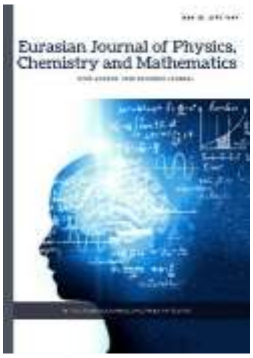

Using Some Transformations for Solving a Type of Fractional Differential Equations

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ABSTRACT

In this article, we presented a new exact solution for Fractional Riccati Differential Equation FRDE $y^{\alpha} = a(x)y^2 + b(x)y + c(x)$. This solution is obtained using many techniques. First, we reduce the equation to a second order linear ordinary differential equation. Next, we will transform it to Bernoulli differential equation, finally, we obtain the solution by supposing a condition on c(x) consists of an arbitrary function. By using the conditions imposed on Riccati differential equation coefficients and choosing the form of the coefficients of the equation. For this case the general solution of the equation is also presented.

Keywords: Fractional Riccati Equation, Integral Condition, Exact Solution,

1. Introduction

The Riccati differential equation was named by an italian nobleman count Jacopo Francesco Riccati (1676-1754), it represents an ordinary first order nonlinear differential equation and this equation plays an important role in the different fields of mathematics and physics, for example the motion of a particle under the influence of a power law central potential. And in classical mechanics the application is given by

$$\frac{dy}{dx} = a(x)y^2 + b(x)y + c(x) \qquad (1.1)$$

where a, b, c are arbitrary real functions of x, and $a, b, c \in C^{\infty}(I)$ which defined on an interval $I \subseteq R$. We know that if a particular solution y_p of a Riccati differential equation (1.1) is given then the general solution of it will written as

$$y(x) = \frac{e^{\int^x b(z)+2a(z)y_p(z)}dz}{W - \int^x a(z)\, e^{\int^z b(\emptyset)+2a(\emptyset)y_p(\emptyset)d\emptyset}dz} + y_p(x)$$

Riccati differential equation (1.1) can be integrated even when a particular solution is unknown and if its coefficients will satisfy some conditions. For example, if

$$a(x) + b(x) + c(x) = 0$$

then (1.1) has a solution as following

$$y(x)=\frac{W+\int(a(x)+c(x))e^{(\int(a(x)-c(x))dx)}dx-e^{(\int(a(x)-c(x))dx)}}{W+\int(a(x)+c(x))e^{(\int(a(x)-c(x))dx)}dx+e^{(\int(a(x)-c(x))dx)}}$$

where W is an arbitrary constant [1].

On other side the transformation $y=\frac{u\prime}{f(x)u}$ lead us to the second order linear ordinary differential equation

$$u''-\left(\frac{f'(x)}{f(x)}+b(x)\right)u'+a(x)c(x)u=0$$

If y_p is known, then the general solution containing an arbitrary constant can be obtained from $y=y_p+\frac{1}{v(x)}$. [9] where $v(x)$ is a solution of the equation

$$\frac{dv}{dx}=-\left(b(x)+2a(x)y_p(x)\right)v-f(x)$$

The FRDE has the form

$$y^{\alpha}=a(x)y^2+b(x)y+\mathrm{c}(x) \qquad (1.2)$$

where y^{α} is the conformable fractional derivative CFD of order $\alpha\in(0,1]$, we must note that the method can be generalized to include any α. Because of the difficulties to get an analytical solution to Riccati differential equation. One should use a numerical method [3,4,5,6,8,10] or must use an approximation method.

In this research we get an analytical solution to FRDE by changing it to Bernoulli differential equation, and easing it to second order linear ordinary differential equation and after reviewing relations between the coefficients of the Riccati equation containing some integral or differential representations, furthermore the existence of some arbitrary functions.

2. Basic Concepts and Definitions

In this section, we will give some basic definitions and properties of fractional calculus theory that are additionally used in this article.

Definition 2.1: [7] The Riemann-Liouville fractional integral operator of order $\alpha>0$, of a function $f\in C\mu,\mu\geq-1$, defined as

$$D_t^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d}{dt}\int_a^t\frac{f(x)}{(t-x)^{\alpha-n+1}}dx,(n-1)\leq\alpha<n$$

Where α is a real number and n is integer.

Definition 2.2: [7] The fractional derivative of $f(t)$ in the sense of Caputo defined by

$$D_t^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t\frac{f^n(x)}{(t-x)^{\alpha+1-n}}dx,(n-1)\leq\alpha<n$$

α is a real number and n is integer.

