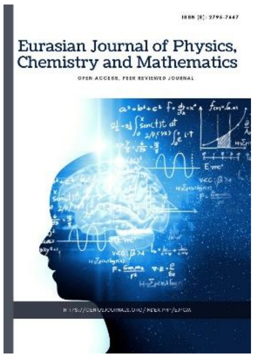

The complete group of structural equations for a nearly Kähler manifold

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ABSTRACT

We consider almost complex and almost Hermitian structures and their associated G -structures. It is proved that the definition of a complex structure on a real linear space is equivalent to the decomposition of its complexification into a direct sum of two complex conjugate subspaces that serve as proper subspaces of this complex structure. It is proved that on every almost complex manifold there exists an almost Hermitian structure. We prove that specifying an almost Hermitian structure on a smooth manifold is equivalent to specifying a G -structure on this manifold with the structure group $G = U(n)$. On the space of the adjoint G -structure, we obtain a complete group of structural equations and give fundamental identities for approximately Kählerian manifolds

Keywords: G-structure, structure equations, nearly Kähler manifold, Riemannian metric, almost complex structure.

Material And Method: The presentation of the material is carried out by the systematic use of Cartan's method of external forms in combination with the method of invariant Koschul calculus. Structural equations are written in a specialized frame, ie on the space of the associated G -structure.

1. Introduction

The concept of an approximately Kählerian manifold is one of the most interesting generalizations of the concept of a Kählerian manifold. It entered the field of geometric research in the second half of the last century and quickly attracted the attention of several leading geometers. This explains the unsettled terminology: along with the term "approximately (nearly) Kählerian manifold", used in the works of A. Gray, J. Wolf, and others, and currently, the most common, synonyms are used: "K-space" (S. Tachibana, Y. Watanabe, S. Koto, and others), as well as "Almost Tachibana Space" (K. Yano, S. Yamaguchi, M. Matsumoto, and others). Interest in the concept of an approximately Kähler manifold emerged after Frölicher proved in 1955 the existence of a canonical almost Hermitian structure on the six-dimensional sphere S^6[1], and Fukami and Ishihara in [2] proved that the fundamental form of this structure is the Killing form (i.e., i.e., its covariant differential is a differential form), which is equivalent to the approximate Kählerian nature of this structure. As an independent geometric object, an approximately Kählerian manifold appears in Tachibana's paper [3] under the name of K -spaces. Further studies of approximately Kahlerian manifolds are associated with the names of A. Gray [4], [5], V.F. Kirichenko [6], [7], [8], Watanabe, and Takamatsu [9], Vanhekke [10], and many others. And at present, the flow of geometric studies of approximately Kahlerian manifolds does not dry out.

The main goalof our work is to obtain a complete group of structural equations, on the space of the associated G -structure.

The work is structured as follows. In Section 2, we define a structure and its almost complex adjoint G -structure and construct a basis adapted to an almost complex structure. In Section 3 we consider the Hermitian structure and construct a modified A -basis. And in the constructed basis we write down the operations of raising and lowering the index for the tensor. In Section 4 we give definitions of an almost Hermitian structure and its adjoint G -structure. In Section 5 we define an approximately Kählerian structure and present the first group of structural equations on the space of the adjoint G -structure. In Section 6, by a differential continuation of the first group, we obtain the complete group of structural equations. And in Section 7 we define a structural tensor of the third kind and prove three fundamental identities for approximately Kählerian manifolds.

2. Almost complex structure and its associated G -structure

Let M be a real differentiable paracompact manifold of dimension $2n$, $\mathcal{X}(M)$be $C^\infty(M)$the -module of smooth vector fields on it.

Definition 2.1 ([11])*An almost complex structure on M is a tensor field of Jtype (1,1) that at each point $m \in M$defines an endomorphism of the tangent space $T_m(M)$such that $J^2 = -id$, where $id$is the identity transformation. A manifold with a fixed almost complex structure is called an almost complex manifold.*

It is known that every almost complex manifold has an even dimension and is orientable ([11]).

