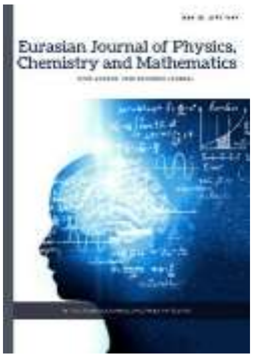

Using An Analytical Method for Solving a Type of Differential - Difference Equations

Safa Issam Ahmed[1] — [1]Department of Mathematics, College of Education for Women, Tikrit University, Tikrit, Iraq, safa.ahmed@st.tu.edu.iq

Muayyad Mahmood Khalil[2] — [2]Department of Mathematics, College of Education for Pure Sciences, Tikrit University, Tikrit, Iraq, medomath80@gmail.com

Nihad Shareef Khalaf[1] — [1]Department of Mathematics, College of Education for Women, Tikrit University, Tikrit, Iraq, nihad.shreef16@tu.edu.iq

ABSTRACT

Laplace decomposition method is predicated on Laplace transform method and Adomian decomposition method. through this paper we show that the technique is applicable to certain successive interval valued linear in addition as nonlinear differential-difference equations of order (1, 2), which means the differential is of order one and the difference order is two. it's also shown that the method gives exact solution for linear problems and suitable approximate solution for nonlinear problems. Three problems are selected to show the applicability of the method.

Keywords: Differential-difference equation, Laplace decomposition method, Laplace transform method.

1. Introduction

In recent times, many researchers [5,6] have an interest in solving a singularly perturbed second order integro-differential-difference equation with one interval condition involving the left extremum of the boundary and another stipulation at the correct extreme of the boundary. The singular perturbation parameter and therefore the delay parameter are selected as small as possible. Such problems play a crucial role in an exceedingly form of physical problems like microscale heat transfer, diffusion in polymers, control of chaotic systems, so on (relevant references quoted in [5]). within the present paper, we formulate a special problem, namely, an integro-differential-difference equation with differential order one and a difference of order two with only interval conditions. this could be done by considering the subsequent integro-differential-difference equation of order (2,2):

$$\varepsilon y''(t) = y'(t) - \left[h(t) + H_1\big(y(t-\omega), y'(t-\omega)\big) + H_2\big(y(t-2\omega), y'(t-2\omega)\big) + \int_0^t G\big(y(t_1 - 2\omega)\big)dt_1\right]$$

We may sometimes allow G to equals zero. Then it will be a differential-difference equation of order (2,2). Further, if we set $\varepsilon = 0$, then it becomes differential-difference equation of order (1,2). Since it is a first order differential-difference equation, we avoid the boundary condition at the right extreme and work with only one interval condition:

$$y(t) = k,\ t \in (0,2\omega).$$

Again, only interval condition is used for integro-differential-difference equation of order (1,2)

In the present article, we apply the Laplace decomposition method for such problems. The method is motivated by the Adomian decomposition method for solving differential equations [3,4,8,7,9,11], Laplace transform method for solving differential difference equations [10], and the Laplace decomposition method as well as Laplace decomposition with Pade approximation for solving integrodifferential equations [2]. It is shown in [1] that the method gives exact solutions for linear problems and suitable approximate solutions for nonlinear problems related to integrodifferential difference equations with both differential and difference of order one as well as one interval condition. The aim of this research is to formulate the interval-valued problem (2.1) and describe the Laplace decomposition method. Then we explain the method with three different problems. Finally, we give concluding remarks about the suitability of the method for both linear and nonlinear problems.

2. Formulation and Description of The Method

Let us consider the following integro-differential-difference equation with differential order one and difference of order two:

$$u'(t) = f(t) + F_1\big(u(t-\omega), u'(t-\omega)\big) + F_2\big(u(t-2\omega), u'(t-2\omega)\big) + \int_0^t G\big(y(t_1-2\omega)\big)dt_1, t > 2\omega \qquad (2.1)$$

and the following interval condition:

$$u(t) = k,\ 0 \le t \le 2\omega \qquad (2.2)$$

In the above equations (2.1) and (2.2), $\omega > 0$ and k are known constants, the functions f, F_1, F_2 and G are either linear or nonlinear functions depending upon the actual problem discussed. so as to use Laplace decomposition method, further, they're selected in such the simplest way that, they'll be approximated by Adomian polynomial suitable for the iterative computation of Laplace transform still as inverse Laplace transform for $u(t)$.

