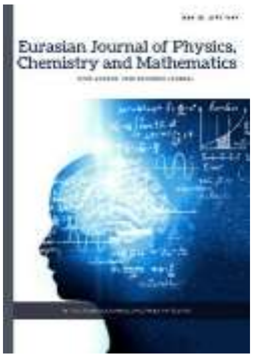

On Non-Informative Robust Fuzzy Bayesian Estimators

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ABSTRACT

In this paper, Suggesting a general method for converting any continuous failure distribution into a fuzzy failure distribution to obtain a more accurate and flexible distribution of inaccurate observations. and suggesting a robust Bayesian method that depends on a non-informational primary distribution that depends on Jeffrey's rule in finding the primary distributions so that the probability of obtaining the observation is conditional on its previous distribution, as the outliers observation will have a probability that differs from the probability of the another of the observations, and applying this method exponential distribution.

Keywords: Bayesian estimation, prior distribution, fuzzy sets, membership function, robustness, Exponential distribution

1. Introduction

In all fields of knowledge the researcher concerns what is the nature of the data, and it is very rare for these data to be directly prepared for the use of statistical methods that give appropriate estimates. The issue of robust estimates in the context of inference is one of the important issues. In (1853) Box put forward the idea of robustness and said that to build an effective model, it must be robust to ensure that there are no risks in it and thus lead to reliable and reliable inferences. Robust statistic is an extension of classic statistic that specifically takes into account the fact that traditional models only provide an approximation of the true basic random mechanism that generates the data. But in practice, the model assumptions are almost completely incompatible with what this random mechanism offers. It can be part of the observations that have patterns that do not share with the bulk of the rest of the data and therefore be outliers. The occurrence of deviations from the model assumptions with atypical values may have unexpected and bad effects on the results of the analysis. If we deal with the concept of robust from the point of view of Bayes theory, we will find that it depends on three main trends, the first depends on the inaccuracy of previous information (Priors), and the second depends on the contamination of the current sample observations or previous observations or the failure to achieve hypotheses of random errors, while the last trend is based on inaccuracy in determining the loss function. The issue of robust estimates in the context of inference is one of the important issues. In (1853) Box put forward the idea of robustness and said that to build an effective model, it must be robust to ensure that there are no risks in it and thus lead to reliable and reliable inferences (Passarin, 2004 ,1). The two Azerbaijani scholars (Lotfi Zadah) and German (D. Klaua) were the first to lay the foundations of the fuzzy sets theory in (1965) when they used the term fuzzy variables on approximate, inaccurate or undefined linguistic expressions and expressions. The fuzzy set is a set of elements in which each element has a degree of affiliation between zero and one that distinguishes it from other elements in the set. It is determined by an affiliation function. (Zadeh, 1965) (Klaua, D., 1965). The researchers (Berger & Berliner) in (1985) were the first to use the idea of the robust Bayesian estimation from two sides, the first depended on the pollution class –contamination ε by defining different pollution rates in the data, and the second relied on the class of maximum likelihood, the second type ML-II for the normal distribution using simulation Monte Carlo. (Berger & Berliner, 1985). After that, it followed many studies and research that dealt with the issue of fuzzy and the issue of robust Bayesian, In 2010, (Karpisek and others) relied on the fuzzy probability distribution and its properties to define the fuzzy reliability, as they described two models of fuzzy reliability using the Fuzzy Kumaraswamy distribution to estimate the fuzzy reliability of concrete structures (Karpisek & et al, 2010), Also. (Kareema) and (Abdul Hameed) (2012) derived the fuzzy probability mass function of the geometric distribution, the fuzzy cumulative distribution function, and some properties of the fuzzy distribution such as the fuzzy mean, the fuzzy variance, and the generation of fuzzy moments. The parameter domain, as well as all formulas that use probability theory, can be fuzzy. (Kareema, 2012 & Abdul Hameed). In 2014, (Safdar) presented a new method for obtaining a fuzzy probability distribution based on the well-known probability density function of the distribution and based on the (Resolution-Identity) property to obtain a fuzzy number and proved the effectiveness and adequacy of this method. (Safdar,2014). In 2018, (Wang & Beli) proposed a robust Bayesian model as an alternative to the standard model that gives protection for data that include outlier values or move away from basic assumptions (Wang &Beli,2018) .In 2019, (Panwar) and others used the robust Bayesian approach to analyze life-times of the Maxwell distribution based on the prior distribution, the class of maximum likelihood, the second type, under a square loss function and a Linux loss function in the case of complete data and data Type I progressive hybrid control (Panwar et al,2019). In (2020) (Entsar & Ahmed) used the standard Bayes method and the robust Bayes method to estimate the parameter (P) and the survival function of the binomial distribution in the case of conflicting previous data for two simulation experiments.

2. Fuzzy principle

Let Ω is Universe of discourse , A subset from it , then each element in A may be belonging or not belonging to A. (H. Garg et al, 2013, 397) (A. Ibrahim, A. Mohammed, 2017, 143)

Let $\mu_A(x)$ is a characteristic function for A give the membership in Ω to A, it is a binary function, {0, 1}, where,

$$\mu_A(x) = \begin{cases} 1, & if\ x \in A \\ 0, & if\ x \notin A \end{cases}$$

If $\mu_A(x)$ = 1, then the element x has full belonging to the set A. If $\mu_A(x)$= 0, then the element x does not belong to the set A. Figure (1) shows the crisp set, as we note in it that belonging to the elements x_r and x_{r+1}equals zero and to the elements x_0 , x_1 x_2 equal to one, and that the elements in it either belong to the set or do not belong to it.

