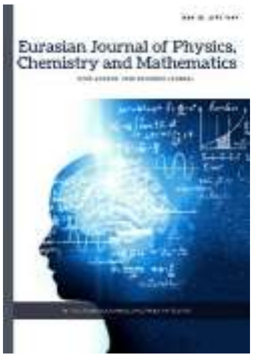

Almost Approximaitly Nearly Semiprime Submodules I

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ABSTRACT

Let Ɍ be a commutative ring with identity and Ḩ be left unitary Ɍ-module. This study deals the concept of almost approximaitly nearly semiprime submodules as new generalization of semiprime submodules, also generalizations of (prime, nearly prime, approximaitly prime and approximaitly semiprime) submodules. We give same basic properties, and characterizations of this concept. Furthermore, we explain the relationships between almost approximaitly nearly semiprime submodules with (semiprime, prime, nearly semiprime, nearly prime, approximaitly semiprime, and approximaitly prime) submodules.

Keywords: Prime submodules, Semi prime submodules, Nearly semi prime submodules, Approximaitly semi prime submodules, Almost approximaitly nearly semi prime submodules

1. Introduction

Prime submodules play an important role in module theory over commutative ring with identity, where a proper submodule F of an Ɍ-module Ḩ is called a prime, if for any $rh \in F$, for $r \in$ Ɍ, $h \in$ Ḩ, implies that either $h \in F$ or $r \in [F:_{Ɍ} Ḩ]$, where $[F:_{Ɍ} Ḩ] = \{r \in Ɍ: rḨ \subseteq F\}$[1]. Many authors have focused on generalizing the concept of a prime submodule such as "Nearly prime and Approximaitly prime" submodules see [2, 3]. The concept semiprime submodule which was first introduced in [1] and extensively studied in [4] is given as a proper submodule F of an Ɍ-module Ḩ is called semiprime submodule, if for any $r^n h \in F$, for $r \in$ Ɍ, $h \in$ Ḩ and $n \in Z^+$, implies that $rh \in F$. [4] characterized semiprime submodules as a proper submodule F of an Ɍ-module Ḩ is semiprime if and only if for any $r^2 h \in F$, for $r \in$ Ɍ, $h \in$ Ḩ implies that $rh \in F$. Recently extensive research has been done on generalizations of semiprime submodules see for example [5, 6]. In this research, we introduce and study a new generalization that we called almost approximaitly nearly semiprime submodules as a proper submodule F of an Ɍ-module Ḩ is almost approximaitly nearly semiprime (simply Alappns-prime) submodule, if for any $r^n h \in F$, for $r \in$ Ɍ, $h \in$ Ḩ, and $n \in Z^+$, implies that $rh \in F + (soc(Ḩ) + J(Ḩ))$. Where $J(Ḩ)$ is the Jacobson radical of Ḩ defined to be the intersection of all maximal submodules of Ḩ. And $soc(Ḩ)$ is the socle of Ḩ defined by the intersection of all essential submodule of Ḩ [7]. Anon-zero submodule F of an Ɍ-module Ḩ is called essential in Ḩ if $F \cap E \neq (0)$ for each non-zero submodule E of Ḩ [8].

2. Preliminaries

This section includes some well-known definitions, remarks and propositions that will be needed for us in our study of the next sections.

Definition (2.1) [9]

An envelope of a submodule F of an Ɽ-module Ⱨ denoted by $E_{Ⱨ}(F)$ defined by $E_{Ⱨ}(F) = \{rh: r \in Ɽ, h \in Ⱨ$ such that $r^n h \in F, n \in Z^+\}$ and $F \subseteq E_{Ⱨ}(F)$.

Definition (2.2) [10]

A submodule F of an Ɽ-module Ⱨ is called small if $F + E = Ⱨ$ implies that $E = Ⱨ$ for any proper submodule E of Ⱨ.

Definition (2.3) [7]

An Ɽ-module Ⱨ is a semi simple if and only if $soc(Ⱨ) = Ⱨ$.

Proposition (2.4) [11, Exe. (12). P 239]

1. Let F be submodule of an Ɽ-module Ⱨ with F is a direct summand of Ⱨ then $J\left(\frac{Ⱨ}{F}\right) = \frac{J(Ⱨ)+F}{F}$.
2. An Ɽ-module Ⱨ is semi simple if and only if for each submodule F of Ⱨ, $soc\left(\frac{Ⱨ}{F}\right) = \frac{soc(Ⱨ)+F}{F}$.

Proposition (2.5) [12, Ex. 12(5). p 242]

A submodule F of an Ɽ-module Ⱨ is maximal and essential if and only if $soc(Ⱨ) \subseteq F$.

Definition (2.6) [10]

A ring Ɽ is called Boolean ring if $r^2 = r$ for every element r of Ɽ.

Proposition (2.7) [11, Theo. (9.1.4)(b)]

Let Ⱨ be an Ɽ-module, and F be a proper submodule of Ⱨ. If $J\left(\frac{Ⱨ}{F}\right) = (0)$ then $J(Ⱨ) \subseteq F$.

Remark (2.8) [4, Rem. and Exam. (1.3)]

Every prime submodule of an R-module M is a semiprime

3. Basic Properties and Characterizations of Almost Approximaitly Nearly Semiprime Submodules

In this section we introduced the definition of almost approximaitly nearly semiprime submodules as new generalization of (semiprime, prime, nearly prime, approximaitly prime and approximaitly semiprime) submodules and we give some basic properties and characterizations of this concept.

Definition (3.1)

A proper submodule F of an Ɽ-module Ⱨ is called almost approximaitly nearly semiprime (simply Alappns-prime) submodule, if for any $r^n h \in F$, for $r \in Ɽ$, $h \in Ⱨ$, and $n \in Z^+$, implies that $rh \in F + (soc(Ⱨ) + J(Ⱨ))$. And an ideal J of a ring Ɽ is called Alappns-prime ideal of Ɽ if J is an Alappns-prime Ɽ-submodule of an Ɽ-module Ɽ. In particular a proper submodule F of an Ɽ-module Ⱨ is Alappns-prime if for any $r^2 h \in F$, for $r \in Ɽ$, $h \in Ⱨ$, implies that $rh \in F + (soc(Ⱨ) + J(Ⱨ))$.