First, we see that

$\int_0^{2\omega} u'(t)e^{-pt}dt = 0$ as a result, we get $\int_{2\omega}^{\infty} u'(t)e^{-pt}dt = L\{u'(t)\}$

Hence multiply both sides of (2.1) by e^{-pt} and integrate between 2ω and ∞ to get

$$\begin{aligned}\int_{2\omega}^{\infty} u'(t)e^{-pt}dt &= \int_{2\omega}^{\infty} f(t)e^{-pt}dt + \int_{2\omega}^{\infty} F_1(u(t-\omega), u'(t-\omega))e^{-pt}dt \\ &+ \int_{2\omega}^{\infty} F_2(u(t-2\omega), u'(t-2\omega))e^{-pt}dt + \int_{2\omega}^{\infty} e^{-pt}\int_0^t G\big(u(t_1-2\omega)\big)dt_1 dt\end{aligned}$$

If we apply a suitable shifting of variables to get

$$\begin{aligned}L\{u'(t)\} &= e^{-2\omega p}L\{f(t+2\omega)\} + e^{-\omega p}L\big\{F_1\big(u(t), u'(t)\big)\big\} - \frac{\lambda e^{-\omega p}}{p}(1-e^{-\omega p}) \\ &+ e^{-2\omega p}L\{F_2(u(t), u'(t))\} + e^{-2\omega p}L\left\{\int_0^t G\big(u(t_1)\big)dt_1\right\}\end{aligned}$$

where $\lambda = F_1(k,0)$ and see that, $\int_{2\omega}^{t+2\omega} G\big(u(t_1-2\omega)\big)dt_1 = \int_0^t G\big(u(t_1)\big)dt_1$.

Finally, we obtain

$$L\{u(t)\} = \frac{k}{p} - \frac{\lambda e^{-\omega p}}{p^2} + \frac{\lambda e^{-2\omega p}}{p^2} + \frac{e^{-2\omega p}}{p} L\{f(t+2\omega)\} + \frac{e^{-\omega p}}{p} L\{F_1(u(t), u'(t))\} + \frac{e^{-2\omega p}}{p} L\{F_2(u(t), u'(t))\} + \frac{e^{-2\omega p}}{p^2} L\{G(u(t))\} \qquad (2.3)$$

In this research, we look forward to the following kind of decomposition for the series solution $u(t)$:

$$u(t) = \sum_{n=0}^{\infty} u_n(t - n\omega) e^{(t-n\omega)} \qquad (2.4)$$

where e^t is a unit step function, given as

$$e^{(t-c)} = 0, \qquad t < c$$
$$e^{(t-c)} = 1, \qquad t > c$$

By using the relation (2.4), $u(t)$ takes the following form in each of the following intervals:

$$u(t) = \sum_{n=0}^{N} u_n(t - n\omega), N\omega \le t \le (N+1)\omega, N = 0,1,2, \dots \qquad (2.5)$$

And by applying Laplace transform, we have the following Laplace decompositions:

$$L\{u(t)\} = \sum_{n=0}^{\infty} e^{-n\omega p} L\{u_n(t)\} \qquad (2.6)$$

$$L\{F_1(u(t), u'(t))\} = \sum_{n=0}^{\infty} e^{-n\omega p} L\{A_n(t)\} \qquad (2.7)$$

$$L\{H_2(u(t), u'(t))\} = \sum_{n=0}^{\infty} e^{-n\omega p} L\{B_n(t)\} \qquad (2.8)$$