In [5] Khalil and others presented a new definition of fractional derivative of order $\alpha\in(0,1]$ which is more reality and effective than prior definitions. Also, they generalized the definition for any α. Moreover, the case $\alpha\in(0,1]$ is the most important case, and the other possibilities become easy when it is approved.

Definition 2.3: [5] Let $f:[0,\infty)\to\mathrm{R}$ be a function then the CFD of f of order $\alpha\in(0,1]$ is defined by

$$T_{\alpha}(f)(t)=\lim_{\varepsilon\to0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon},\forall\ t>0,\alpha\in(0.1]$$

If f is $\alpha-differentiable$ in some regions of interval $(0,\alpha),\alpha>0$ and $\lim_{t\to0^+}f^{\alpha}(t)$ exists, then we define $f^{\alpha}(0)=\lim_{t\to0^+}f^{\alpha}(t)$ for all $t>0,\alpha\in(0,1)$.

The definition of conformable deals with many

The conformable definition treats many inadequacies of previous definitions like the derivative of the constant equals to zero, the product and fractional derivative rules are hold, also chain rule and antiderivative rule are satisfied. In this research, we embrace the CFD definition.

3. Formulation Method of Solution

Many researchers were study the Riccati equation. In this section, we compute the exact solution of FRDE with known particular solution. We will give some theorems which will help us in our work.

Theorem 3.1: We called this theorem (Reduction to second order equation). Let the non-linear FRDE can be reduced into a second order linear ordinary differential equation as the following form:

$$u'' - \left(\frac{\alpha - 1}{x} + R(x)\right)\mathrm{u}' + x^{\alpha-1}S(x)u = 0 \qquad (3.1)$$

Then the solution of above equation results the following solution:

$$y = \frac{-U'(x)x^{1-\alpha}}{a(x)U(x)}$$

Proof: Assume $v = a(x)y, v^{\alpha} = (a(x)y)^{\alpha} = a(x)y^{\alpha} + yx^{1-\alpha}a'(x)$. Where y^{α} denotes FRDE.
From $v = a(x)y$ we have $y = \frac{v}{a}$ then substituting it and do some computing steps we get

$$x^{1-\alpha}v'(x) = v^2 + bv + ca + vx^{1-\alpha}\frac{a'}{a}$$

Collecting the similar terms to obtain

$$v'(x) = x^{1-\alpha}v^2 + \left(x^{1-\alpha}b + \frac{a'}{a}\right)v + x^{1-\alpha}ca \qquad (3.2)$$

Suppose $R(x) = x^{1-\alpha}b + \frac{a'}{a}$ and $S(x) = x^{1-\alpha}ca$ we have

$$v'(x) = x^{1-\alpha}v^2 + vR(x) + S(x)$$

Let

$$x^{1-\alpha}v = \frac{-u'}{u} \qquad (3.3)$$

$$(\alpha - 1)x^{\alpha-2}v + x^{\alpha-1}v' = \frac{-uu'' + (u')^2}{u^2}$$

$$\Rightarrow (\alpha - 1)x^{\alpha-2}v + x^{\alpha-1}v' = \frac{-u''}{u} + v^2(x^{\alpha-1})^2$$

By canceling $x^{\alpha-1}$ from both sides and do some arrangements we get

$$\frac{\alpha - 1}{x}v + x^{1-\alpha}\frac{u''}{u} = x^{\alpha-1}v^2 - v'$$

From the equation (3.2) we have

$$\frac{\alpha - 1}{x}v + x^{1-\alpha}\frac{u''}{u} = -\left(x^{\alpha-1}b + \frac{a'}{a}\right)v - x^{\alpha-1}ca$$

$$\Rightarrow \frac{\alpha - 1}{x}v + x^{1-\alpha}\frac{u''}{u} = -R(x)v - S(x)$$

By combining similar terms to obtain

$$x^{1-\alpha}\frac{u''}{u} + \left(\frac{\alpha - 1}{x} + R(x)\right)v + S(x) = 0$$

Dividing both sides by $x^{1-\alpha}$ and substitute $v = \frac{-u'}{u}x^{1-\alpha}$ we have

$$x^{1-\alpha}\frac{u''}{u} + \left(\frac{\alpha - 1}{x} + R(x)\right)\left(\frac{-u'}{u}x^{1-\alpha}\right) + S(x) = 0$$

$$\Rightarrow \frac{u''}{u} - \left(\frac{\alpha - 1}{x} + R(x)\right)\frac{u'}{u} + x^{1-\alpha}S(x) = 0$$

$$\therefore u'' - \left(\frac{\alpha - 1}{x} + R(x)\right)u' + x^{1-\alpha}S(x)u = 0$$

Finally, this will lead us to

$$y = \frac{v}{a} = \frac{-u'x^{1-\alpha}}{ua}$$

Theorem 3.2: For non-linear FRDE the assumption $v(x) = y(x) - y_1(x)$ will transform FRDE to an ordinary differential equation of first order which is called Bernoulli differential equation, where y_1 is a known particular solution.