Remarks and Examples (3.2)

1. Let $Ⱨ = Z_{72}$, $Ɽ = Z$, the submodule $F = \langle\bar{4}\rangle$ is an Alappns-prime submodule of Z_{72}. Thus for each $r \in Z$, $h \in Z_{72}$, and $n \in Z^+$, if $r^n h \in F$, implies that $rh \in F + \left(soc(Z_{72}) + J(Z_{72})\right) = \langle\bar{4}\rangle + (\langle\overline{12}\rangle + \langle\bar{6}\rangle) = \langle\bar{2}\rangle$. That is if $2^2.\bar{2} = \bar{8} \in F$, for $2 \in Z$, $\bar{2} \in Z_{72}$, implies that $2.\bar{2} \in F + \left(soc(Z_{72}) + J(Z_{72})\right) = \langle\bar{2}\rangle$.
2. It's obvious that every semiprime submodule of an Ɽ-module Ⱨ is an Alappns-prime submodule of Ⱨ, but contrariwise isn't true as in example:

   Let $Ⱨ = Z_{72}$, $Ɽ = Z$, the submodule $F = \langle\bar{4}\rangle$ is an Alappns-prime submodule for Z_{72}[see (1)]. But F is not semiprime submodule for Z_{72}, because $2^2.\bar{1} = \bar{4} \in F$, for $2 \in Z$, $\bar{1} \in Z_{72}$, but $2.\bar{1} \notin F$.
3. It's obvious that every prime submodule of an Ɽ-module Ⱨ is an Alappns-prime submodule of Ⱨ, but contrariwise isn't true as in example:

   Let $Ⱨ = Z_{72}$, $Ɽ = Z$, the submodule $F = \langle\bar{4}\rangle$ is an Alappns-prime submodule for Z_{72}[see (1)]. But F is not prime submodule for Z_{72}, because $2.\bar{2} \in F$, but neither $\bar{2} \in F$ nor $2 \in [F:_Z Z_{72}] = 4Z$.
4. It's obvious that every approximaitly semiprime submodule of an Ɽ-module Ⱨ is an Alappns-prime submodule of Ⱨ, but contrariwise isn't true as in example:

   Let $Ⱨ = Z_{72}$, $Ɽ = Z$ and the submodule $F = \langle\bar{4}\rangle$ is an Alappns-prime submodule for Z_{72}[see (1)]. But F is not approximaitly semiprime submodule for Z_{72}, because $2^2.\bar{1} \in F$, but $2.\bar{1} \notin F + soc(Z_{72}) = \langle\bar{4}\rangle + \langle\overline{12}\rangle = \langle\bar{4}\rangle$.
5. It's obvious that every nearly prime submodule of an Ɽ-module Ⱨ is an Alappns-

prime submodule of Ɥ, but contrariwise isn't true as in example:

Let $Ɥ = Z_{12}$, $Ꞧ = Z$ and the submodule $F = \langle\bar{6}\rangle$ is an Alappns-prime submodule for Z_{12}. Thus for each $r \in Z$, $h \in Z_{12}$ and $n \in Z^+$, if $r^n h \in F$, implies that $rh \in F + \left(soc(Z_{12}) + J(Z_{12})\right) = \langle\bar{6}\rangle + (\langle\bar{2}\rangle + \langle\bar{6}\rangle) = \langle\bar{2}\rangle$. That is if $3^2.\bar{2} \in F$, for $3 \in Z$, $\bar{2} \in Z_{12}$, implies that $3.\bar{2} \in F + \left(soc(Z_{12}) + J(Z_{12})\right) = \langle\bar{2}\rangle$. But F is not nearly prime submodule for Z_{12}, because $3.\bar{2} \in F$, but neither $\bar{2} \in F + J(Z_{12}) = \langle\bar{6}\rangle + \langle\bar{6}\rangle = \langle\bar{6}\rangle$ nor $3 \in [F + J(Z_{12}):_Z Z_{12}] = [\langle\bar{6}\rangle:_Z Z_{12}] = 6Z$.

6. It's obvious that every approximaitly prime submodule of an Ꞧ-module Ɥ is an Alappns-prime submodule of Ɥ, but contrariwise isn't true as in example:

Let $Ɥ = Z_{72}$, $Ꞧ = Z$ and the submodule $F = \langle\bar{4}\rangle$ is an Alappns-prime submodule for Z_{72}[see (1)]. But F is not approximaitly prime submodule for Z_{72}, because $2.\bar{2} \in F$, but $\bar{2} \notin F + soc(Z_{72}) = \langle\bar{4}\rangle + \langle\overline{12}\rangle = \langle\bar{4}\rangle$ and $2 \notin [F + soc(Z_{72}):_Z Z_{72}] = [\langle\bar{4}\rangle:_Z Z_{72}] = 4Z$.

7. The intersection of two Alappns-prime submodules of an Ꞧ-module Ɥ need not be an Alappns-prime submodule of Ɥ. The following example explains that:

Consider the Z-module Z and the submodules $F = 3Z$ and $K = 4Z$ are Alappns-prime submodules of Z, but $F \cap K = 12Z$ is not Alappns-prime submodules of Z, because $2^2.3 \in 12Z$, for $2, 3 \in Z$, but $2.3 \notin 12Z + (soc(Z) + J(Z)) = 12Z$.

8. The residual of Alappnq-prime submodule of an Ꞧ-module Ɥ is not Alappnq-prime ideal of Ꞧ. The following example explains that:

Let $Ɥ = Z_{72}$, $Ꞧ = Z$, the submodule $F = \langle\bar{4}\rangle$. F is an Alappnq-prime submodule of Z_{72} [see (1)]. But $[F:_Z Z_{72}]$=$[\langle\bar{4}\rangle:_Z Z_{72}] = 4Z$ is not an Alappnq-prime ideal of Z because $2.2.1 \in 4Z$ for $2,1 \in Z$ and $2.1 \notin 4Z + (soc(Z) + J(Z)) = 4Z + (0) = 4Z$.

Now, we introduce many characterizations of Alappns-prime submodules.

Proposition (3.3)

Let Ɥ be an Ꞧ-module, and $F \subsetneq Ɥ$. Then F is an Alappns-prime submodule of Ɥ if and only if for any $I^n L \subseteq F$, for I is an ideal of Ꞧ, L is a submodule of Ɥ and $n \in Z^+$, implies that $IL \subseteq F + (soc(Ɥ) + J(Ɥ))$.

Proof

(⟹) Let $x \in IL$, implies that $x = r_1x_1 + r_2x_2 + \cdots + r_nx_n$, where $r_i \in J$, $x_i \in L$, $i = 1,2,3, \dots\dots, n$, that is $r_ix_i \in JL$ for all $i = 1,2,3, \dots\dots, n$, it follows that ${r^n}_i x_i \in J^n L \subseteq F$. That is ${r^n}_i x_i \in F$, but F is an Alappns-prime submodule of Ɥ, implies that $r_ix_i \in F + (soc(Ɥ) + J(Ɥ))$ for each $i = 1,2, \dots, n$. That is $x \in F + (soc(Ɥ) + J(Ɥ))$. Hence $JL \subseteq F + (soc(Ɥ) + J(Ɥ))$.