$$L\{G(u(t))\} = \sum_{n=0}^{\infty} e^{-n\omega p} L\{C_n(t)\} \qquad (2.9)$$

In the equation (2.7), $A_n{}'s$ are the n^{th} Adomian Polynomials [2] of $F_1(u(t), u'(t))$ as given below:

$$A_0(t) = F_1(x, y)\Big|_{(u_{0(t)},\, u'_{0(t)})}$$

$$A_1(t) = \frac{\partial F_1}{\partial x}\Big|_{(u_0(t), u'_0(t))} u_1(t) + \frac{\partial F_1}{\partial y}\Big|_{(u_0(t), u'_0(t))} u'_1(t)$$

$$A_2(t) = \frac{\partial F_1}{\partial x}\Big|_{(u_0(t), u'_0(t))} u_2(t) + \frac{\partial F_1}{\partial y}\Big|_{(u_0(t), u'_0(t))} u'_2(t) + \frac{1}{2!}\left[\frac{\partial^2 F_1}{\partial x^2}\Big|_{(u_0(t),\, u'_0(t))} u_1^2(t) + 2\frac{\partial^2 F_1}{\partial x \partial y}\Big|_{(u_0(t), u'_0(t))} u_1(t) u'_1(t) + \frac{\partial^2 F_1}{\partial y^2}\Big|_{(u_{0(t)},\, u'_{0(t)}} (u'_1(t))^2\right]$$

and so on. In eqaution (2.8), $B_n{}'s$ are the n^{th} Adomian Polynomials [2] of $F_2(u(t), u'(t))$. Let us note that $B_0, B_1, B_2, \dots$ are same as $A_0, A_1, A_2, \dots$ except for the fact that F_1 should be replaced by F_2 throughout. In (2.9), $C_n{}'s$ are the n^{th} Adomian Polynomials [3] of $G(u(t))$ as given below:

$$C_0(t) = G(u_0(t))$$
$$C_1(t) = G'\big((u_0(t))\big) u_1(t)$$
$$C_2(t) = G'\big((u_0(t))\big) u_2(t) + \frac{1}{2!} G''\big((u_0(t))\big) u_1^2(t)$$

and so on. Applying the Laplace decompositions $(2.6) - (2.9)$ and substitute in (2.3), we get

$$\sum_{n=0}^{\infty} e^{-n\omega p} L\{u_n(t)\}$$
$$= \frac{k}{p} - \frac{\lambda e^{-\omega p}}{p^2} + \frac{\lambda e^{-2\omega p}}{p^2} + \frac{e^{-2\omega p}}{p} L\{f(t+2\omega)\} + \frac{e^{-\omega p}}{p} \sum_{n=0}^{\infty} e^{-n\omega p} L\{A_n(t)\}$$
$$+ \frac{e^{-2\omega p}}{p} \sum_{n=0}^{\infty} e^{-n\omega p} L\{B_n(t)\} + \frac{e^{-2\omega p}}{p^2} \sum_{n=0}^{\infty} e^{-n\omega p} L\{C_n(t)\} \qquad (2.10)$$

We may compute $L\{u_n(t)\}$ iteratively as follows:

$$L\{u_0(t)\} = \frac{k}{p}$$
$$L\{u_1(t)\} = -\frac{k}{p^2} + \frac{1}{p} L\{A_0(t)\}$$
$$L\{u_2(t)\} = \frac{\lambda}{p^2} + \frac{1}{p} L\{f(t+2\omega)\} + \frac{1}{p} L\{A_1(t)\} + \frac{1}{p} L\{B_0(t)\} + \frac{1}{p^2} L\{C_0(t)\}$$
$$\vdots$$
$$L\{u_{n+1}(t)\} = \frac{1}{p} L\{A_n(t)\} + \frac{1}{p} L\{B_{n-1}(t)\} + \frac{1}{p^2} L\{C_{n-1}(t)\} \quad , n = 2,3,4, \dots$$

We can obtain the approximate or exact solutions by applying inverse Laplace transform.

3. Explanatory examples

In this section, we work on 3 illustrative examples to explain the procedure of the method.