Proof: we will derive the relation $v(x) = y(x) - y_1(x)$ to an order α. So, by some arrangement it will becomes

$$y^{\alpha}(x) = v^{\alpha}(x) + y_1^{\alpha}(x) \qquad (3.4)$$

Because $y_1(x)$ is a solution for FRDE, then it must be as follows

$$y_1^{\alpha} = a(x)y_1^2 + b(x)y_1 + c(x) \qquad (3.5)$$

Substituting (3.5) in (1.2) we get

$$v^{\alpha}(x) + y_1^{\alpha}(x) = a(x)[v + y_1]^2 + b(x)[v + y_1] + c(x) \qquad (3.6)$$

Where, $y^{\alpha}(x) = v^{\alpha}(x) + y_1^{\alpha}(x)$ and $y(x) = v + y_1$

Now, (3.6) becomes

$$x^{1-\alpha}v'(x) + ay_1^2 + by_1 + c = av^2 + 2avy_1 + ay_1^2 + bv + by_1 + c$$

$$\Rightarrow x^{1-\alpha}v'(x) = av^2(x) + 2ay_1v(x) + bv(x)$$

$$\Rightarrow v'(x) = ax^{1-\alpha}v^2(x) + 2ax^{1-\alpha}y_1v(x) + bx^{1-\alpha}v(x)$$

$$\Rightarrow v'^{(x)} + [-2x^{1-\alpha}a(x)y_1 - x^{1-\alpha}b(x)]v = ax^{1-\alpha}v^2(x) \qquad (3.7)$$

The equation (3.7) represents the form of Bernoulli differential equation with value of $n = 2$, we can reduce it to a first order linear differential equation by assuming

$$u = v^{-1}$$

Differentiation above assumption with respect to x to obtain

$$\frac{du}{dx} = -v^{-2}\frac{dv}{dx}$$

Multiply the equation (3.7) by $-v^{-2}$ we have

$$-v^{-2}v' + [2x^{\alpha-1}ay_1 + x^{\alpha-1}b]v^{-2}v = -ax^{\alpha-1}$$

$$\Rightarrow v' + [2x^{\alpha-1}ay_1 + x^{\alpha-1}b]v = ax^{\alpha-1} \qquad (3.8)$$

Finally, the solution of the equation is given by

$$v = \frac{\int \mu(x)q(x).dx + c(x)}{\mu(x)} \qquad (3.9)$$

Where $\mu(x)$ is an integrating factor and written by $\mu(x) = e^{\left(\int[2x^{\alpha-1}ay_1 + x^{\alpha-1}b]dx\right)}$.

Theorem 3.3: suppose y_1 be a solution of the equation (1.2), and let $z = \frac{1}{y-y_1}$, then the solution of the FRDE is

$$z = e^{-I(2ay_1+b)}I_{\alpha}e^{I(2ay_1+b)}\left(\left(-a(x)\right)\right) \qquad (3.10)$$

Proof: let $y_1$be a solution of the equation (1.2), and assume that $z = \frac{1}{y-y_1}$, then

$$z(y - y_1) = 1 \Rightarrow y = \frac{1}{z} + y_1 \qquad (3.11)$$

By Applying the α-derivative definition to both sides of the equation (3.11) we get

$$T_{\alpha}y = T_{\alpha}\left(\frac{1}{z}\right) + T_{\alpha}y_1 \Rightarrow T_{\alpha}y = -z^{-1-\alpha}z` + T_{\alpha}y_1$$

Substituting above equation into the original FRDE we get

$$-z^{-1-\alpha}z' + T_{\alpha}y_1 = a\left[\frac{1}{z} + y_1\right]^2 + b\left[\frac{1}{z} + y_1\right] + c$$

$$\Rightarrow -z^{-1-\alpha}z' = a\left[\frac{1}{z^2} + \frac{2y_1}{z} + y_1^2\right] + b\left[\frac{1}{z} + y_1\right] + c - T_{\alpha}y_1$$

