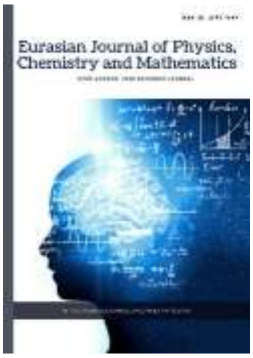

Topological Conjugate for Some Maps

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ABSTRACT

Topological conjugation in dynamical systems is of great interest in the study. We have shown that there are chaotic characteristics that can be transferred from one function to another if there is a topological conjugate between the two functions, which serves us in studying some chaotic functions. As the conjugacy between the Tent map , quadratic map and shift map .There is also topological conjugate between other maps, for example, the Bernoulli map and the logistic map are topological conjugate. And the shift map on the space of two –sided sequences in two symbols is topologically conjugate or semi conjugate to the Henon map.

Keywords: Topological conjugate between the Tent map , quadratic map, shift map, Henon map, horseshoe map, Bernoulli map

1-Introduction:

The mathematician Edward Lorenz first discovered chaotic behavior in the 1950 s. He discovered how a small difference in initial values can result in a large difference in long-term results, in 1986 Robert L.Devaney is the first to formally define chaos.Iterated functions is necessary, since in one iterative function can be determined, so at for a conjugacy of function, this also known as topological equivalence [1]. Let A and B are iterated functions, and there exists a homeomorphism C satisfy:

$$B = C^{-1} \circ A \circ C$$

A and B are topological conjugate. Also satisfy:

$$B^{n} = C^{-1} \circ A^{n} \circ C$$

Topological conjugate is iterated function ,where o denotes the composition. Topological conjugacy between maps is a very powerful tool in the study of dynamical systems. By study the dynamical properties of a system, we can comment on the dynamical properties of other systems which are topologically conjugate to the first system. This is due to the fact that most of the dynamical properties of a system are retained under topological conjugation [2]. In order to other to prove that such maps are chaotic, we show that they are topologically conjugate to other maps which we know are chaotic[3].In this paper we denoted topological transitive (TT) and sensitive dependence on initial condition(SDIC).

2- Some important definition and theories in research:

Definition (2.1): Let A: X → X , B: Y → Y, where C: X → Y are topological spaces and coninuous, A is topologically semi conjugate to B means , that C is a surjection such that $AoC = CoB$.

Definition (2.2)[4]: Let G: A → A continuous function of A be metric space is called **chaotic(Devaney)** then satisfy :

- SDIC
- TT
- Periodic point are dense in A

Let M, K are nonempty open sets in Y then (Y, G) is **TT** if satisfy $G^n(M) \cap K \neq \phi$, G is called **topologically mixing** if for any M, K are nonempty open sets in Y $\exists$ $N > 0$ such that $G^n(M) \cap K \neq \phi$ for $n \geq N$.
The $G: A \to A$ is called **SDIC** when $\exists$ $\varepsilon > 0$, $\forall \delta > 0$, $\forall$ $y_o \in A$ and any open set $M \subset A$ containing y_o there exists $y_o \in M$ and $n \in Z^+$ such that $(y_o, z_o) < \delta$ then $d(f^n(y_o), f^n(z_o)) > \varepsilon$ [4] **.**
The $G: A \to A$ is **minimal** if all points are transitive (each orbit is dense that is G has no periodic points.[5]
Proposition (2.3)[2]**:** Let the topological dynamical system $A: X \to X$ is topological conjugate to the topological dynamical system $B: Y \to Y$ by the conjugacy map $C: X \to Y$, then

- A is TT iff B is TT.
- A is topologically mixing iff B is topologically mixing.
- A is topologically weakly mixing iff B is topologically weakly mixing.
- A is topologically exact iff B is topologically exact.
- A is topologically minimal iff B is topologically minimal.
- A is Devaney chaoic iff B is Devaney chaoic.
- A is exact Devaney chaoic iff B is exact Devaney chaoic.
- A is mixing Devaney chaoic iff B is mixing Devaney chaoic.
- A is weakly mixing Devaney chaoic iff B is weakly mixing Devaney chaoic.