(⟸) Let $r^n h \in F$, for $r \in Ꞧ$, $h \in Ɥ$ and $n \in Z^+$, implies that $\langle r\rangle^n Ꞧh \subseteq F$. Thus by hypothesis we have $\langle r\rangle Ꞧh \subseteq F + (soc(Ɥ) + J(Ɥ))$, that is $rh \in \langle r\rangle Ꞧh \subseteq F + (soc(Ɥ) + J(Ɥ))$, implies $rh \in F + (soc(Ɥ) + J(Ɥ))$. Hence Ɥ is an Alappns-prime submodule of Ɥ.

The following corollaries are direct consequence of proposition (3.3).

Corollary (3.4)

Let Ɥ be an Ꞧ-module, and $F \subsetneq Ɥ$. Then F is an Alappns-prime submodule of Ɥ if and only if for any $I^2 L \subseteq F$, for I is an ideal of Ꞧ, L is a submodule of Ɥ, implies that $IƗ \subseteq F + (soc(Ɥ) + J(Ɥ))$.

Corollary (3.5)

Let Ɥ be an Ꞧ-module, and $F \subsetneq Ɥ$. Then F is an Alappns-prime submodule of Ɥ if and only if for any $I^2 h \subseteq F$, for I is an ideal of Ꞧ, $h \in Ɥ$, implies that $Ih \subseteq F + (soc(Ɥ) + J(Ɥ))$.

Proposition (3.6)

Let Ɥ be an Ꞧ-module, and $F \subsetneq Ɥ$. Then F is an Alappns-prime submodule of Ɥ if and only if $\left[F:_Ɥ r^n\right] \subseteq \left[F + (soc(Ɥ) + J(Ɥ)):_Ɥ r\right]$, for $r \in Ꞧ$ and $n \in Z^+$

Proof

(⟹) Let $h \in \left[F:_Ɥ r^n\right]$, implies that $r^n h \in F$. But F is an Alappns-prime submodule of Ɥ, then $rh \in F + (soc(Ɥ) + J(Ɥ))$, that is $h \in \left[F + (soc(Ɥ) + J(Ɥ)):_Ɥ r\right]$. Thus $\left[F:_Ɥ r^n\right] \subseteq \left[F + soc(Ɥ) + J(Ɥ):_Ɥ r\right]$.

(⟸) Suppose that $r^n h \in F$, for $r \in Ꞧ$, $h \in Ɥ$, and $n \in Z^+$, it follows that $h \in \left[F:_Ɥ r^n\right] \subseteq \left[F + (soc(Ɥ) + J(Ɥ)):_Ɥ r\right]$, implies that $h \in \left[F + (soc(Ɥ) + J(Ɥ)):_Ɥ r\right]$, so $rh \in F +$

$(soc(Ḥ) + J(Ḥ))$. Therefore F is an Alappns-prime submodule of Ḥ.

As a direct can sequence of proposition (3.6) we get the following corollaries.

Corollary (3.7)

Let Ḥ be an Ɍ-module, and $F \subsetneq$ Ḥ. Then F is an Alappns-prime submodule of Ḥ if and only if for any $r^n L \subseteq F$, for $r \in$ Ɍ, L is a submodule of Ḥ, and $n \in Z^+$, implies that $rL \subseteq F + (soc(Ḥ) + J(Ḥ))$.

Corollary (3.8)

Let Ḥ be an Ɍ-module, and $F \subsetneq$ Ḥ. Then F is an Alappns-prime submodule of Ḥ if and only if for any $r^2 L \subseteq F$, for $r \in$ Ɍ, L is submodule of Ḥ, implies that $rL \subseteq F + (soc(Ḥ) + J(Ḥ))$.

Proposition (3.9)

Let F be a proper submodule of an Ɍ-module Ḥ. Then F is an Alappns-prime submodule of Ḥ if and only if $E_{Ḥ}(F) \subseteq F + (soc(Ḥ) + J(Ḥ))$.

Proof

($\Longrightarrow$) Let $x \in E_{Ḥ}(F)$, implies that $x = rh$, for $r \in$ Ɍ, $h \in$ Ḥ such that $r^n h \in F$ for some $n \in Z^+$. But F is an Alappns-prime submodule of Ḥ, then $rh \in F + (soc(Ḥ) + J(Ḥ))$, that is $x \in F + (soc(Ḥ) + J(Ḥ))$ so $E_{Ḥ}(F) \subseteq F + (soc(Ḥ) + J(Ḥ))$.

($\Longleftarrow$) Suppose that $r^n h \in F$, for $r \in$ Ɍ, $h \in$ Ḥ, and $n \in Z^+$. Since $r^n h \in F$ then $rh \in E_{Ḥ}(F) \subseteq F + (soc(Ḥ) + J(Ḥ))$[by hypothesis]. It follows that $rh \in F + (soc(Ḥ) + J(Ḥ))$. Hence F is an Alappns-prime submodule of Ḥ.

Proposition (3.10)

Let Ḥ be an Ɍ-module, and F be a submodule of Ḥ with $soc(Ḥ) + J(Ḥ) \subseteq F$. Then F is an Alappns-prime submodule of Ḥ if and only if $[F :_{Ḥ} I]$ is an Alappns-prime submodule of Ḥ, for every nonzero ideal I of Ɍ.

Proof

($\Longrightarrow$) Suppose that F is an Alappns-prime submodule of Ḥ, and let $r^n h \in [F :_{Ḥ} I]$, for $r \in$ Ɍ, $h \in$ Ḥ, $n \in Z^+$, and I is an ideal of Ɍ, then $r^n(hI) \subseteq F$. But F is an Alappns-prime submodule of Ḥ, then $r(hI) \subseteq F + soc(Ḥ) + J(Ḥ)$, but $soc(Ḥ) + J(Ḥ) \subseteq F$, implies that $F + soc(Ḥ) + J(Ḥ) = F$. Thus $r(hI) \subseteq F$, it follows that $rh \in [F :_{Ḥ} I]$. That is $rh \in [F :_{Ḥ} I] + soc(Ḥ) + J(Ḥ)$. Hence $[F :_{Ḥ} I]$ is an Alappns-prime submodule of Ḥ.

($\Longleftarrow$) Follows by put $I =$ Ɍ.

Proposition (3.11)

Let F be a proper submodule of an Ɍ-module Ḥ. Then $F + (soc(Ḥ) + J(Ḥ))$ is an Alappns-prime submodule of Ḥ if and only if $[F + (soc(Ḥ) + J(Ḥ)) :_{R} Ḥ]$ is a semiprime ideal of Ɍ.