As for the fuzzy set, it is a set of ambiguous boundaries, each element in the fuzzy set has a certain degree of membership, and the fuzzy set is characterized by a membership function that assigns each element in the set a degree of membership in the interval [0, 1]. In which the element or object is allowed to belong partly. (Pak, 2017, 504)

Let Ω is Universe of discourse , a fuzzy subset $\widetilde{A}$ from it that distinguished with the membership function $\mu_{\widetilde{A}}(x)$ which produce values in the interval [0, 1] for each values in the fuzzy sample space, then the fuzzy set is , (Danyaro & et al., 2010, 240)

$$\widetilde{A} = \{(x_i, \mu_{\widetilde{A}}(x_i)), x \in \Omega, i = 1,2,3, \dots\dots n, 0 < \mu_{\widetilde{A}}(x) < 1\} \quad \dots (1)$$

Figure (3) shows the fuzzy set, as we note in it that the membership to the elements a, c can fall between zero and one, and the element b has a degree of membership equal to one, and that the elements can belong to set A with different degrees of membership.

3. α-cut

The principle of cutting in the fuzzy set was first introduced by (Zadeha, 1971), and α is defined as the lowest degree of belonging to any element in the fuzzy group A˜ and its value falls within the period [0 1]. (H. Garg et al, 2013, 398) , which represents the degree of affiliation of the important elements, since the important affiliation is limited to two values (a_1, a_m) on the fulcrum of the fuzzy group (Support $\widetilde{A}$), and except for those values, it is of little importance and outside the scope of work (cut-out) (Auji, 2015).

4. Suggested Fuzzy Probability Distribution

Let we have a failure time $t_1, t_2, \dots, t_n$ where $t \in T$ inaccurate, uncertain, and expressed in fuzzy numbers $\tilde{t} \in \widetilde{T}$, where $\tilde{t} = \{[0, \infty), \mu_{\tilde{t}}(t)\}$. The crisp sample observations vector that we can get from the fuzzy set, which represents all the elements that have a degree of membership greater or equal to the alpha-cut (α-cut), which represents the degree of membership of the elements we are interested in and expresses those elements as the set $A^{(\alpha)}$

$A^{(\alpha)} = \{\tilde{t} = [0, \infty) \in \widetilde{T}, \mu_{\tilde{t}}(t) = \alpha \,;\, \mu_{\tilde{t}}(t) \geq \alpha\}$ $, 0 < \alpha < 1$ ... (2)

$\mu_{\tilde{t}}(t)$ is a membership function through which a degree of membership is generated for each failure time in the sample space and can take any form of membership functions, then $\tilde{t}_{A^{(\alpha)}}$ is Borel Measurable which will represent the fuzzy sample space and the events represent the smallest sigma-borel field (σ-Borel).Then the fuzzy cumulative distribution function $\widetilde{CDF}$ ism=,

$$\tilde{F}\left(\tilde{t}_{A^{(\alpha)}}\right) = \int_0^{\tilde{t}_{A^{(\alpha)}}} f(u)du \quad \dots (3)$$

By deriving the equation (32-2) for ($\tilde{t}_{A^{(\alpha)}}$) we get the fuzzy probability distribution as follows:

$$\tilde{f}(\tilde{t}) = \frac{\partial \tilde{F}\left(\tilde{t}_{A^{(\alpha)}}\right)}{\partial \tilde{t}_{A^{(\alpha)}}} = \frac{\partial}{\partial \tilde{t}_{A^{(\alpha)}}}\left[\int_0^{\tilde{t}_{A^{(\alpha)}}} f(u)du\right] \quad ; \quad 0 < \tilde{t}_{A^{(\alpha)}} < \infty \quad \dots (4)$$

5. Fuzzy Exponential distribution:

The probability density function for a crisp Exponential distribution is:

$$f(t, \lambda) = \lambda e^{-\lambda t} \quad ; \quad t > 0 \quad \dots (5)$$

From (5) we obtain,

$$\tilde{F}\left(\tilde{t}_{A^{(\alpha)}}\right) = \int_0^{\tilde{t}_{A^{(\alpha)}}} f(u)du = 1 - e^{\lambda \tilde{t}_{A^{(\alpha)}}} = F\left(\tilde{t}_{A^{(\alpha)}}\right) \quad \dots (6)$$

The probability density function for the fuzzy Exponential distribution can be obtained as follows:

$$\tilde{f}\left(\tilde{t}_{A^{(\alpha)}}\right) = \frac{\partial \tilde{F}\left(\tilde{t}_{A^{(\alpha)}}\right)}{\partial \tilde{t}_{A^{(\alpha)}}} = \frac{\partial}{\partial \tilde{t}}\left[1 - e^{\lambda \tilde{t}_{A^{(\alpha)}}}\right] = \lambda e^{-\lambda \tilde{t}_{A^{(\alpha)}}} = f(\tilde{t}_{A^{(\alpha)}}) \quad \dots (7)$$

6. Fuzzy Weibull distribution:

The probability density function for a crisp Weibull distribution is:

$$f(t, \theta, \lambda) = \theta p t^{\theta - 1} e^{-p t^{\theta}} \quad ; \quad t, \theta, p > 0 \quad \dots (8)$$

From (4) we obtain,

$$\tilde{F}\left(\tilde{t}_{A^{(\alpha)}}\right) = \int_0^{\tilde{t}_{A^{(\alpha)}}} f(u)du = 1 - e^{-p \tilde{t}_{A^{(\alpha)}}{}^{\theta}} = F\left(\tilde{t}_{A^{(\alpha)}}\right) \quad \dots (9)$$