Definition 2.2 ([12])*A complexification $\mathcal{X}(M)$is a tensor product $\mathcal{X}^{\boldsymbol{C}}(M) = \mathcal{X}(M) \otimes \boldsymbol{C} = \{\sum z_k X_k \mid z_k \in \boldsymbol{C}, X_k \in \mathcal{X}(M)\}$. Any element of the complexification can be represented as $\sum z_k X_k = \sum X_k Y_k + \sqrt{-1}\, X_k Y_k = X + \sqrt{-1} Y$, where $X, Y \in \mathcal{X}(M)$.*

In $\mathcal{X}^{\boldsymbol{C}}(M)$a natural way, an involutive automorphism is defined: $\tau: \mathcal{X}^{\boldsymbol{C}}(M) \to \mathcal{X}^{\boldsymbol{C}}(M)$called the *complex conjugation*of vectors and acting according to the formula: if $X = \sum_k z_k X_k$, then $\tau(X) = \sum_k \bar{z}_k X_k$, where $\bar{z}_k$is the usual operation of complex conjugation.

Let be (M, J)an almost complex manifold. We define in $\mathcal{X}^{\boldsymbol{C}}(M)$two operators $\sigma$and $\bar{\sigma}$, acting as follows:

$\sigma = \frac{1}{2}\left(id - \sqrt{-1} J^{\boldsymbol{C}}\right), \bar{\sigma} = \frac{1}{2}\left(id + \sqrt{-1} J^{\boldsymbol{C}}\right)$,

where $J^{\boldsymbol{C}}$is the complexification of the operator J, namely:

$J^{\boldsymbol{C}}(\sum_k z_k X_k) = \sum_k z_k J(X_k)$.

In the future, allowing freedom of speech, $J^{\boldsymbol{C}}$we will simply denote endomorphism J. It is easy to show that σin $\bar{\sigma}$mutually complementary projectors, i.e., a) $\sigma + \bar{\sigma} = id$; b) $\sigma^2 = \sigma$. In addition $J \circ \sigma = \frac{1}{2}\left(J + \sqrt{-1} id\right) = \frac{\sqrt{-1}}{2}\left(id - \sqrt{-1} J\right) = \sqrt{-1}\sigma$, which means $Im\, \sigma \subset D_J^{\sqrt{-1}}$. (Here and in what follows, the symbol D_F^{λ}denotes the proper subspace of the endomorphism F corresponding to the eigenvalue λ).

Conversely, if $X \in D_J^{\sqrt{-1}}$, then $\sigma X = \frac{1}{2}\left(X - \sqrt{-1} J X\right) = \frac{1}{2}(2X) = X$, in particular, $X \in Im\, \sigma$. Thus, $Im\, \sigma = D_J^{\sqrt{-1}}$. Likewise, $Im\, \bar{\sigma} = D_J^{-\sqrt{-1}}$. Since $\mathcal{X}^{\boldsymbol{C}}(M) = D_J^{\sqrt{-1}} \oplus D_J^{-\sqrt{-1}}$, we get:

Theorem 2.1.*$C^\infty(M)$-module of smooth vector fields onM^{2n} $\mathcal{X}^{\boldsymbol{C}}(M)$decomposes into a direct sum of eigenspaces of the endomorphism Jcorresponding to the eigenvalues $\sqrt{-1}$and $-\sqrt{-1}$, i.e., $\mathcal{X}^{\boldsymbol{C}}(M) = D_J^{\sqrt{-1}} \oplus D_J^{-\sqrt{-1}}$, and the endomorphisms $\sigma$and $\bar{\sigma}$are projections onto the subspaces $D_J^{\sqrt{-1}}$and $D_J^{-\sqrt{-1}}$, respectively.*

Theorem 2.2. *Specifying a complex structure on an $\boldsymbol{R}$ -linear space is $\mathcal{X}(M)$equivalent to splitting $\mathcal{X}^{\boldsymbol{C}}(M)$into a direct sum of two complex conjugate subspaces that serve as proper subspaces of this complex structure.*