Proof

($\Longrightarrow$) Suppose that $F + (soc(Ḥ) + J(Ḥ))$ is an Alappns-prime submodule of Ḥ, to prove that $\sqrt{[F + (soc(Ḥ) + J(Ḥ)) :_{R} Ḥ]} = [F + (soc(Ḥ) + J(Ḥ)) :_{R} Ḥ]$. It is well-known that $[F + (soc(Ḥ) + J(Ḥ)) :_{R} Ḥ] \subseteq \sqrt{[F + (soc(Ḥ) + J(Ḥ)) :_{R} Ḥ]}$, to prove only $\sqrt{[F + (soc(Ḥ) + J(Ḥ)) :_{R} Ḥ]} \subseteq [F + (soc(Ḥ) + J(Ḥ)) :_{R} Ḥ]$. Let $r \in \sqrt{[F + (soc(Ḥ) + J(Ḥ)) :_{R} Ḥ]}$ implies that $r^n \in [F + (soc(Ḥ) + J(Ḥ)) :_{R} Ḥ]$ for some $n \in Z^+$, it follows that $r^n Ḥ \subseteq F + (soc(Ḥ) + J(Ḥ))$, then $r^n h \in F + (soc(Ḥ) + J(Ḥ))$ for all $h \in$ Ḥ. But $F + (soc(Ḥ) + J(Ḥ))$ is an Alappns-prime submodule of Ḥ, then $rh \in F + (soc(Ḥ) + J(Ḥ))$ for all $h \in$ Ḥ. That is $rḤ \subseteq F + (soc(Ḥ) + J(Ḥ))$, hence $r \in [F + (soc(Ḥ) + J(Ḥ)) :_{R} Ḥ]$. Hence $\sqrt{[F + (soc(Ḥ) + J(Ḥ)) :_{R} Ḥ]} \subseteq [F + (soc(Ḥ) + J(Ḥ)) :_{R} Ḥ]$. Thus $\sqrt{[F + (soc(Ḥ) + J(Ḥ)) :_{R} Ḥ]}$ $=[F + (soc(Ḥ) + J(Ḥ)) :_{R} Ḥ]$. Therefore $[F + (soc(Ḥ) + J(Ḥ)) :_{R} Ḥ]$ is a semiprime ideal of Ɍ.

($\Longleftarrow$) Let $r^n h \in F + soc(Ḥ) + J(Ḥ)$, for $r \in$ Ɍ, $h \in$ Ḥ and $n \in Z^+$, implies that $r^n Ḥ \subseteq F + (soc(Ḥ) + J(Ḥ))$, that is $r^n \in [F + (soc(Ḥ) + J(Ḥ)) :_{R} Ḥ]$, implies that $r \in \sqrt{[F + soc(Ḥ) + J(Ḥ) :_{R} Ḥ]}$, but $[F + (soc(Ḥ) + J(Ḥ)) :_{R} Ḥ]$ is a semiprime ideal of Ɍ then $r \in [F + (soc(Ḥ) + J(Ḥ)) :_{R} Ḥ]$, that is $rḤ \subseteq F + (soc(Ḥ) + J(Ḥ))$, hence $rh \in F + (soc(Ḥ) + J(Ḥ))$ for all $h \in$ Ḥ. That is $F + (soc(Ḥ) + J(Ḥ))$ is an Alappns-prime submodule of Ḥ.

Now, we give some basic properties of Alappns-prime submodules.

Proposition (3.12)

Let F be semi simple submodule of Ɍ-module Ḥ. If $soc(Ḥ)$ is a semiprime submodule of Ḥ, then F is an Alappns-prime submodule of Ḥ.

Proof

Let $r^n h \in F$, for $r \in$ Ɍ, $h \in$ Ḥ, and $n \in Z^+$. Since F is semi simple submodule of Ḥ then $F \subseteq Ḥ = soc(Ḥ)$, so $r^n h \in soc(Ḥ)$. But $soc(Ḥ)$

is semiprime submodule of Ⱨ then $rh \in soc(Ⱨ) \subseteq F + (soc(Ⱨ) + J(Ⱨ))$. Hence $rh \in F + (soc(Ⱨ) + J(Ⱨ))$. Therefore F is an Alappns-prime submodule of Ⱨ.

Proposition (3.13)

Let F be a small submodule of an Ɍ-module Ⱨ. If $J(Ⱨ)$ is a semiprime submodule of Ⱨ, then F is an Alappns-prime submodule of Ⱨ.

Proof

Let $r^n h \in F$, for $r \in Ɍ$, $h \in Ⱨ$, and $n \in Z^+$. Since F is small submodule then $F \subseteq J(Ⱨ)$ by definition of $J(Ⱨ)$, so $r^n h \in J(Ⱨ)$. But $J(Ⱨ)$ is a semiprime submodule of $rh \in J(Ⱨ) \subseteq F + (soc(Ⱨ) + J(Ⱨ))$. Hence $rh \in F + (soc(Ⱨ) + J(Ⱨ))$. Thus F is Alappns-prime submodule of Ⱨ.

Proposition (3.14)

Let Ⱨ be an Ɍ-module, and F, L are submodules of Ⱨ such that $L \subsetneq F$, and F is a proper submodule of Ⱨ .If F is an Alappns-prime submodule of Ⱨ, then $\frac{F}{L}$ is an Alappns-prime submodule of $\frac{Ⱨ}{L}$ **.**

Proof

Suppose that F is an Alappns-prime submodule of Ⱨ, and let $r^2(h+L) = r^2 h + L \in \frac{F}{L}$, for $r \in Ɍ$, $h + L \epsilon \frac{Ⱨ}{L}$, $h \in Ⱨ$. Then $r^2 h \in F$. But F is an Alappns-prime submodule of Ⱨ, implies that $rh \in F + (soc(Ⱨ) + J(Ⱨ))$. It follows that $rh + L \in \frac{F + (soc(Ⱨ) + J(Ⱨ))}{L}$, that is $rh + L \in \frac{F}{L} + \frac{F + soc(Ⱨ)}{L} + \frac{F + J(Ⱨ)}{L} \subseteq \frac{F}{L} + soc\left(\frac{Ⱨ}{L}\right) + J\left(\frac{Ⱨ}{L}\right)$. Thus $rh + L \in \frac{F}{L} + soc\left(\frac{Ⱨ}{L}\right) + J\left(\frac{Ⱨ}{L}\right)$. Hence $\frac{F}{L}$ is an Alappns-prime submodule of $\frac{Ⱨ}{L}$**.**

Now, we give the converse of proposition (3.14).