The solution $y_1$satisfies the FRDE, so

$$-z^{-1-\alpha}z' = \frac{a}{z^2} + \frac{2y_1 a}{z} + ay_1^2 + \frac{b}{z} + by_1 + c - ay^2 - by_1 - c$$

Combining similar terms and divide both sides by $-z^{-1-\alpha}$

$$z' = -(2ay_1 + b)z^{\alpha} - az^{\alpha-1}$$

Then

$$z' + (2ay_1 + b)z^{\alpha} = -az^{\alpha-1} \qquad (3.12)$$

Multiplying both sides of an equation (3.12) by $z^{1-\alpha}$ we obtain

$$z^{1-\alpha}z' + (2ay_1 + b)z = -a$$

$$z^{\alpha} + (2ay_1 + b)z = -a \qquad (3.13)$$

which is Abel's formula.

Hence, the solution is

$$z = e^{-I(2ay_1+b)} I_{\alpha}\left(e^{I(2ay_1+b)}\big(-a(x)\big)\right) \qquad (3.14)$$

FRDE maybe considered a quadratic equation in y and write it by

$$a(x)y^2 + b(x)y + c(x) - y^{\alpha} = 0$$

Then, the particular solution of the equation is written by the form

$$y_p(x) = \frac{-b(x) \pm \sqrt{b^2(x) - 4a(x)c(x) + 4a(x)y^{\alpha}}}{2a(x)}$$

And the analytical solution of Ricatti differential equation will satisfying some conditions were given by [2]. The following theorem explain the analytical solution of RDE satisfies an integral condition with arbitrary function.

Theorem 3.4: Suppose that the coefficients $a(x), b(x), c(x)$ of FRDE (1.2), if $c(x)$ satisfies the following integral condition

$$c(x) = \frac{f_1(x) - \left\{b(x) + a(x)\left[\int^x \frac{f_1(\phi) - b^2(\phi)}{2a(\phi)} - a_1\right]\right\}^2}{4a} \qquad (3.15)$$

where a_1 is an integration constant. And f_1 is the new generating function satisfying the differential condition:

$$b^2(x) + 4a(x)x^{1-\alpha}\frac{dy_p}{dx} = f_1(x) \qquad (3.16)$$

Then the general solution is given by:

$$y(x) = \frac{1}{e^{-I(2ay_1+b)} I_{\alpha}\left(e^{I(2ay_1+b)}\big(-a(x)\big)\right)} + \frac{1}{2}\left[\int^x \frac{f_1(\phi) - b^2(\phi)}{2a(\phi)}d\phi - a_1\right] \qquad (3.17)$$

Proof: Let the arbitrary functions $a(x), b(x), f_1(x)$ satisfying the condition (3.15) then the particular solution is written by

$$y_{p^{\pm}}(x) = \frac{-b \pm \sqrt{f_1 - 4ac}}{2a}$$

$$= \frac{-b \pm \sqrt{f_1 - 4a\frac{f_1(x) - \left\{b(x) + a(x)\left[\int^x \frac{f_1(\phi) - b^2(\phi)}{2a(\phi)} - a_1\right]\right\}^2}{4a}}}{2a}$$

$$= \frac{-b \pm \sqrt{f_1 - f_1(x) - \left\{b(x) + a(x)\left[\int^x \frac{f_1(\phi) - b^2(\phi)}{2a(\phi)} - a_1\right]\right\}^2}}{2a}$$

$$= \frac{-b + b(x) + a(x)\left[\int^x \frac{f_1(\phi) - b^2(\phi)}{2a(\phi)} - a_1\right]}{2a}$$

$$= \frac{a(x)\left[\int^{x}\frac{f_1(\phi)-b^2(\phi)}{2a(\phi)}-a_1\right]}{2a}$$
$$= \frac{1}{2}\left[\int^{x}\frac{f_1(\phi)-b^2(\phi)}{2a(\phi)}-a_1\right]$$

So,

$$y_{p^{\pm}}(x) = \frac{-b\pm\sqrt{f_1-4ac}}{2a} = \frac{1}{2}\left[\int^{x}\frac{f_1(\phi)-b^2(\phi)}{2a(\phi)}-a_1\right] \qquad (3.18)$$

Hence, the general solution of FRDE is given by

$$y(x) = \frac{1}{e^{-I(2ay_p+b)}I_\alpha\left(e^{-I(2ay_p+b)}\left(-a(x)\right)\right)} + \frac{1}{2}\left[\int^{x}\frac{f_1(\phi)-b^2(\phi)}{2a(\phi)}-a_1\right]$$

4. Illustrative Example

Example 4.1: find the solution of FRDE

$$y^{(\frac{1}{2})} = (y-2\sqrt{x})^2+1\,, y(0)=1 \qquad (4.1)$$

Where $y_1(x)=2\sqrt{x}$ is a known solution for it.