Example 3.1: Let we have the following linear differential-difference equation with order (1,2):

$$2u'(t) - u(t-\omega) = u(t-2\omega),\ t > 2\omega \qquad (3.1)$$

with a condition

$$u(t) = 1,\ 0 \le t \le 2\omega \qquad (3.2)$$

By applying the steps of the method for the equations (3.1) and (3.2), we have directly:

$$L\{u(t)\} = \frac{1}{p} - \frac{e^{-\omega p}}{2p^2} + \frac{e^{-2\omega p}}{2p^2} + \frac{e^{-\omega p}}{2p} L\{u(t)\} + \frac{e^{-2\omega p}}{2p} L\{u(t)\} \qquad (3.3)$$

Now, apply the equation (2.6) in (3.3), we have

$$\sum_{n=0}^{\infty} e^{-n\omega p} L\{u_n(t)\}$$
$$= \frac{1}{p} - \frac{e^{-\omega p}}{2p^2} + \frac{e^{-2\omega p}}{2p^2} + \frac{e^{-\omega p}}{2p} \sum_{n=0}^{\infty} e^{-n\omega p} L\{u_n(t)\}$$
$$+ \frac{e^{-2\omega p}}{2p} \sum_{n=0}^{\infty} e^{-n\omega p} L\{u_n(t)\} \qquad (3.4)$$

By Equating the terms with coefficient of $e^{-n\omega p}$ on both sides of (3.4) we obtain $L\{u_n(t)\}$.
Taking the inverse Laplace transform will result $u_n(t)$:

$$u_0(t) = 1; u_1(t) = 0$$

And for $n \ge 2$ we get,

$$u_n(t) = \sum_{r=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-r-1}{r-1} \frac{t^{n-r}}{2^{n-r-1} \cdot (n-r)!},\ n \ge 2 \qquad (3.5)$$

Now, by using the equation (3.5) we have,

$$u(t) = 1 + \sum_{n=2}^{\infty} \sum_{r=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-r-1}{r-1} \frac{(t-n\omega)^{n-r} e^{(t-n\omega)}}{2^{n-r-1} \cdot (n-r)!}, t > 0 \qquad (3.6)$$

Furthermore, through an equation (3.6) we can get the exact solution of (3.1) in the interval wise:

$$u(t) = 1 + \sum_{n=2}^{N} \sum_{r=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-r-1}{r-1} \frac{(t-n\omega)^{n-r}}{2^{n-r-1} \cdot (n-r)!}, N\omega \le t \le (N+1)\omega$$

$$N = 2,3,4, \dots .$$

When $\omega \to 0$, the equation (3.1) becomes an ordinary first order differential equation and when we use (3.6), the solution becomes

$$u(t) = 1 + \sum_{n=1}^{\infty} \left[\binom{n-1}{0} + \binom{n-1}{0} + \dots + \binom{n-1}{n-1} \right] \frac{t^n}{2^{n-1} \cdot n!}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} = e^t$$

Example 3.2: Let the following nonlinear differential-difference equation

$$u'(t) = 2 - u(t-\omega) + au^3(t-2\omega),\ t > 2\omega \qquad (3.7)$$

with

$$u(t) = 1,\ 0 \le t \le 2\omega \qquad (3.8)$$

By applying the method on above equation, we obtain

$$L\{u(t)\} = \frac{1}{p} + \frac{e^{-\omega p}}{p^2} + \frac{e^{-2\omega p}}{p^2} - \frac{e^{-\omega p}}{p} L\{u(t)\} - a\frac{e^{-2\omega p}}{p} L\{u^3(t)\}.$$

Now, we will compute Laplace decomposition series

$$L\{u^3(t)\} = \sum_{n=0}^{\infty} e^{-n\omega p} L\{A_n(t)\}$$

where $A_i\ 's$ are Adomian Polynomials,

$$\begin{aligned} A_0(t) &= u_0^3(t) \\ A_1(t) &= 3u_0^2(t)u_1(t) \\ A_2(t) &= 3u_0^2(t)u_2(t) + 3u_0(t)u_1^2(t) \\ A_3(t) &= 3u_0^2(t)u_3(t) + 3u_0(t)u_1(t)u_2(t) + u_1^3(t) \text{ and so on.} \end{aligned}$$