Proposition (3.15)

Let Ⱨ be a semi simple Ɍ-module, and F, L are submodules of Ⱨ such that L is containd in F, and F is a proper submodule of Ⱨ. If L and $\frac{F}{L}$ are Alappns-prime submodules of Ⱨ and $\frac{Ⱨ}{L}$ respectively, then F is an Alappns-prime submodule of Ⱨ**.**

Proof

Suppose that L and $\frac{F}{L}$ are Alappns-prime submodules of Ⱨ and $\frac{Ⱨ}{L}$ respectively, and let $r^n h \in F$, for $r \in Ɍ$, $h \in Ⱨ$ and $n \in Z^+$. So $r^n(h+L) = r^n h + L \in \frac{F}{L}$. If $r^n h \in L$ and L is an Alappns-prime submodule of Ⱨ, implies that $r^n h \in L + (soc(Ⱨ) + J(Ⱨ)) \subseteq F + (soc(Ⱨ) + J(Ⱨ))$, implies that F is an Alappns-prime submodule of Ⱨ. So, we may assume that $r^n h \notin L$. It follows that $r^n(h+L) \in \frac{F}{L}$, but $\frac{F}{L}$ is an Alappns-prime submodule of $\frac{Ⱨ}{L}$ **,** implies that $r(h+L) \in \frac{F}{L} + soc\left(\frac{Ⱨ}{L}\right) + J\left(\frac{Ⱨ}{L}\right)$. Since Ⱨ is a semi simple then by proposition (2.4) $r(h+L) \in \frac{F}{L} + \frac{L + soc(Ⱨ)}{L} + \frac{L + J(Ⱨ)}{L}$. Since $L \subseteq F$, it follows that $L + soc(Ⱨ) \subseteq F + soc(Ⱨ)$ and $L + J(Ⱨ) \subseteq F + J(Ⱨ)$, hence $\frac{F}{L} + \frac{L + soc(Ⱨ)}{L} + \frac{L + J(Ⱨ)}{L} \subseteq \frac{F}{L} + \frac{F + soc(Ⱨ)}{L} + \frac{F + J(Ⱨ)}{L}$. Thus $(h+L) \in \frac{F + (soc(Ⱨ) + J(Ⱨ))}{L}$, that is $rh \in F + (soc(Ⱨ) + J(Ⱨ))$. Hence F is an Alappns-prime submodule of Ⱨ.

The following proposition shows that the intersection of two Alappn-prime submodules is an Alappn-prime submodule under certain condition.

Proposition (3.16)

Let Ⱨ be an Ɍ-module with either F or K are maximal essential submodule of Ⱨ and $K \nsubseteq F$. If F and K are Alappns-prime submodules of Ⱨ then $F \cap K$ is Alappns-prime submodule of Ⱨ.

Proof

Let $r^n h \in F \cap K$, for $r \in Ɍ$, $h \in Ⱨ$, and $n \in Z^+$, then $r^n h \in F$ and $r^n h \in K$. But F and K are Alappns-prime submodules of Ⱨ, then $rh \in F + (soc(Ⱨ) + J(Ⱨ))$ and $rh \in K + (soc(Ⱨ) + J(Ⱨ))$. Thus $rh \in (F + (soc(Ⱨ) + J(Ⱨ))) \cap (K + (soc(Ⱨ) + J(Ⱨ)))$. Since F or K are maximal essential in Ⱨ then by proposition (2.5) $soc(Ⱨ) \subseteq F$ or $soc(Ⱨ) \subseteq K$ and since F or K are maximal then $J(Ⱨ) \subseteq F$ or $J(Ⱨ) \subseteq K$. Suppose F is a maximal essential in Ⱨ so $soc(Ⱨ) \subseteq F$ and $J(Ⱨ) \subseteq F$ and hence $(soc(Ⱨ) + J(Ⱨ)) \subseteq F$, it follows that $rh \in F \cap \left(K + (soc(Ⱨ) + J(Ⱨ))\right)$. Therefor by Modular law we have $rh \in (F \cap K) + (soc(Ⱨ) + J(Ⱨ))$. Thus $F \cap K$ is an Alappns-prime submodule of Ⱨ.

Similar arguments follows if K is maximal essential.

Proposition (3.17)

Let $f: Ⱨ \to Ⱨ'$ be an Ɍ-epimorphism, and $\ker f$ is a small submodule of Ⱨ. If F is an

Alappns-prime submodule of Ḥ′, then $f^{-1}(F)$ is an Alappns-prime submodule of Ḥ.

Proof

It is obvious that $f^{-1}(F)$ is a proper submodule of Ḥ. Now, suppose $r^n h \in f^{-1}(F)$, for $r \in$ Ꞧ, $h \in$ Ḥ, and $n \in Z^+$, implies that $r^n f(h) \in F$. But F is an Alappns-prime submodule of Ḥ′, then $rf(h) \in F + (soc(Ḥ') + J(Ḥ'))$. It follows that $rh \in f^{-1}(F) + f^{-1}soc(Ḥ') + f^{-1}J(Ḥ') \subseteq f^{-1}(F) + soc(Ḥ) + J(Ḥ)$. That is $rh \in f^{-1}(F) + (soc(Ḥ) + J(Ḥ))$. Therefore $f^{-1}(F)$ is an Alappns-prime submodule of Ḥ.

Proposition (3.18)

Let $f: Ḥ \to Ḥ'$ be an Ꞧ-epimorphism, and $\ker f$ is a small submodule of Ḥ. If F is an Alappns-prime submodule of Ḥ with $\ker f \subseteq F$. Then $f(F)$ is an Alappns-prime submodule of Ḥ′.

Proof

It is obvious that $f(F)$ is a proper submodule of Ḥ′. Let $I^2 h' \subseteq f(F)$, for I is an ideal of Ꞧ, $h' \in$ Ḥ′, implies $I^2 f(h) \subseteq f(F)$ for some $h \in$ Ḥ (since f is onto), that is $f(I^2 h) \subseteq f(F)$, $f(I^2 h) = f(n)$ for some $n \in F$ that is $f(n - I^2 h) = 0$, so $n - I^2 h \in Ker\, f \subseteq F$, implies that $I^2 h \subseteq F$. But F is an Alappns-prime submodule of Ḥ, then by corollary (3.5) $Ih \subseteq F + soc(Ḥ) + J(Ḥ)$. Hence $Ih' = If(h) \subseteq f(F) + fsoc(Ḥ) + fJ(Ḥ) \subseteq f(F) + soc(Ḥ') + J(Ḥ')$. Thus $f(F)$ is an Alappns-prime submodule of Ḥ′.

4. Sufficient Conditions Alappns-prime Submodules to be (Semiprime, Prime, Nearly semiprime, Nearly prime, Approximaitly semiprime, and Approximaitly prime) Submodules.

In this section we introduced the relationships between semiprime, prime, nearly semiprime, nearly prime, approximaitly semiprime, approximaitly prime and Alappns-prime submodules.

We start by this remark that establishes the relationship of semiprime and Alappns-prime submodules.

Remark (4.1) [Remarks and Examples (3.2) (2)]

Every semiprime submodule of an Ꞧ-module Ḥ is Alappns-prime, but the converse isn’t true.

The following results showed that under certain conditions the reverse implication is holds.