Proof. Necessity follows from Theorem 2.1. Let now $\mathcal{X}^{\boldsymbol{C}}(M) = D \oplus \tau D$. Then $\forall X \in \mathcal{X}^{\boldsymbol{C}}(M) \Longrightarrow X = X_1 + X_2$; $X_1 \in D, X_2 \in \tau D$. We construct an endomorphism $\mathcal{J}: \mathcal{X}^{\boldsymbol{C}}(M) \longrightarrow \mathcal{X}^{\boldsymbol{C}}(M)$by setting $\mathcal{J}(X) = \sqrt{-1}(X_1 - X_2)$. Obviously, $\tau(X) = \tau(X_1) + \tau(X_2)$, and $\tau(X_1) \in \tau D, \tau(X_2) \in D$. Therefore $(\mathcal{J} \circ \tau)(X) = \sqrt{-1}\big(\tau(X_2) - \tau(X_1)\big)$. On the other hand, due to the antilinearity of the operator τ, $(\tau \circ \mathcal{J})(X) = -\sqrt{-1}\big(\tau(X_1) - \tau(X_2)\big) = \sqrt{-1}(\tau X_2 - \tau X_1)$. Thus, $\mathcal{J} \circ \tau = \tau \circ \mathcal{J}$. So, $\mathcal{J} = J^{\boldsymbol{C}}$for some R -linear endomorphism $J: \mathcal{X}(M) \to \mathcal{X}(M)$. Obviously, $\mathcal{J}^2 = -id$, in particular, $J^2 = -id$, i.e. $J$is the complex structure on $\mathcal{X}(M)$. If $X \in D$, then $X = X_1$, which means $\mathcal{J}(X) = \sqrt{-1}X_1 = \sqrt{-1}X$. Therefore, $D \subset D_J^{\sqrt{-1}}$. Conversely, if $X \in D_J^{\sqrt{-1}}$, then $\sqrt{-1}(X_1 - X_2) = \mathcal{J}(X) = \sqrt{-1}X = \sqrt{-1}(X_1 + X_2)$, whence $X_2 = 0$, and hence $X \in D$. Therefore, $D_J^{\sqrt{-1}} \subset D$, i.e. $D_J^{\sqrt{-1}} = D$. Likewise, $D_J^{-\sqrt{-1}} = \tau D$.

□

Lemma 2.1. *In the introduced notation,*
1) $\tau \circ \sigma = \bar{\sigma} \circ \tau$,
2) $\tau \circ \bar{\sigma} = \sigma \circ \tau$.

Proof. Taking into account the antilinearity of the mapping $\tau$and using the fact that a $\boldsymbol{C}$ -linear operator $F: \mathcal{X}^{\boldsymbol{C}}(M) \to \mathcal{X}^{\boldsymbol{C}}(M)$is a linear extension of some $\boldsymbol{R}$ -linear operator $f: \mathcal{X}(M) \to \mathcal{X}(M)$if and only if, $\tau \circ F = F \circ \tau$we have: $\tau \circ \sigma(X) = \frac{1}{2}\tau\big(X - \sqrt{-1}JX\big) = \frac{1}{2}\big(\tau X + \sqrt{-1}\tau \circ JX\big) = \frac{1}{2}\big(\tau X + \sqrt{-1}J \circ \tau X\big) = \sigma \circ \tau(X)$; $X \in \mathcal{X}^{\boldsymbol{C}}(M)$.The second relation is proved similarly.

□

Theorem 2.3. *The mappings* $\sigma|_V: V \to D_J^{\sqrt{-1}}$*and* $\bar{\sigma}|_V: V \to D_J^{-\sqrt{-1}}$*are, respectively, an isomorphism and an anti-isomorphism of* $\boldsymbol{C}$ *-linear spaces.*

Proof. The additivity of the mappings $\sigma|_{\mathcal{X}(M)}$and $\bar{\sigma}|_{\mathcal{X}(M)}$is obvious. Let now $z = \alpha + \sqrt{-1}\beta \in \boldsymbol{C}, X \in \mathcal{X}(M)$. As already seen, $\sigma \circ J = J \circ \sigma = \sqrt{-1}\sigma, \bar{\sigma} \circ J = J \circ \bar{\sigma} = -\sqrt{-1}\sigma$. Therefore $\sigma(zX) = \sigma(\alpha X + \beta JX) = \alpha\sigma X + \beta\sqrt{-1}\sigma X = z(\sigma X)$. Similarly, $\bar{\sigma}(zX) = \bar{z}(\bar{\sigma}X)$, and thus the maps $\sigma|_{\mathcal{X}(M)}$and $\bar{\sigma}|_{\mathcal{X}(M)}$are, respectively, a homomorphism and an antihomomorphism of $\boldsymbol{C}$ -linear spaces.