By using the equations (2.6) and (3.10) in the equation (3.9), we have

$$\sum_{n=0}^{\infty} e^{-n\omega p} L\{u_n(t)\}$$

$$= \frac{1}{p} + \frac{e^{-\omega p}}{p^2} + \frac{e^{-2\omega p}}{p^2} - \frac{e^{-\omega p}}{p} \sum_{n=0}^{\infty} e^{-n\omega p} L\{u_n(t)\}$$

$$- a\frac{e^{-2\omega p}}{p} \sum_{n=0}^{\infty} e^{-n\omega p} L\{A_n(t)\} \qquad (3.11)$$

Simply, by equating the terms with coefficient of $e^{-n\omega p}$ on both sides of the equation (3.11) we obtain $L\{u_n(t)\}$. By taking the inverse Laplace transform will lead us to $u_n(t)$.

For $4\omega \le t \le 5\omega$, the approximate solution is

$$u(t) = \sum_{n=0}^{4} u_n(t - n\omega)$$

$$= 1 + (1-a)(t-2\omega) - (1-a)\frac{(t-3\omega)^2}{2!}$$
$$+(1-a)\frac{(t-4\omega)^3}{3!} + 3a(1-a)\frac{(t-4\omega)^2}{2!} \qquad (3.12)$$

When $\omega \to 0$ and $a = 0$, the equation (3.7) becomes an ordinary linear first order differential equation,

$$u'(t) = 2 - u(t) \qquad (3.13)$$

and the exact solution for it will be given by $2 - e^{-t}$.

Applying Laplace decomposition method to the equation (3.13), we have

$$u_n(t) = (-1)^n \frac{t^{n-1}}{(n-1)!}, n \geq 2$$

Hence,

$$u(t) = 1 + \sum_{n=2}^{\infty} u_n(t)$$
$$= 1 + t - \frac{t^2}{2!} + \frac{t^3}{3!} - \frac{t^4}{4!} + \cdots + (-1)^n \frac{t^{n-1}}{(n-1)!} + \cdots$$
$$= 2 - e^{-t}$$

Example 3.3: Suppose the following integrodifferential difference equation with differential order one and difference of order two.

$$u'(t) = u(t-\omega)u'(t-\omega) + \int_0^t \sin\big(u(t_1 - 2\omega)\big)dt_1\,,\ t > 2\omega \qquad (3.14)$$

with an interval condition:

$$u(t) = 1,\ 0 \leq t \leq 2\omega \qquad (3.15)$$

Following the steps of the method for the equations (3.14) − (3.15), we directly get

$$L\{u(t)\} = \frac{1}{p} + \frac{e^{-\omega p}}{p} L\{u(t)u'(t)\} + \frac{e^{-2\omega p}}{p^2} L\big\{\sin\big(u(t)\big)\big\} \qquad (3.16)$$

The next step is to evaluate the next Laplace decomposition series for $L\{u(t)u'(t)\}$ and $L\{\sin\,(u(t))\}$:

$$L\{u(t)u'(t)\} = \sum_{n=0}^{\infty} e^{-n\omega p} L\{B_n(t)\}$$

where $B_i\,'s$ are Adomian Polynomials,

$$B_n(t) = u_0(t)u_n'(t) + u_1(t)u_{n-1}'(t) + \cdots + u_n(t)u_0'(t), \text{ for } n \geq 0$$

And

$$L\big\{\sin\big(u(t)\big)\big\} = \sum_{n=0}^{\infty} e^{-n\omega p} L\{C_n(t)\} \qquad (3.18)$$

where $C_i\,'s$ are Adomian Polynomials given below,

$$C_0(t) = \sin\,(u_0(t))$$
$$C_1(t) = u_1(t)\cos\,(u_0(t))$$
$$C_2(t) = u_2(t)\cos\,(u_0(t)) - \frac{1}{2}u_1^2(t)\sin\,(u_0(t))$$
$$C_3(t) = u_3(t)\cos\,(u_0(t)) - u_1(t)u_2(t)\sin\,(u_0(t)) - \frac{1}{6}u_1^3(t)\cos\,(u_0(t))$$

and so on.