The probability density function for the fuzzy Weibull distribution can be obtained as follows:

$$\tilde{f}\left(\tilde{t}_{A^{(\alpha)}}\right) = \frac{\partial \tilde{F}\left(\tilde{t}_{A^{(\alpha)}}\right)}{\partial \tilde{t}_{A^{(\alpha)}}} = \frac{\partial}{\partial \tilde{t}}\left[1 - e^{-p \tilde{t}_{A^{(\alpha)}}{}^{\theta}}\right] = p\theta \tilde{t}_{A^{(\alpha)}}{}^{\theta - 1} e^{-p \tilde{t}_{A^{(\alpha)}}{}^{\theta}} = f(\tilde{t}_{A^{(\alpha)}}) \quad \dots (10)$$

7. Fuzzy Kumaraswamy distribution:

The probability density function for a crisp Kumaraswamy distribution is:

$$f(t, \theta, \beta) = \theta\beta\, t^{\beta - 1}\left[1 - t^{\beta}\right]^{\theta - 1} \quad 0 < t < 1 \quad \dots (11)$$

From (4) we obtain,

$$\tilde{F}\left(\tilde{t}_{A^{(\alpha)}}\right) = \int_0^{\tilde{t}_{A^{(\alpha)}}} f(u)du = 1 - \left[1 - \tilde{t}_{A^{(\alpha)}}{}^{\beta}\right]^{\theta} = F\left(\tilde{t}_{A^{(\alpha)}}\right) \quad \dots (12)$$

The probability density function for the fuzzy Kumaraswamy distribution can be obtained as follows:

$$\tilde{f}(\tilde{t}_{A(\alpha)}) = \frac{\partial \tilde{F}(\tilde{t}_{A(\alpha)})}{\partial \tilde{t}_{A(\alpha)}} = \frac{\partial}{\partial \tilde{\boldsymbol{t}}}\left[1 - \left[\mathbf{1} - \tilde{t}_{A(\alpha)}{}^{\beta}\right]^{\theta}\right] = \theta\beta\tilde{t}_{A(\alpha)}{}^{\beta-1}\left[\mathbf{1} - \tilde{t}_{A(\alpha)}{}^{\beta}\right]^{\theta-1} = f(\tilde{t}_{A(\alpha)}) \quad \ldots (13)$$

8. Proposed Robust Bayesian method

Bayesian modeling takes into account the inaccuracy of the unknown parameters in a statistical model (Gelman et al., 2014). Therefore, the Bayesian model uses a set of sample data t_i Which is represented by the likelihood function of the current observations, as we have the original distribution of the items of the current sample, which represents the probability density function of the data $\varphi(t_i/\underline{\theta})$ with parameter vector $\underline{\theta}$ and prior distribution $\pi(\underline{\theta}/\underline{\vartheta})$ with hyper- parameters $\underline{\vartheta}$.

$$\{t_i/\ \underline{\theta} \sim iid\ \varphi(t_i/\underline{\theta})\ ,\underline{\theta} \sim \pi(\underline{\theta}/\underline{\vartheta})\ \}, \qquad i = 1,2,\ldots\ldots,n \qquad \ldots (14)$$

To find the Joint posterior distribution,

$$h(\underline{\theta}/t_i/\underline{\vartheta}) = \frac{\pi(\underline{\theta}/\underline{\vartheta}) \prod_{i=1}^{n} \varphi(t_i/\underline{\theta})}{\int_{\forall\underline{\theta}} \pi(\underline{\theta}/\underline{\vartheta}) \prod_{i=1}^{n} \varphi(t_i/\underline{\theta})} \quad \ldots (15)$$

We note in Model (2-68) that for the parameter estimated from the observations of the sample as a whole, there is one primary distribution, which is $\pi(\underline{\theta}/\underline{\vartheta})$ with hyper- parameters $\underline{\vartheta}$ his does not achieve robustness in the estimation because all the items of the current sample data will have a common initial distribution so that the vocabulary of the same format and the abnormal vocabulary will have the same previous probability. In order to make the model (68-2) enjoy robustness, we will suggest that for each of the parameters to be estimated at each item of the sample vector t_i drawing from $\varphi(t_i/\theta_i)$ there is preliminary information represented by an initial distribution $\pi(\theta_i/\underline{\vartheta})$ for parameter θ_i with hyper- parameters $\underline{\vartheta}$,

$$t_i/\theta_i \sim iid\ \varphi(t_i/\theta_i)\ ,\theta_i \sim iid\ \pi(\theta_i/\underline{\vartheta})\ , \qquad i = 1,2,\ldots\ldots,n \qquad \ldots (16)$$

The robust posterior joint distribution of $(\underline{\theta}/t_i)$ with parameters $\underline{\theta} = (\theta_1, \theta_2 \ldots., \theta_n)$ as following:

$$\mathbb{H}(\underline{\theta}/t_i/\underline{\vartheta}) = \frac{\prod_{i=1}^{n} \pi(\theta_i/\underline{\vartheta})\ \varphi(t_i/\theta_i)}{\int_{\forall\theta_i} \prod_{i=1}^{n} \pi(\theta_i/\underline{\vartheta})\ \varphi(t_i/\theta_i)} \quad \ldots (17)$$

And the model (17) will include that each observation of the sample is completely independent from the other observation and is conditional on the estimation of the parameter (θ_i). In other words, the sample data will be completely independent of each other.