Solution: First we need to verify that $y_1 = 2\sqrt{x}$ is a solution to the equation (4.1)

By changing the variables after we assume that $y = v + 2\sqrt{x}$ and $y^{(\frac{1}{2})} = v^{(\frac{1}{2})} + 1$ which will lead us to the following equation

$$v^{(\frac{1}{2})} + 1 = (v+2\sqrt{x}-2\sqrt{x})^2+1$$

The above equation can be reduced to a Bernoulli differential equation for the variable v and it will take the form

$$v' = x^{-\frac{1}{2}}v^2 \qquad (4.2)$$

Assume that

$$u = v^{-1} \qquad (4.3)$$

Differentiation (4.3) with respect to x we have

$$u' = -v^{-2}v' \qquad (4.4)$$

Multiply equation (4.2) by the term $-v^{-2}$ to get

$$-v^{-2}v' = -x^{-\frac{1}{2}}v^{-2}v^2$$

By making some arrangements we will get

$$u' = -x^{-\frac{1}{2}} \Rightarrow u' = \frac{-1}{\sqrt{x}} \Rightarrow \frac{du}{dx} = \frac{-1}{\sqrt{x}} \Rightarrow du = \frac{-1}{\sqrt{x}}dx$$
$$\Rightarrow u = \int\frac{-1}{\sqrt{x}}.dx = -2\sqrt{x}+c$$
$$\frac{1}{v} = -2\sqrt{x}+c \qquad (u=v^{-1})$$
$$v = \frac{1}{-2\sqrt{x}+c}$$

Reversing the substitution $y = v+2\sqrt{x}$ then

$$y = \frac{1}{-2\sqrt{x}+c}+2\sqrt{x}$$

Using the initial condition $y(0)=1$ to find value of c which is $c=1$ then the general solution is

$$y = \frac{3}{-4\sqrt{x^3}+1}-\frac{x^2}{2}$$

5. Conclusions

In this paper, we found an exact solution of FRDE by using the CFD which is more simple and more efficient than the previous derivatives. The new definition explains a natural extension of a classical derivative for solving FDE. So, we presented some theorems which lead us to find a second solution when a particular solution is known.

REFERENCES

1. E. Kamke, (1959-65), Differentialgleichungen, Losungs-methoden und Losungen, New York, Chelsea Pub. Co.
2. Harko, T., Lobo, F. and Mak, M. K. (2013), Analytical Solutions of the Riccati Equation with Coefficients Satisfying Integral or Differential Conditions with Arbitrary Functions, Universal Journal of Applied Mathematics, 2: 109-118.
3. He, J. H. (1999), Variational Iteration Method-a Kind of Non-Linear Analytical Technique: Some Examples, International Journal of Non-Linear Mechanics, 34: 699-708.
4. Khader, M. M., EL Danaf, T. S. and Hendy, A. S. (2012), Efficient Spectral Collocation Method for Solving Multi-Term Fractional Differential Equations Based on the Generalized Laguerre Polynomials, Journal of Fractional Calculus and Applications, 3(13), 1-14.
5. Khalil, Roshdi, Horani, M., Yousef, Abdelrahman and Sababheh, M. (2014), A New Definition of Fractional Derivative, Computational and Applied Mathematics, 246: 65-70.
6. Momani, Shaher and Shawagfeh, Nabil (2006), Decomposition Method for Solving Fractional Riccati Differential Equations, Applied Mathematics and Computation, 182(2): 1083-1092.
7. Mahmood Khalil, M. (2019). On a unique solution of fractional differential system. Tikrit Journal of Pure Science, 24(3), 128-132.
8. Odibat, Zaid and Momani, Shaher (2008), Modified Homotopy Perturbation Method: Application to Quadratic Riccati Differential Equation of Fractional Order, Chaos, Solutions & Fractals, 36(1), 167-174.
9. Weisstein, Eric W. "Riccati Differential Equation." From *MathWorld*--A Wolfram Web Resource, https://mathworld.wolfram.com/RiccatiDifferentialEquation.html.
10. Wu, Fei and Huang, Lan-Lan. (2014), Approximate Solutions of Fractional Riccati Equations using the Adomain decomposition Method, Abstract and Applied, Volume 2014, Article ID 957590, 4 pages.