Now, by using the equations (2.6), (3.17) and (3.18) and substituting in the equation (3.16), we obtain,

$$\sum_{n=0}^{\infty} e^{-n\omega p} L\{u_n(t)\} = \frac{1}{p} + \frac{e^{-\omega p}}{p}\sum_{n=0}^{\infty} e^{-n\omega p} L\{B_n(t)\} + \frac{e^{-2\omega p}}{p^2}\sum_{n=0}^{\infty} e^{-n\omega p} L\{C_n(t)\}$$

By equally the terms with the coefficient of $e^{-n\omega p}$ on both sides of an equation (3.19) we get $L\{u_n(t)\}$.

Take inverse Laplace transform will result $u_n(t)$. For $4\omega \leq t \leq 5\omega$, the approximate solution is

$$u(t) \approx \sum_{n=0}^{4} u_n(t-n\omega)$$
$$= 1 + \sin(1)\frac{(t-2\omega)^2}{2!} + \sin(1)\frac{(t-3\omega)^2}{2!}$$
$$+\sin(1)\frac{(t-4\omega)^2}{2!} + \sin(1)\cos(1)\frac{(t-4\omega)^4}{4!}$$

In this procedure, we can continue and workout higher level approximate solutions.

4. Conclusion

The above three illustrative examples clearly demonstrate the fact that Laplace decomposition method transforms a differential-difference equation or a integro-differential-difference equation with differential order one and difference order two with a given interval condition into an algebraic equation suitable for applying inverse Laplace transformation technique. Finally, this results into a series expression involving unit step functions which represents the solution. It is interesting to note that one can get exact solution in the case of a linear problem. However, in the case of a nonlinear problem, one can compute iteratively approximate solutions without any hassles. If the nonlinear problem has a closed form solution, then after certain stage, every iteration leads to the same exact solution. Hence this method is suitable for both linear and nonlinear problems.

References

[1] A. M. Wazwaz, A first course in integral equations, World Scientific, Singapore, 1997.

[2] D. Bahuguna, A. Ujlayan and D. N. Pandey, (2009). A comparative study of numerical methods for solving an integro-differential equation, Computers and Mathematics with Applications, 57(9),1485 − 1493.

[3] G. Adomian, (1991). A review of the decomposition method and some recent results for nonlinear equation, Computers. Math. Aplic., 21(5), 101-127

[4] G. Adomian, (1994). Solving Frontier Problems of Physics: The Decomposition Method, Kluwer Academic Publishers, .

[5] J. I. Ramos, (2006). Exponential methods for singularly perturbed ordinary differential-diffrence equations, Applied Mathematics and Computation, 182(2),1528 − 1541.

[6] M. K. Kadalbajoo and K. K. Sharma, (2005). Numerical treatment for singularly perturbed nonlinear differential difference equations with negative shift, Nonlinear Analysis, 63 (5-7), 19091924.

[7] M. Rahaman,(2007). Integral Equations and their Applications, WIT Press, Southompton Boston.

[8] P. J. Collins, (2010). Differential and Integral Equations, Oxford University Press.

[9] Razouki, Hawazen Issa, Muayyad Mahmood Khalil, and Ghassan Ezaldeen. (2021). On Analytical Solutions of Some Types of Ordinary Differential Difference Equations. Turkish Journal of Computer and Mathematics Education 12.14: 4020-4031.

[10] R. Bellman and K. L. Cooke, (1963). Differential-Difference Equations, Academic press, New York.

[11] R. Rangarajan and S. R. Ananth Kumar, (2012). A Laplace decomposition method for solving an integro-differential-difference equation, IFRSA'S International journal of computing, **2**(4),819 − 829.