Proposition (4.2)

Let Ḥ be Ꞧ-module and F is an $e$ssential submodule of Ḥ with $J(Ḥ) \subseteq F$. Then F is semiprime if and only if F is Alappns-prime submodule of Ḥ.

Proof

(⟾) Direct.

(⟽) Let $r^n h \in F$ for $r \in$ Ꞧ, $h \in$ Ḥ, and $n \in Z^+$, hence $rh \in F + (soc(Ḥ) + J(Ḥ))$. Since F is $e$ssential submodule of Ḥ, then $soc(Ḥ) \subseteq F$ and by hypotheses $J(Ḥ) \subseteq F$, we get $F + (soc(Ḥ) + J(Ḥ)) = F$. Hence $rh \in F$. Therefore F is semiprime submodule of Ḥ.

The following corollaries are direct $c$onsequence of proposition (4.2).

Corollary (4.3)

Let Ḥ be Ꞧ-module and F is a proper submodule of Ḥ with $soc(Ḥ) + J(Ḥ) \subseteq F$. Then F is semiprime if and only if F is Alappns-prime submodule of Ḥ.

Corollary (4.4)

Let Ḥ be Ꞧ-module and F is maximal submodule of Ḥ with $soc(Ḥ) \subseteq F$. Then F is semiprime if and only if F is Alappns-prime submodule of Ḥ.

Corollary (4.5)

Let Ḥ be Ꞧ-module and F is maximal and essential submodule of Ḥ. Then F is semiprime if and only if F is Alappns-prime submodule of Ḥ.

Corollary (4.6)

Let Ḥ be Ꞧ-module with $soc(Ḥ) + J(Ḥ) = (0)$ and $F \subset$ Ḥ. Then F is semiprime if and only if F is Alappns-prime submodule of Ḥ.

Remark (4.7) [Remarks and Examples (3.2) (3)]

Every prime submodule of an Ꞧ-module Ḥ is an Alappns-prime submodule of Ḥ, but contrariwise isn't true.

The following results showed that under certain conditions the reverse implication is holds.

Proposition (4.9)

Let Ḥ be an Ꞧ-module over Boolean ring and F is proper submodule of Ḥ with $J\left(\frac{Ḥ}{F}\right) = (0)$ and $soc(Ḥ) \subseteq F$. Then F is prime if and only if F is Alappns-prime submodule of Ḥ.

Proof

($\Longleftarrow\!\!\!\Longrightarrow$) Direct.

($\Longleftarrow$) Since $J\left(\frac{Ḩ}{F}\right) = (0)$, then by proposition (2.7) we get $J(Ḩ) \subseteq F$, but $soc(Ḩ) \subseteq F$, hence $F + soc(Ḩ) = F$ and $F + \big(soc(Ḩ) + J(Ḩ)\big) = F + J(Ḩ) = F$. Let $rh \in F$ for $r \in Ɍ$, $h \in Ḩ$ with $h \notin F$. Since Ɍ is Boolean ring then $r^2h = rh \in F$. But F is Alappns-prime, then $rh \in F + (soc(Ḩ) + J(Ḩ)) = F$. Thus $rḨ \subseteq F$. Hence F is prime submodule of Ḩ.

Proposition (4.10)

Let Ḩ be an Ɍ-module over Boolean ring and F is an $e$ssential submodule of Ḩ with $J(Ḩ) \subseteq F$. Then F is prime if and only if F is Alappns-prime submodule of Ḩ.

Proof

($\Longrightarrow$) Direct.

($\Longleftarrow$) Let $rh \in F$ for $r \in Ɍ$, $h \in Ḩ$ with $h \notin F$, Since Ɍ is Boolean ring then $r^2h \in F$. But F is Alappns-prime, then $rh \in F + (soc(Ḩ) + J(Ḩ))$. But F is $e$ssential submodule of Ḩ, then $soc(Ḩ) \subseteq F$ and by hypotheses $J(Ḩ) \subseteq F$, we get $F + (soc(Ḩ) + J(Ḩ)) = F$. Therefore $rh \in F$, and hence $rḨ \subseteq F$.

The following corollaries are direct $c$onsequence of proposition (4.9) and proposition (4.10).

Corollary (4.11)

Let Ḩ be an Ɍ-module over Boolean ring and F is a proper submodule of Ḩ with $soc(Ḩ) + J(Ḩ) \subseteq F$. Then F is prime if and only if F is Alappns-prime submodule of Ḩ.

Corollary (4.12)

Let Ḩ be an Ɍ-module over Boolean ring and F is maximal submodule of Ḩ with $soc(Ḩ) \subseteq F$. Then F is prime if and only if F is Alappns-prime submodule of Ḩ.

Corollary (4.13)

Let Ḩ be an Ɍ-module over Boolean ring and F is maximal and essential submodule of Ḩ. Then F is prime if and only if F is Alappns-prime submodule of Ḩ.

Remark (4.14)

Every nearly semiprime submodule of an Ɍ-module Ḩ is an Alappns-prime submodule of Ḩ, but contrariwise isn't true.

Proof

Let F is nearly semiprime submodule of an Ɍ-module Ḩ, and $r^n h \in F$ for $r \in Ɍ$, $h \in Ḩ$, and $n \in Z^+$. Since F is nearly semiprime then $rh \in F + J(Ḩ) \subseteq F + (soc(Ḩ) + J(Ḩ))$. Hence $rh \in F + (soc(Ḩ) + J(Ḩ))$. Therefore F Alappns-prime submodule of Ḩ.

Proposition (4.15)

Let Ḩ be Ɍ-module, $soc(Ḩ) \subseteq J(Ḩ)$ and $F \subset Ḩ$. Then F is nearly semiprime if and only if F is Alappns-prime submodule of Ḩ.

Proof

($\Longrightarrow$) Direct.

($\Longleftarrow$) Let $r^n h \in F$ for $r \in Ɍ$, $h \in Ḩ$, and $n \in Z^+$. Since F is Alappns-prime, then $rh \in F + (soc(Ḩ) + J(Ḩ))$. Since $soc(Ḩ) \subseteq J(Ḩ)$, then $soc(Ḩ) + J(Ḩ) = J(Ḩ)$, thus $rh \in F + J(Ḩ)$. Hence F is nearly semiprime submodule of Ḩ.

Proposition (4.16)

Let Ḩ be Ɍ-module with $F \subset Ḩ$ and $soc(Ḩ) \subseteq F$. Then F is nearly semiprime if and only if F is Alappns-prime submodule of Ḩ.

Proof

($\Longrightarrow$) Direct.

($\Longleftarrow$) Let $r^n h \in F$ for $r \in Ɍ$, $h \in Ḩ$, and $n \in Z^+$. Since F is Alappns-prime, then $rh \in F + (soc(Ḩ) + J(Ḩ))$. Since $soc(Ḩ) \subseteq F$, then $F + soc(Ḩ) = F$, so $\big(F + soc(Ḩ) + J(Ḩ)\big) = F + J(Ḩ)$. Thus $rh \in F + J(Ḩ)$. Hence F is nearly semiprime submodule of Ḩ.