3- Topological conjugacy between of the maps:

Tent map is $T_c(x) = \begin{cases} cx & 0 \leq x \leq \frac{1}{2} \\ c(1-x) & \frac{1}{2} \leq x \leq 1 \end{cases}$, a straightforward calculation shows that T can be conjugated to Quadratic map(logistic map) $Q_\mu = \mu x(1-x)$. different values of the parameters μ and c are studied of the dynamics . The tent and the quadratic families maps have chaotic in values $\mu > 3$ and $c > 1$.
When $c > 2$ and $\mu > 4$ are the Cantor sets that occur as non-wandering sets.[5]

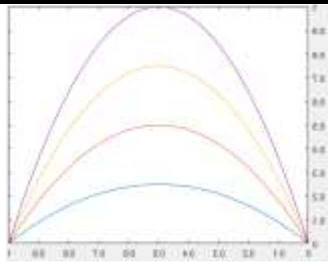


Figure (1) the graph of the quadratic family $\mu = 1,2,3,4$

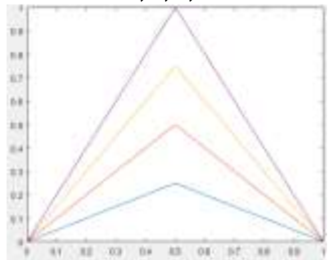


Figure (2) the graph of the tent family when $c = 0.5, 1, 1.5, 2$

Edward Lorenz is the first clear up chaotic behavior in the 1950s, he discovered how a small difference in initial values can result in a large difference in long-term results.
Also Robert L.Devaney who the first to formally define chaos in 1986. A non-wandering set consists of all points that never leave the interval
Proposition (3.1): there is a topological conjugacy between Q_4 and T_2 as in Figures (1),(2)
The shift map is an important map in symbolic dynamics, it shifts the entire sequence $\mathcal{S}(x)$ one place to the left in such a way that the first entry, s_0 is forgotten, it is define as follows:
The **shift** map $\sigma: \Sigma_2 \to \Sigma_2$ is given by $\sigma = (s_0 s_1 s_2 \dots) = (s_1 s_2 s_3 \dots.)$, where Σ_2 is an cantor set

Proposition (3.2): there are a topological conjugacies between Q_μ and σ and between T_c and σ so-called commutative .So the following is true:
(1) $\mathcal{S} \circ Q_\mu = \sigma \circ \mathcal{S}$
(2) $\mathcal{S} \circ T_c = \sigma \circ \mathcal{S}$
Proof: prove the Proposition in [1].

Since Γ_Q and Γ_T are Cantor sets, all topological properties are preserved by such a topological conjugacy. For example, if x_q is a fixed point for Q_μ then $\mathcal{S}(x_q)$ is a fixed point for σ:

$\mathcal{S}(x_q) = \mathcal{S}(Q_\mu(x_q)) = \sigma(\mathcal{S}(x_q))$

In the same way, if Q_μ contain n period of x_q lead to the shift map has $\mathcal{S}(x_q)$ of period n:

$\mathcal{S}(x_q) = \mathcal{S}(Q_\mu^n x_q) = \sigma \circ \mathcal{S} \circ Q^{n-1}(x_q) = \sigma^2 \circ \mathcal{S} \circ Q^{n-2}(x_q) = \cdots = \sigma^n(\mathcal{S}(x_q))$

And Since of the conjugacy between Q_μ and σ all the topological properties satisfy of σ also hold of Q_μ, we can summarize this in the following proposition:

Proposition(3.3) [4]: If $Q_\mu(x) = \mu x(1-x)$ with $\mu > 2 + \sqrt{5}$ then:

- Q_μ has a dense orbit in Γ_Q.
- The cardinality of $per_n(Q_\mu)$ is 2^n.
- $per(Q_\mu)$is dense in Γ_Q.

Since there is topological conjugacy between T_c and σ, so the same properties that hold in proposition(3.3) hold of T_c as well:

Proposition (3.4): If T_c is the Tent map family with $c > 2$ then:

- T_c has a dense orbit in Γ_T.
- The cardinality of $per_n(T_c)$ is 2^n.
- $per(T_c)$is dense in Γ_T.

Proposition (3.5): Let $Q_\mu(x) = \mu x(1-x)$ satisfy Devaney definition when $\mu > 2 + \sqrt{5}$ on the cantor set Γ_Q.

