4N\left(\frac{zv_\epsilon}{v_\pi-1}I_x(v_{\epsilon+1},\ v_{\pi-1}) + 1 - I_x(v_\epsilon,\ v_\pi)\right)\right\}$$

Theorem 9: Let an arbitrary z > 0. Let the function $g_3(u) = z^2u^2 - 2zu$. Then

$$E\left\{g_3\left(\frac{s_b}{1-s_b}\right)\middle| N\right\} = z^2\frac{v_\epsilon(v_\epsilon+1)}{(v_b-2)(v_b-1)}I_x(v_\epsilon+2, v_b-2) - 2z\frac{v_\epsilon}{v_b-1}I_x(v_\epsilon+1, v_b-1) - 1 \quad (59)$$

Proof: using result (36) and let $f_3(x) = min\{zx, 1\}\ and\ g_3(x) = {f_3}^2(x) - 2f_3(x)$ $, (x > 0)$, we have

$$E\left\{g_3\left(\min\{\frac{s_b}{1-s_b}, 1\}\right)\middle| N\right\} = E\left\{min\left\{\frac{z^2s_b^2}{(1-s_b)^2},1\right\} - min\left\{\frac{2zs_b}{1-s_b},1\right\}\middle| N\right\}$$

$$= \frac{z^2}{\beta(v_\epsilon,v_\pi)}\int_0^x\frac{s_b^2}{(1-s_b)^2}s_b^{v_\epsilon-1}(1-s_b)^{v_b-1}ds_b + 1 - \frac{2z}{\beta(v_\epsilon,v_b)}\int_0^x\frac{s_b}{1-s_b}s_b^{v_\epsilon-1}(1-s_b)^{v_b-1}ds_b - 2$$

$$= \frac{z^2\beta(v_\epsilon+2,v_b-2)}{\beta(v_\epsilon,v_b)}I_x(v_\epsilon+2, v_b-2) - \frac{2z\beta(v_\epsilon+1,v_b-1)}{\beta(v_\epsilon,v_b)}I_x(v_\epsilon+1, v_b-1) - 1$$

$$E\left\{g_3\left(\frac{s_b}{1-s_b}\right)\middle| N\right\} = z^2\frac{v_\epsilon(v_\epsilon+1)}{(v_b-2)(v_b-1)}I_x(v_\epsilon+2, v_b-2) - 2z\frac{v_\epsilon}{v_b-1}I_x(v_\epsilon+1, v_b-1) - 1$$

Theorem 10: Let $\hat{\mu}_{2,z;abc}$ be the corresponding Type 2 estimator of θ_{abc} $(a = 1, \dots, I; b = 1, .., J; c = 1, .., K)$ and z > 0, under the model (1), we have that,

$$CMSE\left[\left(\hat{\theta}_{2;abc},\theta_{abc}\right)\right]=\frac{\sigma_\epsilon^2}{K}\left\{\frac{(K+I-1)}{I}+E\left[\left(J(I-1)(K-2)+2N\right)z^2\frac{v_\epsilon(v_\epsilon+1)}{(v_b-2)(v_b-1)}I_x(v_\epsilon+2,v_b-2)-2z\frac{v_\epsilon}{v_b-1}I_x(v_\epsilon+1,v_b-1)-1+2\pi^*_{a(b)}(\boldsymbol{\pi}^{*T}\boldsymbol{\pi}^*)^{-1}\left(z^2\frac{v_\epsilon(v_\epsilon+1)}{(v_b-2)(v_b-1)}I_x(v_\epsilon+2,v_b-2)-2z\frac{v_\epsilon}{v_b-1}I_x(v_\epsilon+1,v_b-1)-1+2N\left(\frac{v_\epsilon}{(v_\pi-1)}I_x(v_\epsilon+1,v_\pi-1)\right)\right)\right]\right\}\quad if\ \boldsymbol{\pi}^*\neq\mathbf{0}$$

$I,K>1\quad(a=1,\dots,I;b=1,\dots,J;c=1,\dots,K)$ (60)

Proof: using corollary (1),

$$CMSE\left[\left(\hat{\theta}_{2;abc},\theta_{abc}\right)\right]=\frac{\sigma_\epsilon^2}{p}\{\frac{(K+I-1)}{I}+E[(J(I-1)(K-2)+2N)g_2\left(min\left\{\frac{s_b}{1-s_b},1\right\}\right)+2\pi^*_{a(b)}(\boldsymbol{\pi}^{*T}\boldsymbol{\pi}^*)^{-1}\left(Ng_2\left(min\left\{\frac{s_b}{1-s_b},1\right\}\right)+2Nf_2\left(min\left\{\frac{s_\pi}{1-s_\pi},1\right\}\right)\right)]\}$$