The probability for each of the independent and identically distributed data (iid) can be obtained as follows:

$$\varphi(t_i/\underline{\vartheta}) = \int \pi(\theta_i/\underline{\vartheta})\ \varphi(t_i/\theta_i) d\theta_i \quad \ldots (18)$$

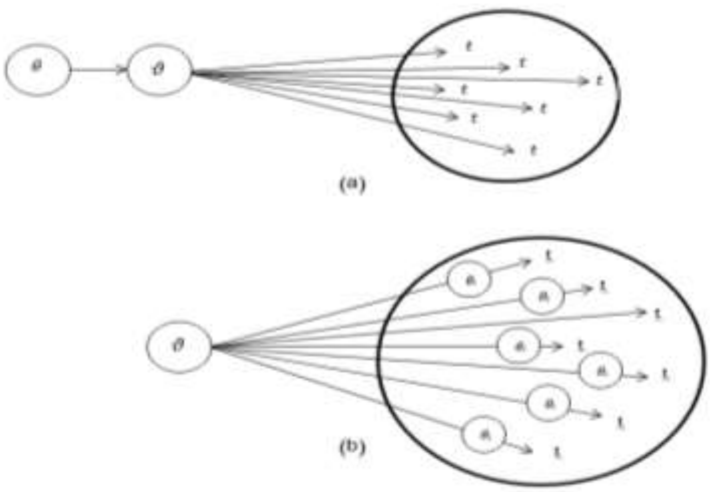

Figure (1) Graphic representation (a) Standard Bayes model (b) The proposed robust Bayes model

6. General Formula of Proposed Fuzzy Robust Bayesian method

When we substitute the fuzzy probability distribution in the formula (4) instead of the traditional probability distribution in the proposed robust Bayes formula (17), we get the following:

$$\tilde{h}\left(\underline{\theta}/\tilde{t}_{A(\alpha)_i}/\hat{\vartheta}\right) = \frac{\prod_{i=1}^{n} \pi(\theta_i/\hat{\vartheta})\, \tilde{\varphi}\left(\tilde{t}_{A(\alpha)_i}/\theta_i\right)}{\int \prod_{i=1}^{n} \pi(\theta_i/\hat{\vartheta})\, \tilde{\varphi}\left(\tilde{t}_{A(\alpha)_i}/\theta_i\right) d\theta_i} \quad \dots (18)$$

And the formula (18) represents the fuzzy robust posterior probability distribution of the fuzzy sample data from which the fuzzy robust Bayes estimator $\underline{\hat{\hat{\theta}}}_{BRF}$ can be found at any loss function.

9. Exponential Non-Informative Robust Fuzzy Bayesian Estimator

Suppose we have failure times $t_1, t_2, \dots, t_n$ whrer $\boldsymbol{t} \in \mathbf{T}$ has an exponential distribution with parameter (λ), then the conventional probability density function is:

$$f(\mathrm{t}) = \lambda e^{-\lambda t} \quad ; \ t, \lambda > 0 \quad \dots (19)$$

The possibility function of the exponential distribution can be written as:

$$L_{exp} = \prod_{i=1}^{n} f(\mathrm{t}_i, \lambda)$$

$$= \lambda^n e^{-\lambda \sum_{i=1}^{n} \mathrm{t}_i} \quad \dots (20)$$

According to Jeffery's rule, the previous distribution of the parameter to be estimated (λ) is as follows:

$$\pi(\lambda) \propto \sqrt{\mathrm{I}(\lambda)}$$

$$\pi(\lambda) = \mathrm{Constant} \sqrt{\mathrm{I}(\lambda)}$$

$$= \mathrm{K} \sqrt{\mathrm{I}(\lambda)}$$

$$\mathrm{I}(\lambda) = -\tilde{n} E\left[\frac{\partial^2 Lnf(\mathrm{t})}{\partial \lambda^2}\right]$$

$$Lnf(\mathrm{t}) = Ln\left(\lambda e^{-\lambda t}\right) = Ln(\lambda) - \lambda t$$

$$\frac{\partial Lnf(\mathrm{t})}{\partial \lambda} = \frac{1}{\lambda} - t$$

$$\frac{\partial^2 Lnf(\mathrm{t})}{\partial \lambda^2} = -\frac{1}{\lambda^2}$$

$$E\left[\frac{\partial^2 Lnf(\mathrm{t})}{\partial \lambda^2}\right] = E\left[-\frac{1}{\lambda^2}\right] =$$

$$\therefore \ \mathrm{I}(\lambda) = -nE\left[\frac{\partial^2 Lnf(\mathrm{t})}{\partial \lambda^2}\right] = \frac{n}{\lambda^2}$$

Therefore, the previous distribution of the parameter (λ) is as follows:

$$\pi(\lambda) = \mathrm{K} \sqrt{\mathrm{I}(\lambda)}$$

$$= \mathrm{K} \sqrt{\frac{n}{\lambda^2}} = \mathrm{K} \frac{\sqrt{n}}{\lambda} \quad \dots (21)$$

Therefore, the joint probability density function for the two variables t, λ is:

$$\mathbf{G}(\mathrm{t}_i, \lambda) = \pi(\lambda) \prod_{\boldsymbol{i=1}}^{\tilde{n}} f(\mathrm{t}_i)$$

$$= \mathrm{K}\sqrt{n} \lambda^{n-1} e^{-\lambda \sum_{i=1}^{n} \mathrm{t}_i} \quad \dots (22)$$