The proofs of the following results are direct.

Proposition (4.17)

Let Ḩ be Ɍ-module with $F \subset Ḩ$, and $soc(Ḩ) = (0)$. Then F is nearly semiprime if and only if F is Alappns-prime submodule of Ḩ.

Proposition (4.18)

Let Ḩ be Ɍ-module and F is an $e$ssential submodule of Ḩ. Then F is nearly semiprime if and only if F is Alappns-prime submodule of Ḩ.

Remark (4.19) [Remarks and Examples (3.2) (5)]

Every nearly prime submodule of an Ɍ-module Ḩ is an Alappns-prime submodule of Ḩ, but contrariwise isn't true.

The following results showed that under certain conditions the reverse implication is holds.

Proposition (4.20)

Let Ḩ be an Ɍ-module over Boolean ring, $soc(Ḩ) \subseteq J(Ḩ)$ and $F \subset Ḩ$. Then F is nearly prime if and only if F is Alappns-prime submodule of Ḩ.

Proof

($\Longmapsto$) Direct.

($\Longleftarrow$) Let $rh \in F$ for $r \in Ɍ$, $h \in Ḥ$ with $h \notin F + J(Ḥ)$. Since Ɍ is Boolean ring then $r^2h \in F$. But F is Alappns-prime, then $rh \in F + (soc(Ḥ) + J(Ḥ))$. Since $soc(Ḥ) \subseteq J(Ḥ)$, then $soc(Ḥ) + J(Ḥ) = J(Ḥ)$, it follows that $rh \in F + J(Ḥ)$. Thus $rḤ \subseteq F + J(Ḥ)$. Hence F is nearly prime submodule of Ḥ.

Proposition (4.21)

Let Ḥ be an Ɍ-module over Boolean ring with $F \subset Ḥ$ and $soc(Ḥ) \subseteq F$. Then F is nearly prime if and only if F is Alappns-prime submodule of Ḥ.

Proof

($\Longmapsto$) Direct.

($\Longleftarrow$) Let $rh \in F$ for $r \in Ɍ$, $h \in Ḥ$ with $h \notin F + J(Ḥ)$. Since Ɍ is Boolean ring then $r^2h \in F$. But F is Alappns-prime, then $rh \in F + (soc(Ḥ) + J(Ḥ)))$. But $soc(Ḥ) \subseteq F$, then $F + soc(Ḥ) = F$, so $F + \left(soc(Ḥ) + J(Ḥ)\right) = F + J(Ḥ)$, it follows that $rh \in F + J(Ḥ)$. Thus $rḤ \subseteq F + J(Ḥ)$. Hence F is nearly prime submodule of Ḥ.

The proofs of the following results are direct.

Proposition (4.22)

Let Ḥ be an Ɍ-module over Boolean ring with F is proper of Ḥ, and $soc(Ḥ) = (0)$. Then F is nearly prime if and only if F is Alappns-prime submodule of Ḥ.

Proposition (4.23)

Let Ḥ be an R-module over Boolean ring and F is an $e$ssential submodule of Ḥ. Then F is nearly prime if and only if F is Alappns-prime submodule of Ḥ.

Remark (4.24) [Remarks and Examples (3.2) (4)]

Every approximaitly semiprime submodule of an Ɍ-module Ḥ is an Alappns-prime submodule of Ḥ, but contrariwise isn't true.

The following results showed that under certain conditions the reverse implication is holds.

Proposition (4.25)

Let Ḥ be Ɍ-module with $F \subset Ḥ$ and $J(Ḥ) \subseteq F$. Then F is approximaitly semiprime if and only if F is Alappns-prime submodule of Ḥ.

Proof

($\Longmapsto$) Direct.

($\Longleftarrow$) Let $r^n h \in F$ for $r \in Ɍ$, $h \in Ḥ$, and $n \in Z^+$. Since F is Alappns-prime, then $rh \in F + (soc(Ḥ) + J(Ḥ))$. Since $J(Ḥ) \subseteq F$, then $F + J(Ḥ) = F$, so $F + \left(soc(Ḥ) + J(Ḥ)\right) = F + soc(Ḥ)$. Thus $rh \in F + soc(Ḥ)$. Hence F is approximaitly semiprime submodule of Ḥ.

Proposition (4.26)

Let Ḥ be Ɍ-module with $J(Ḥ) \subseteq soc(Ḥ)$, and F is a proper submodule of Ḥ. Then F is approximaitly semiprime if and only if F is Alappns-prime submodule of Ḥ.

Proof

($\Longmapsto$) Direct.

($\Longleftarrow$) Since $J(Ḥ) \subseteq soc(Ḥ)$, then $soc(Ḥ) + J(Ḥ) = soc(Ḥ)$, so $F + \left(soc(Ḥ) + J(Ḥ)\right) = F + soc(Ḥ)$. Let $r^n h \in F$ for $r \in Ɍ$, $h \in Ḥ$, and $n \in Z^+$. Since F is Alappns-prime, then $rh \in F + \left(soc(Ḥ) + J(Ḥ)\right) = F + soc(Ḥ)$. Thus $rh \in F + soc(Ḥ)$. Hence F is approximaitly semiprime submodule of Ḥ.

The proofs of the following results are direct.

Proposition (4.27)

Let Ḥ be Ɍ-module with $J(Ḥ) = soc(Ḥ) = (0)$, and F is a proper submodule of Ḥ. Then F is approximaitly semiprime if and only if F is Alappns-prime submodule of Ḥ.

Proposition (4.28)

Let F be a proper submodule of an Ɍ-module Ḥ with Ḥ has no maximal submodule. Then F is approximaitly semiprime if and only if F is Alappns-prime submodule of Ḥ.

Proposition (4.29)

Let Ḥ be Ɍ-module with $soc(Ḥ) \subseteq F$, $J(Ḥ) \subseteq F$ and $F \subset Ḥ$. Then the following concepts are equivalent:

1. F is semiprime submodule of Ḥ.
2. F is approximaitly semiprime submodule of Ḥ.
3. F is Alappns-prime submodule of Ḥ.
4. F is nearly semiprime submodule of Ḥ.

Proof

(1) $\Longrightarrow$ (2) Let $r^n h \in F$, for $r \in Ɍ$, $h \in Ḥ$, and $n \in Z^+$. Since F is semiprime submodule of Ḥ, then $rh \in F \subseteq F + soc(Ḥ)$. Thus $rh \in F + soc(Ḥ)$. Hence F is approximaitly semiprime submodule of Ḥ.

(2) $\Leftrightarrow$ (3) It follows by proposition (4.25).