Proof: We needed to prove all three condition of Definition (2.2)

Now, let B_o be the set as in the Cantor set . Let $\varepsilon < B_o$ and $p, q \in \Gamma_Q$, if $p \neq q$, then $\mathcal{S}(p) \neq \mathcal{S}(q)$ which means the itineraries of p and q must differ in at least one spot. Suppose they differ in the n^{th} spot, then $Q_\mu^n(p)$ lies in J_O while $Q_\mu^n(q)$ lies in J_1or vice versa. Then$\left|Q_\mu^n(p) - F_\mu^n(q)\right| > \varepsilon$. Q_μ on the Cantor set Γ_Q is a function $Q_\mu(x): \Gamma_Q \to \Gamma_Q$ consider p in any open set in Γ_Q, there is topological conjugacy as shown in Proposition(3.2), this means that $Q^k(p) = \mathcal{S}^{-1} \circ \sigma^k \circ \mathcal{S}(p)$ since Γ_Q is a Cantor set for $\mu > 2 + \sqrt{5}$ we know it is a perfect set, so there exists a dense orbit is enough to prove topological transitivity, according to Proposition(3.3), Q_μ has a dense orbit in Γ_Q

By Proposition (3.3) shows that the periodic points are dense in Γ_Q.

Not only chaotic for $\mu > 2 + \sqrt{5}$ of the quadratic family but for all values $\mu > 4$.

Also the Tent family has the same reasoning as Proposition (3.4) for $c > 2$,by Proposition (3.2).

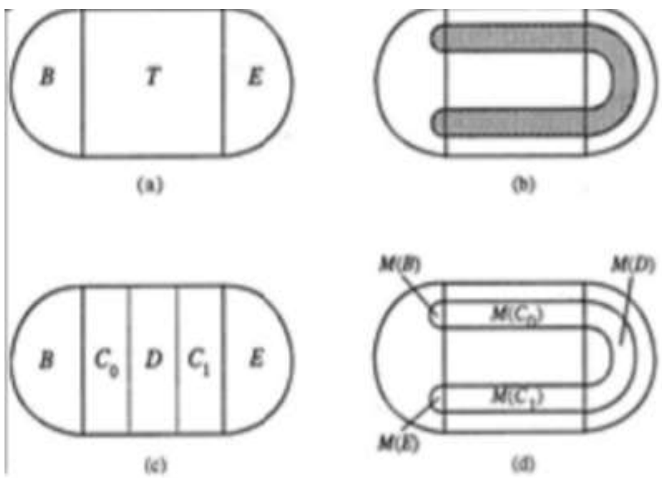
Since T_c with $c > 2$ is topological conjugate of σ and σ is chaotic (Devaney) then T_c is chaotic (Devaney) . .

In the Hirsch, Smale and Devaney the chaotic character of the Logistic map is proved by constructing a semi conjugation of the tent and logistic maps:

The tent and logistic maps from [0,1] to itself have fixed points there is two, one of them 0, and other is repelling, and Q^2 has four fixed points, two are fixed points and two of them has cycle 2 of Q , all unstable, Q^3 have eight fixed points, the fixed points are two and the rest form two the cycles.

Now we will discuss the topological conjugate between the horseshoe and the shift maps:

The horseshoe map is the first described in the 1960s by the Stephen Smale(1967) in R^2, the function H shrinks S(S contains its boundary is composed of the unit square $[0,1] \times [0,1]$)) vertically by a factor $a < \frac{1}{3}$ and expanding S *horezontally* by a factor b=3 with the semicircles B and E as shown in figure (),



Figure() The a horseshoe map [5]

Theorem (3.6): The horseshoe map (S, H) is topologically mixing.(where S is horseshoe set).

Proof: Since The a horseshoe and the full two shift maps are topological conjugate, so look at the proof in [6].

There are also topologically conjugacy between other maps, for example, the Bernoulli map and the logistic map are topological conjugate [7]. And the shift map on the space of two –sided sequences in two symbols is topologically conjugate or semi conjugate to the Henon map[4].

Conclution:

Cantor Set is the non-wandering set is the points that never leave interval J, we have looked at symbolic dynamics. For $\mu > 4$ and $c > 2$, we observed that certain points of J leave the interval after the first iteration. There is topological conjugacy between the shift map and both Q_μ and T_c. We used symbolic dynamics to see that the Tent map family and quadratic family are chaotic for $c > 2$ and Q_μ on there Cantor sets. Topological conjugacy is an equivalence relation on any given collection of dynamical endomorphism . We write A~B denoted that A is topologically conjugate to B, a dynamical property of a system is one which is preserved under topological conjugacy. There is also topological conjugate between other maps, for example, the Bernoulli map and the logistic map are topological conjugate. And the shift map on the space of two –sided sequences in two symbols is topologically conjugate or semi conjugate to the Henon map.

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