From equation (22), we find the marginal function of the variable $\mathrm{t_i}$, as follows:

$$\mathrm{M}(\mathrm{t_i}) = \int_0^\infty \mathrm{K}\sqrt{\mathrm{n}} \lambda^{\mathrm{n}-1} \mathrm{e}^{-\lambda \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{t_i}}\, \mathrm{d}\lambda$$

$$= \frac{\mathrm{K}\sqrt{\mathrm{n}}\,\Gamma(\mathrm{n})}{\left(\sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{t_i}\right)^{\mathrm{n}}} \quad \dots (23)$$

Therefore, the conditional posterior distribution is as follows:

$$\mathrm{h}(\lambda |\, \mathrm{t_i}) = \frac{\mathrm{G}(\mathrm{t_i}, \lambda)}{\mathrm{M}(\mathrm{t_i})}$$

$$= \frac{\left(\sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{t_i}\right)^{\mathrm{n}}}{\Gamma(\mathrm{n})} \lambda^{\mathrm{n}-1} \mathrm{e}^{-\lambda \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{t_i}} \quad \dots (24)$$

We note that the previous distribution is nothing but a gamma distribution with the two parameters $(\alpha = \mathrm{n}, \beta = \sum_{i=1}^{n} \mathrm{t}_i)$.

The non-informative standard Bayes estimator in light of a squared loss function is nothing but an expectation of the subsequent distribution of the parameter to be estimated as follows:

$$\hat{\lambda}_{NISBexp} = \frac{n}{\sum_{i=1}^{n} t_i} \quad \dots (25)$$

The proposed non-informational, fuzzy standard Bayesian estimator for the exponential distribution:
Suppose we have a failure times $t_1, t_2, \dots, t_n$ where $t \in T$ has an exponential distribution with the parameter (λ) then the fuzzy set at the cutoff α, $\widetilde{A}_\alpha = \{\tilde{t}_1, \tilde{t}_2, \dots, \tilde{t}_{\tilde{n}}\}$, where $\tilde{t} \in \widetilde{T}$, $\tilde{t} = \{[\mathbf{0}, \infty), \mu_{\tilde{t}}(t)\}$ has a fuzzy exponential distribution with the parameter λ with the following fuzzy probability density function:

$$\tilde{f}(\tilde{t}_{A(\alpha)}) = \lambda e^{-\lambda \tilde{t}_{A(\alpha)}} \quad \tilde{t}_{A(\alpha)}, \lambda > 0 \quad \dots (26)$$

According to Jeffery's rule, the prior distribution of the parameter to be estimated (λ) is as follows:

$$\pi(\lambda) = \text{Constant} \sqrt{I(\lambda)}$$
$$= K \sqrt{I(\lambda)}$$

$$I(\lambda) = -\tilde{n} E \left[\frac{\partial^2 Lnf(t)}{\partial \lambda^2} \right]$$

$$Ln\tilde{f}(\tilde{t}_{A(\alpha)}) = Ln\left(\lambda e^{-\lambda \tilde{t}_{A(\alpha)}}\right) = Ln(\lambda) - \lambda \tilde{t}_{A(\alpha)}$$

$$\frac{\partial Ln\tilde{f}(\tilde{t}_{A(\alpha)})}{\partial \lambda} = \frac{1}{\lambda} - \tilde{t}_{A(\alpha)}$$

$$\frac{\partial^2 Ln\tilde{f}(\tilde{t}_{A(\alpha)})}{\partial \lambda^2} = -\frac{1}{\lambda^2}$$

$$E\left[\frac{\partial^2 Ln\tilde{f}(\tilde{t}_{A(\alpha)})}{\partial \lambda^2}\right] = E\left[-\frac{1}{\lambda^2}\right]$$

$$\therefore \ I(\lambda) = -\tilde{n} E\left[\frac{\partial^2 Ln\tilde{f}(\tilde{t}_{A(\alpha)})}{\partial \lambda^2}\right] = \frac{\tilde{n}}{\lambda^2}$$

Therefore, the prior distribution of the parameter (λ) is as follows:

$$\pi(\lambda) = K \sqrt{I(\lambda)}$$
$$= K \sqrt{\frac{\tilde{n}}{\lambda^2}} = K \frac{\sqrt{\tilde{n}}}{\lambda} \quad \dots (27)$$

According to the impenetrable fuzzy Bayes method proposed in the formula (27), the previous distribution of each parameter estimated from each of the observations of the fuzzy sample will be as follows:

$$\pi(\lambda_i) = K \frac{\sqrt{\tilde{n}}}{\lambda_i} \quad \dots (28)$$

Therefore, the joint probability density function for the two variables $\tilde{t}_{A(\alpha)}$, λ is:

$$\mathbf{G}\left(\tilde{t}_{A(\alpha)_i}, \lambda_i\right) = \prod_{i=1}^{\tilde{n}} \pi(\lambda_i) \ \tilde{f}\left(\tilde{t}_{A(\alpha)_i}\right)$$
$$= \left(K\sqrt{\tilde{n}}\right)^{\tilde{n}} \prod_{i=1}^{\tilde{n}} e^{-\lambda_i \tilde{t}_{A(\alpha)_i}} \quad \dots (29)$$