(3) $\Leftrightarrow$ (4) It follows by proposition (4.16).

(4) $\Longrightarrow$ (1) Since $J(\text{Ḫ}) \subseteq F$, then $F + J(\text{Ḫ}) = F$, $r^n h \in F$, for $r \in \text{Ꞧ}$, $h \in \text{Ḫ}$, and $n \in Z^+$. Since F is nearly semiprime, then $rh \in F + J(\text{Ḫ}) = F$. Thus $rh \in F$. Therefore F is semiprime submodule of Ḫ.

Remark (4.30) [Remarks and Examples (3.2) (6)]

Every approximaitly prime submodule of an Ꞧ-module Ḫ is an Alappns-prime submodule of Ḫ, but contrariwise isn't true.

The following results showed that under certain conditions the reverse implication is holds.

Proposition (4.31)

Let Ḫ be an Ꞧ-module over Boolean ring, $J(\text{Ḫ}) \subseteq soc(\text{Ḫ})$ and $F \subset \text{Ḫ}$. Then F is approximaitly prime if and only if F is Alappns-prime submodule of Ḫ.

Proof

($\Longrightarrow$) Direct.

($\Longleftarrow$) Let $rh \in F$ for $r \in \text{Ꞧ}$, $h \in \text{Ḫ}$ with $h \notin F + soc(\text{Ḫ})$. Since Ꞧ is Boolean ring then $r^2 h \in F$. But F is Alappns-prime, then $rh \in F + (soc(\text{Ḫ}) + J(\text{Ḫ}))$. Since $J(\text{Ḫ}) \subseteq soc(\text{Ḫ})$, then $soc(\text{Ḫ}) + J(\text{Ḫ}) = soc(\text{Ḫ})$, it follows that $rh \in F + soc(\text{Ḫ})$. Thus $r\text{Ḫ} \subseteq F + soc(\text{Ḫ})$. Hence F is approximaitly prime submodule of Ḫ.

Proposition (4.32)

Let Ḫ be an Ꞧ-module over Boolean ring with $F \subset \text{Ḫ}$ and $J(\text{Ḫ}) \subseteq F$. Then F is approximaitly prime if and only if F is Alappns-prime submodule of Ḫ.

Proof

($\Longrightarrow$) Direct.

($\Longleftarrow$) Let $rh \in F$ for $r \in \text{Ꞧ}$, $h \in \text{Ḫ}$ with $h \notin F + soc(\text{Ḫ})$. Since Ꞧ is Boolean ring then $r^2 h \in F$. But F is Alappns-prime, then $rh \in F + (soc(\text{Ḫ}) + J(\text{Ḫ}))$. But $J(\text{Ḫ}) \subseteq F$, then $F + J(\text{Ḫ}) = F$, so $F + soc(\text{Ḫ}) + J(\text{Ḫ}) = F + soc(\text{Ḫ})$, it follows that $rh \in F + soc(\text{Ḫ})$. Thus $r\text{Ḫ} \subseteq F + soc(\text{Ḫ})$. Hence F is approximaitly prime submodule of Ḫ.

The proofs of the following results are direct.

Proposition (4.33)

Let Ḫ be an Ꞧ-module over Boolean ring and F is a maximal submodule of Ḫ. Then F is approximaitly prime if and only if F is Alappns-prime submodule of Ḫ.

Proposition (4.34)

Let Ḫ be an Ꞧ-module over Boolean ring and $F \subset \text{Ḫ}$ with $J\left(\frac{\text{Ḫ}}{F}\right) = (0)$. Then F is approximaitly prime if and only if F is Alappns-prime submodule of Ḫ.

Proposition (4.35)

Let Ḫ be an Ꞧ-module over Boolean ring with $soc(\text{Ḫ}) \subseteq F$, $J(\text{Ḫ}) \subseteq F$ and $F \subset \text{Ḫ}$. Then the following concepts are equivalent:

1. F is prime submodule of Ḫ.
2. F is semiprime submodule of Ḫ.
3. F is approximaitly semiprime submodule of Ḫ.
4. F is nearly semiprime submodule of Ḫ.
5. F is Alappns-prime submodule of Ḫ.
6. F is nearly prime submodule of Ḫ.
7. F is approximaitly prime submodule of Ḫ.

Proof

(1) $\Longrightarrow$ (2) It follows by Remark (2.8).

(2) $\Longrightarrow$ (3) Let F be semiprime submodule of Ḫ and $r^n h \in F$, for $r \in \text{Ꞧ}$, $h \in \text{Ḫ}$ and $n \in Z^+$. Since F is semiprime submodule of Ḫ, then $rh \in F \subseteq F + soc(\text{Ḫ})$. Thus $rh \in F + soc(\text{Ḫ})$. Hence F is an approximaitly semiprime submodule of Ḫ.

(3) $\Longrightarrow$ (4) Let F be approximaitly semiprime submodule of an Ꞧ-module Ḫ and $r^2 h \in F$, for $r \in \text{Ꞧ}$, $h \in \text{Ḫ}$. Since F is approximaitly semiprime submodule of Ḫ, then $rh \in F + soc(\text{Ḫ})$. But $soc(\text{Ḫ}) \subseteq F$, then $F + soc(\text{Ḫ}) = F$. Thus $rh \in F \subseteq F + J(\text{Ḫ})$. Hence F is nearly semiprime submodule of Ḫ.

(4) $\Leftrightarrow$ (5) It follows by proposition (4.16).

(5) $\Leftrightarrow$ (6) It follows by proposition (4.21).

(6) $\Longrightarrow$ (7) Let F be nearly prime submodule of an Ꞧ-module Ḫ and $rh \in F$, for $r \in \text{Ꞧ}$, $h \in \text{Ḫ}$. Since F is nearly prime submodule of Ḫ, then either $h \in F + J(\text{Ḫ})$ or $r\text{Ḫ} \subseteq F + J(\text{Ḫ})$. But $J(\text{Ḫ}) \subseteq F$, then $F + J(\text{Ḫ}) = F$. Thus either $h \in F \subseteq F + soc(\text{Ḫ})$ or $r\text{Ḫ} \subseteq F \subseteq F + soc(\text{Ḫ})$. Hence F is approximaitly prime submodule of Ḫ.

(7) $\Longrightarrow$ (1) Since $soc(\text{Ḫ}) \subseteq F$, then $F + soc(\text{Ḫ}) = F$, Let $rh \in F$ for $r \in \text{Ꞧ}$, $h \in \text{Ḫ}$. But F is approximaitly prime, then either $h \in F + soc(\text{Ḫ}) = F$ or $r\text{Ḫ} \subseteq F + soc(\text{Ḫ}) = F$. Thus either $h \in F$ or $r\text{Ḫ} \subseteq F$. Therefore F is prime submodule of Ḫ.

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