Using result(59) and

$$E\left[min\left\{\frac{s_\pi}{1-s_\pi},1\right\}\middle|N\right]=\int_0^x\frac{1}{\beta(v_\epsilon,v_\pi)}s_\pi^{(v_\epsilon+1)-1}(1-s_\pi)^{(v_\pi-1)-1}ds_\pi$$

$$=\frac{\beta_x(v_\epsilon+1,v_\pi-1)}{\beta(v_\epsilon,v_\pi)}=\frac{v_\epsilon}{(v_\pi-1)}I_x(v_\epsilon+1,v_\pi-1)\qquad(61)$$

we have that

$$CMSE\left[\left(\hat{\theta}_{2;abc},\theta_{abc}\right)\right]=\frac{\sigma_\epsilon^2}{K}\left\{\frac{(K+I-1)}{I}+E\left[\left(J(I-1)(K-2)+2N\right)z^2\frac{v_\epsilon(v_\epsilon+1)}{(v_b-2)(v_b-1)}I_x(v_\epsilon+2,v_b-2)-2z\frac{v_\epsilon}{v_b-1}I_x(v_\epsilon+1,v_b-1)-1+2\pi^*_{a(b)}(\boldsymbol{\pi}^{*T}\boldsymbol{\pi}^*)^{-1}\left(z^2\frac{v_\epsilon(v_\epsilon+1)}{(v_b-2)(v_b-1)}I_x(v_\epsilon+2,v_b-2)-2z\frac{v_\epsilon}{v_b-1}I_x(v_\epsilon+1,v_b-1)-1+2N\left(\frac{v_\epsilon}{(v_\pi-1)}I_x(v_\epsilon+1,v_\pi-1)\right)\right)\right]\right\}\quad if\ \boldsymbol{\pi}^*\neq\mathbf{0}$$

$I,K>1\quad(a=1,\dots,I;b=1,\dots,J;c=1,\dots,K)$

Theorem 11: Let $\hat{\theta}_{2,z;abc}$ be the corresponding type 2 estimator of θ_{abc} $(a=1,\dots,n;j=1,..,q;k=1,..,p)$ and z > 0. Then, under the one-way repeated measurement random model, we have that,

$$CE\left[\left(\hat{\theta}_{2,z;abc}-\theta_{abc}\right)\left(\hat{\theta}_{2;a'b'c'}-\theta_{a'b'c'}\right)\right]=\frac{(K+I-1)}{IK}\sigma_\epsilon^2+\pi^*_{a(b)}\pi^*_{a'(b')}E\left[z^2\frac{v_\epsilon(v_\epsilon+1)}{(v_b-2)(v_b-1)}I_x(v_\epsilon+2,v_b-2)-2z\frac{v_\epsilon}{v_b-1}I_x(v_\epsilon+1,v_b-1)-1+\frac{2\sigma_\epsilon^2}{K}(\boldsymbol{\pi}^{*T}\boldsymbol{\pi}^*)^{-1}N\left(\frac{v_\epsilon}{(v_\pi-1)}I_x(v_\epsilon+1,v_\pi-1)\right)\right]\quad if\ \boldsymbol{\pi}^*\neq\mathbf{0}$$

$$=\frac{(K+I-1)}{IK}\sigma_\epsilon^2+\pi^*_{a(b)}\pi^*_{a'(b')}E\left[z^2\frac{v_\epsilon(v_\epsilon+1)}{(v_b-2)(v_b-1)}I_x(v_\epsilon+2,v_b-2)-2z\frac{v_\epsilon}{v_b-1}I_x(v_\epsilon+1,v_b-1)-1\right]\quad if\ \boldsymbol{\pi}^*=\mathbf{0}$$

$I,K>1\quad(a=1,\dots,I;b=1,\dots,J;c=1,\dots,K)$ (62)

Proof: Since

$$CE\left[\left(\hat{\theta}_{3;abc}-\theta_{abc}\right)\left(\hat{\theta}_{3;a'b'c'}-\theta_{a'b'c'}\right)\right]=\frac{(K+I-1)}{IK}\sigma_\epsilon^2+\pi^*_{a(b)}\pi^*_{a'(b')}E\left[g_3\left(\frac{s_b}{1-s_b}\right)+\frac{2\sigma_\epsilon^2}{K}(\boldsymbol{\pi}^{*T}\boldsymbol{\pi}^*)^{-1}Nf_3\left(\frac{s_\pi}{1-s_\pi}\right)\right]$$