From equation (30), we find the marginal function of the variable $\tilde{t}_{A(\alpha)_i}$ as follows:

$$M(\tilde{t}_{A(\alpha)_i}) = \left(K\sqrt{\tilde{n}}\right)^{\tilde{n}} \int_0^\infty \left[\prod_{i=1}^{\tilde{n}} e^{-\lambda_i \tilde{t}_{A(\alpha)_i}}\right] d\lambda_i$$
$$= \left(K\sqrt{\tilde{n}}\right)^{\tilde{n}} \int_0^\infty \left[e^{-\lambda_1 \tilde{t}_{A(\alpha)_1}} . e^{-\lambda_2 \tilde{t}_{A(\alpha)_2}} \ \dots \ e^{-\lambda_{\tilde{n}} \tilde{t}_{A(\alpha)_{\tilde{n}}}} \right] d\lambda_i$$
$$= \left(K\sqrt{\tilde{n}}\right)^{\tilde{n}} \left[\int_0^\infty e^{-\lambda_1 \tilde{t}_{A(\alpha)_1}} \ d\lambda_1 . \int_0^\infty e^{-\lambda_1 \tilde{t}_{A(\alpha)_1}} \ d\lambda_2 \dots \int_0^\infty e^{-\lambda_1 \tilde{t}_{A(\alpha)_1}} \ d\lambda_{\tilde{n}}\right]$$
$$= \left(K\sqrt{\tilde{n}}\right)^{\tilde{n}} \prod_{i=1}^{\tilde{n}} \left[\int_0^\infty e^{-\lambda_i \tilde{t}_{A(\alpha)_i}} \ d\lambda_i\right]$$

$$= \left(K\sqrt{\tilde{n}}\right)^{\tilde{n}} \prod_{i=1}^{\tilde{n}} \frac{1}{\tilde{t}_{A(\alpha)_i}} \left(\int_0^{\infty} e^{-u}\ du\right)$$

$$= \frac{\left(K\sqrt{\tilde{n}}\right)^{\tilde{n}}}{\prod_{i=1}^{\tilde{n}} \tilde{t}_{A(\alpha)_i}} \qquad \dots (31)$$

Therefore, the conditional fuzzy post hoc distribution is as follows:

$$\boldsymbol{h}\left(\lambda/\tilde{t}_{A(\alpha)_i}\right) = \frac{G(\tilde{t}_{A(\alpha)_i}, \lambda_i)}{M(\tilde{t}_{A(\alpha)_i})}$$

$$= \frac{\left(K\sqrt{\tilde{n}}\right)^{\tilde{n}} \prod_{i=1}^{\tilde{n}} e^{-\lambda_i \tilde{t}_{A(\alpha)_i}}}{\frac{\left(K\sqrt{\tilde{n}}\right)^{\tilde{n}}}{\prod_{i=1}^{\tilde{n}} \tilde{t}_{A(\alpha)_i}}}$$

$$= \prod_{i=1}^{\tilde{n}} \tilde{t}_{A(\alpha)_i}\, e^{-\lambda_i \tilde{t}_{A(\alpha)_i}} \qquad \dots (32)$$

And to prove that the posterior distribution in equation (32) is a probability function as follows:

$$\int_0^{\infty} \prod_{i=1}^{\tilde{n}} \tilde{t}_{A(\alpha)_i}\, e^{-\lambda_i \tilde{t}_{A(\alpha)_i}}\ d\lambda_i$$

$$\left(\tilde{t}_{A(\alpha)_i}\right)^{\tilde{n}} \prod_{i=1}^{\tilde{n}} \int_0^{\infty} e^{-\lambda_i \tilde{t}_{A(\alpha)_i}}\, d\lambda_i$$

$$\therefore \left(\tilde{t}_{A(\alpha)_i}\right)^{\tilde{n}} \prod_{i=1}^{\tilde{n}} \int_0^{\infty} e^{-u} \frac{1}{\tilde{t}_{A(\alpha)_i}} du = 1$$

The non-informational fuzzy standard Bayes estimator under a quadratic loss function is nothing but the expectation of the post-distribution, that is:

$$\hat{\lambda}_{\text{NRFSBexp}} = \int_0^{\infty} \lambda_i \prod_{i=1}^{\tilde{n}} \tilde{t}_{A(\alpha)_i}\, e^{-\lambda_i \tilde{t}_{A(\alpha)_i}}\, d\lambda_i$$

$$= \left(\tilde{t}_{A(\alpha)_i}\right)^{\tilde{n}} \prod_{i=1}^{\tilde{n}} \left(\frac{1}{\tilde{t}_{A(\alpha)_i}}\right)^{\frac{1}{\tilde{n}}+1} \Gamma\left(\frac{1}{\tilde{n}}+1\right) \qquad \dots (33)$$

9. Simulation experiments

The Monte-Carlo Simulation method was adopted for the purpose of comparing the Bayes estimators for crisp data and the proposed robust fuzzy bass estimators the exponential distribution, Non-informative prior at a squared error loss function. The theoretical values for the parameter of the distribution were obtained empirically from conducting several experiments and selecting the values, and then the Bayes estimates were stable and gave the best results:

Parameter	1	2	3
λ	1	1.5	4

The crisp data was generated that the distributions represented by the vector t from each distribution by using inverse cumulative distribution function by applying the inverse transformation method according to $t_i = -\frac{\ln(1-u)}{\lambda}$. Then the crisp data vector has been polluted with outlier values by finding the arithmetic mean and standard deviation of the crisp sample vector and adding the outlier values to it according to the equation $t\_Outlier = mean(t{:}\,i) + 3(SD{:}\,i)$. The crisp sample vector $\underline{t\_Outlier} = (t_1, t_2, \dots, t_n)'$ is transformed from each distribution to the fuzzy by finding the degree of membership corresponding to each of the observations of the polluted crisp sample vector using a triangular membership function as follows:

$$\mu_A(t) = \begin{cases} 0 & if\ \ t < a \\ \frac{t-a}{b-a} & if\ \ a \le t \le b \\ 1 & if\ \ t > b \end{cases}$$

... (33)

As a represents the lowest value of the observations values of the crisp sample and b represents the largest value of the observations values of the traditional sample vector, which results in us a fuzzy sample vector $\underline{\tilde{t}} = \tilde{t}_1, \tilde{t}_2, \dots, \tilde{t}_n$ includes each observation and its corresponding degree of membership which :

$\tilde{t}_i = \{(t_i,\ \mu_A(t_1)), (t_2,\ \mu_A(t_2)), \dots, (t_{\tilde{n}}, \mu_A(t_n))\}$ ... (34)

After that, the fuzzy set is obtained at the cutoff α $\tilde{A}_\alpha = \{\tilde{t}_1, \tilde{t}_2, \dots, \tilde{t}_{\tilde{n}}\}$ for the studied distribution by choosing the elements in the fuzzy set that have a degree of belonging greater or equal to the cut, α that is $\tilde{A}_\alpha = \{\tilde{t} \in T;\ \mu_{\tilde{A}}(t) \geq \alpha\}$ by choosing $\alpha - \text{cut} = 0.2, 0.4, 0.5, 0.7, 0.9$. The Estimation methods were compared using the mean squared error criterion (MSE) by using Matlab 2015

First: When the data contains one outlier:

Table (1) Estimation of parameters and mean square error of MSE in the crisp and proposed Bayesian methods at cutoff coefficients α-cut=0.2,0.4,0.5,0.7,0.9 and at default value of exponential distribution parameter $\boldsymbol{\lambda} = 1$ and one outlier.

Distribution		**Exponential**		**Best**
cut	**Method**	**Estimation**	**MSE**	
0.2	**NSB**	**1.65962**	**0.16740**	**NRFES**
	NRFES	**1.33242**	**0.12127**	
0.4	**NSB**	**1.52244**	**0.14246**	**NRFES**
	NRFES	**1.26246**	**0.11784**	
0.5	**NSB**	**1.32574**	**0.12238**	**NRFES**
	NRFES	**1.22257**	**0.11238**	
0.7	**NSB**	**1.22772**	**0.11531**	**NRFES**
	NRFES	**1.21356**	**0.11085**	
0.9	**NSB**	**1.22156**	**0.09531**	**NRFES**
	NRFES	**1.22111**	**0.01457**	

Table (2) Estimation of parameters and mean square error of MSE in the crisp and proposed Bayesian methods at cutoff coefficients α-cut=0.2,0.4,0.5,0.7,0.9 and at default value of exponential distribution parameter $\lambda = 1.5$ and one outlier.

Distribution		**Exponential**		**Best**
cut	**Method**	**Estimation**	**MSE**	
0.2	**NSB**	**1.89321**	**0.67363**	**NRFES**
	NRFES	**1.72195**	**0.21723**	
0.4	**NSB**	**1.73166**	**0.44597**	**NRFES**
	NRFES	**1.63966**	**0.11580**	
0.5	**NSB**	**1.71135**	**0.41244**	**NRFES**
	NRFES	**1.61246**	**0.10577**	
0.7	**NSB**	**1.61238**	**0.23248**	**NRFES**
	NRFES	**1.52145**	**0.04351**	
0.9	**NSB**	**1.5672**	**0.07833**	**NRFES**
	NRFES	**1.51214**	**0.03113**	

Table (2) Estimation of parameters and mean square error of MSE in the crisp and proposed Bayesian methods at cutoff coefficients α-cut=0.2,0.4,0.5,0.7,0.9 and at default value of exponential distribution parameter $\boldsymbol{\lambda} = 4$ and one outlier

Distribution		**Exponential**		**Best**
cut	**Method**	**Estimation**	**MSE**	
0.2	**NSB**	**4.52195**	**0.53155**	**NRFES**
	NRFES	**4.34215**	**0.33253**	
0.4	**NSB**	**4.32555**	**0.23465**	**NRFES**
	NRFES	**4.22143**	**0.13668**	
0.5	**NSB**	**4.321354**	**0.21142**	**NRFES**
	NRFES	**4.21433**	**0.12645**	
0.7	**NSB**	**4.23167**	**0.11127**	**NRFES**

	NRFES	**4.13583**	**0.11232**	
0.9	**NSB**	**4.22125**	**0.10754**	**NRFES**
	NRFES	**4.12355**	**0.10753**	

Second: When the data contains three outlier

Table (4) Estimation of parameters and mean square error of MSE in the crisp and proposed Bayesian methods at cutoff coefficients α-cut=0.2,0.4,0.5,0.7,0.9 and at default value of exponential distribution parameter $\lambda = 1$ and three outlier.

Distribution		**Exponential**		**Best**
cut	**Method**	**Estimation**	**MSE**	
0.2	**NSB**	**5.6466**	**4.48655**	**NRFES**
	NRFES	**1.32444**	**0.00434**	
0.4	**NSB**	**5.11224**	**4.23144**	**NRFES**
	NRFES	**1.21535**	**0.00316**	
0.5	**NSB**	**1.14354**	**0.00296**	**NRFES**
	NRFES	**2.18941**	**1.42555**	
0.7	**NSB**	**1.66854**	**0.21854**	**NRFES**
	NRFES	**1.12311**	**0.00136**	
0.9	**NSB**	**1.34133**	**0.08918**	**NRFES**
	NRFES	**1.11076**	**0.00136**	

Table (5) Estimation of parameters and mean square error of MSE in the crisp and proposed Bayesian methods at cutoff coefficients α-cut=0.2,0.4,0.5,0.7,0.9 and at default value of exponential distribution parameter $\lambda = 1.5$, and three outlier.