$(a>a'=1,\dots,I;\ b>b'=1,..,J);\ c>c'=1,..,K)$

let $f_3(x)=\min\{zx,1\}$, $x>0$, we have

$$CE\left[\left(\hat{\theta}_{2,z;abc}-\theta_{abc}\right)\left(\hat{\theta}_{2;a'b'c'}-\theta_{a'b'c'}\right)\right]=\frac{(K+I-1)}{IK}\sigma_\epsilon^2+\pi^*_{a(b)}\pi^*_{a'(b')}E\left[\min\{\frac{s_b}{1-s_b},1\}+\frac{2\sigma_\epsilon^2}{p}(\boldsymbol{\pi}^{*T}\boldsymbol{\pi}^*)^{-1}N\min\{\frac{s_\pi}{1-s_\pi},1\}\right]$$

using result (59) and (61), we have

$$CE\left[\left(\hat{\theta}_{2,z;abc}-\theta_{abc}\right)\left(\hat{\theta}_{2;a'b'c'}-\theta_{a'b'c'}\right)\right]=\frac{(K+I-1)}{IK}\sigma_\epsilon^2+\pi^*_{a(b)}\pi^*_{a'(b')}E\left[z^2\frac{v_\epsilon(v_\epsilon+1)}{(v_b-2)(v_b-1)}I_x(v_\epsilon+2,v_b-2)-2z\frac{v_\epsilon}{v_b-1}I_x(v_\epsilon+1,v_b-1)-1+\frac{2\sigma_\epsilon^2}{K}(\boldsymbol{\pi}^{*T}\boldsymbol{\pi}^*)^{-1}N\left(\frac{v_\epsilon}{(v_\pi-1)}I_x(v_\epsilon+1,v_\pi-1)\right)\right] \quad if\ \boldsymbol{\pi}^*\neq\mathbf{0}$$

$$=\frac{(K+I-1)}{IK}\sigma_\epsilon^2+\pi^*_{a(b)}\pi^*_{a'(b')}E\left[z^2\frac{v_\epsilon(v_\epsilon+1)}{(v_b-2)(v_b-1)}I_x(v_\epsilon+2,v_b-2)-2z\frac{v_\epsilon}{v_b-1}I_x(v_\epsilon+1,v_b-1)-1\right] \quad if\ \boldsymbol{\pi}^*=\mathbf{0}$$

$I,K>1 \quad (a=1,\dots,I;b=1,\dots,J;c=1,\dots,K)$

Reference

1. AL-Mouel, A. H. S. and Al-Isawi A J, (2018) .Best Quadratic unbiased Estimator for Variance Component of One-Way Repeated Measurement Model, *Journal of Advance of Mathematics 14, No. 01, pp.7615-7623.*
2. AL-Mouel, A. H. S. and Al-Isawi, J M, (2019) .Minimum-quadratic unbiased estimator of variance components for the repeated measurements model by a new approach, *Journal of Heliyon 6, pp.1-8.*
3. Al - Mouel, A. H. S, and Batto, M. B. (2010). Analysis of Nested Repeated Measures Model with applications. *Journal of Al-Qadisiyah for Computer Science and Mathematics*, *2*(2), pp. 151-161.
4. Al-Mouel , A. H. S. and Faker A. S. (2012). Time-Independent Covariates in Two-Way Multivariate Repeated Measurements Model. *Al-Kut Journal of Economics Administrative Sciences, Volume 1, pp. 125-146.*
5. AL-Mouel, A. H. S and Kori H. A., (2021). Conditional and Unconditional of Repeated Measurements Model. *J. Phys.: Conf. Ser.* **1818** 012107, pp.1-11.
6. AL-Mouel, A. H. S., Mohaisen A.J. and Khawla (2017). Bayesian One-Way Repeated Measurements Model Based on Bayes Quadratic Unbiased Estimator, *Journal of Advance in Mathematics, Vol. 13, No. 2, pp. 7176-7182.*
7. AL-Mouel, A. H. S. and Al-shmailawi, H. R., (2016) .A Note on Estimation of Variance Components by Maximization, *IJPRET 5 (3), pp.1-10.*
8. Al-Mouel , A. H. S. and Wang, J. L., (2004) .One –Way Multivariate Repeated Measurements Analysis of Variance Model. *Applied Mathematics a Journal of Chinese Universities* **19(4)**,pp.435-448.
9. Kori H. A. and AL-Mouel, A. H. S, (2021). Bayesian Estimation of the Expected Mean Square Rate of Repeated Measurements Model**,** Turkish Journal of Computer and Mathematics Education Vol.12 No.7 (2021), 331-339.
10. Kori H. A. and AL-Mouel, A. H. S, (2021). Expected mean square rate estimation of repeated measurements model, Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 75-83**.**
11. Naji H. Z., Dawood R. and Faik, R. R., (2011) .Analysis of variance and covariance Repeated Measurement Model, *Journal of Basrah Researches* **37**, No. 1. 15 , pp.32-38.
12. Mohaisen, A. J., and Swadi, K. A. R., (2014). A note on Bayesian One- Way Repeated Measurements Model, *Journal of Mathematical Theory and Modeling, 4, pp. 106-114.*
13. Vonesh, E.F. and Chinchilli, V.M., (1997). Linear and Nonlinear Models for the Analysis of Repeated Measurements, *Marcel Dakker, Inc., New York.*