Distribution		**Exponential**		**Best**
cut	**Method**	**Estimation**	**MSE**	
0.2	**NSB**	**3.32455**	**4.45632**	**NRFES**
	NRFES	**1.59533**	**0.16322**	
0.4	**NSB**	**2.12686**	**2.12253**	**NRFES**
	NRFES	**1.55462**	**0.11347**	
0.5	**NSB**	**2.12121**	**2.11197**	**NRFES**
	NRFES	**1.55227**	**0.10463**	
0.7	**NSB**	**2.11057**	**2.10875**	**NRFES**
	NRFES	**1.54352**	**0.09081**	
0.9	**NSB**	**1.78576**	**1.44633**	**NRFES**
	NRFES	**1.52243**	**0.04767**	

Table (6) Estimation of parameters and mean square error of MSE in the crisp and proposed Bayesian methods at cutoff coefficients α-cut=0.2,0.4,0.5,0.7,0.9 and at default value of exponential distribution parameter $\lambda = 4$, and three outlier.

Distribution		**Exponential**		**Best**
cut	**Method**	**Estimation**	**MSE**	
0.2	**NSB**	**7.5844**	**4.79446**	**NRFES**
	NRFES	**4.67333**	**0.04543**	
0.4	**NSB**	**5.79544**	**3.53744**	**NRFES**
	NRFES	**4.43768**	**0.03122**	
0.5	**NSB**	**4.56366**	**1.45881**	**NRFES**
	NRFES	**4.24359**	**0.03111**	
0.7	**NSB**	**4.41128**	**1.11046**	**NRFES**
	NRFES	**4.11464**	**0.00456**	
0.9	**NSB**	**4.35663**	**1.00463**	**NRFES**
	NRFES	**4.03533**	**0.00045**	

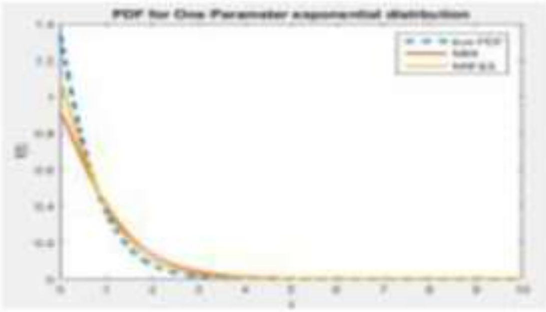

Figure (4) the curve of the probability density function for the exponential distribution at the traditional and proposed Bayesian estimation method at the cutoff 0.9

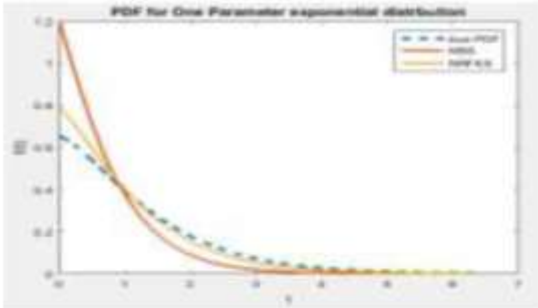

Figure (5) the curve of the probability density function for the exponential distribution at the traditional and proposed Bayesian estimation method at the cutoff 0.7

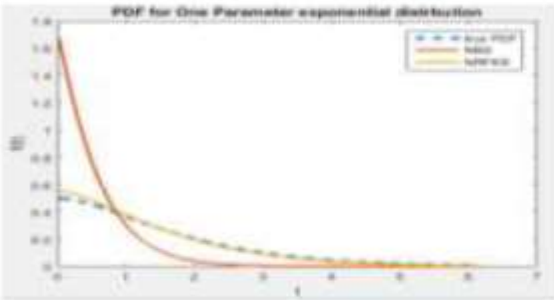

Figure (6) the curve of the probability density function for the exponential distribution at the traditional and proposed Bayesian estimation method at the cutoff 0.5

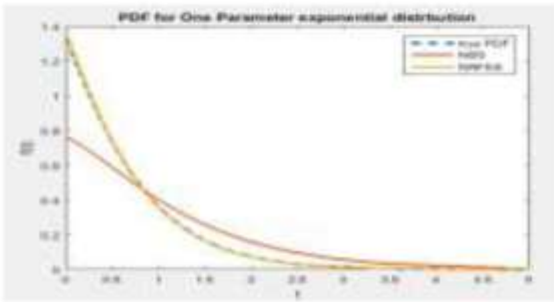

Figure (7) the curve of the probability density function for the exponential distribution at the traditional and proposed Bayesian estimation method at the cutoff 0.4

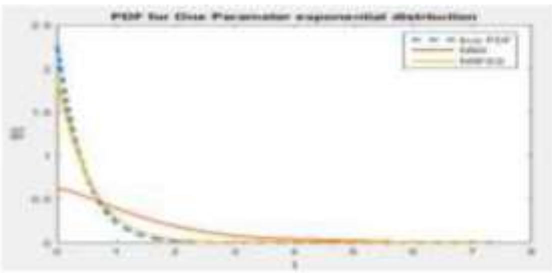

Figure (7) the curve of the probability density function for the exponential distribution at the traditional and proposed Bayesian estimation method at the cutoff 0.2

10. Results and discussion:

It is clear from Tables (1) to (6) the proposed Robust fuzzy Bayes method based on an Non-informational prior distribution is superior to the traditional Bayes method under outliers' observations. The greater the cutoff α, the less the mean of the squares of error and the greater the accuracy of the estimates extracted according to the fuzzy robust Bayesian method and for all simulation experiments.

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