Let $\exists X \in \mathcal{X}(M)$and $\sigma X = 0$. Applying the operator to both parts of this identity τ, taking into account Lemma 2.1, we obtain that $\bar{\sigma}X = 0$, and hence $X = \sigma X + \bar{\sigma}X = 0$. Therefore, $\ker\sigma|_{\mathcal{X}(M)} = \{0\}$. Similarly, $\ker\bar{\sigma}|_{\mathcal{X}(M)} = \{0\}$, i.e.,$\sigma$and $\bar{\sigma}$are monomorphism and antimonomorphism, respectively.

Let, finally $X \in D_J^{\sqrt{-1}}$. Consider the vector $Y = X + \tau X$. Then $Y \in \mathcal{X}(M)$. On the other hand, since, $X \in Im\,\sigma = \ker\bar{\sigma}$taking into account Lemma 2.1, we have: $\sigma Y = \sigma X + (\tau \circ \sigma)X = X + (\tau \circ \bar{\sigma})X = X$. Similarly, if $X \in D_J^{-\sqrt{-1}}$, then $\bar{\sigma}Y = X$, and, thus, $\sigma|_{\mathcal{X}(M)}$and $\bar{\sigma}|_{\mathcal{X}(M)}$are an epimorphism and an anti-epimorphism, respectively.

□

Let, in particular, V be a finite-dimensional R -linear space, $\dim M = 2n$, and let $b = \{e_1, \dots, e_n\}$be its basis as a $\boldsymbol{C}$ -module. Consider a system of vectors $b_A = \{\varepsilon_1, \dots, \varepsilon_n, \varepsilon_{\hat{1}}, \dots, \varepsilon_{\hat{n}}\}$, where $\varepsilon_a = \sigma(e_a), \varepsilon_{\hat{a}} = \bar{\sigma}(e_a); a = 1, \dots, n$. By Theorem 2.3, vectors $\{\varepsilon_1, \dots, \varepsilon_n\}$form a basis of a $\boldsymbol{C}$ -linear space $D_J^{\sqrt{-1}}$, and vectors form $\{\varepsilon_{\hat{1}}, \dots, \varepsilon_{\hat{n}}\}$a basis of a ***C* -linear** space $D_J^{-\sqrt{-1}}$, and, by virtue of Lemma 2.1, $\tau\varepsilon_a = (\tau \circ \sigma)e_a = (\bar{\sigma} \circ \tau)e_a = \bar{\sigma}e_a = \varepsilon_{\hat{a}}$. Moreover, because of Theorem 2.1, the system of vectors $b_A = \{\varepsilon_1, \dots, \varepsilon_n, \varepsilon_{\hat{1}}, \dots, \varepsilon_{\hat{n}}\}$forms a basis of the space V^C, characterized by the fact that the endomorphism matrix Jin this basis has the form

$$\big(J_j^i\big) = \begin{pmatrix} \sqrt{-1}I_n & 0 \\ 0 & -\sqrt{-1}I_n \end{pmatrix}, \tag{2 .1}$$

Let's call such a basis *adapted to the complex structure*, in short *A-basis*.

3. Hermitian structures

Definition 3.1. *Let V be a real linear space. A Hermitian structure on V is a pair $(J, g = \langle\cdot,\cdot\rangle)$, where $J$is a complex structure on V, $g = \langle\cdot,\cdot\rangle$is a (pseudo) Euclidean structure, and*

$$\langle JX, JY\rangle = \langle X, Y\rangle, X, Y \in V. \quad (3.1)$$

Let be $(J, g = \langle\cdot,\cdot\rangle)$a Hermitian structure on V. Let us construct a mapping $\Omega: V \times V \to R$by setting $\Omega(X, Y) = \langle X, JY\rangle, X, Y \in V$. Obviously $\Omega(Y, X) = \langle Y, JX\rangle = \langle JY, J^2X\rangle = -\langle JY, X\rangle = -\langle X, JY\rangle = -\Omega(X, Y)$. Thus, $\Omega$is an outer 2-form on V . It is called the *fundamental form*of structure. Obviously, its skew-symmetry is equivalent to the identity

$$\langle JX, Y\rangle = -\langle X, JY\rangle;\ X, Y \in V, \quad (3.2)$$

which, in turn, is equivalent to (3.1). An obvious consequence of this identity is the important relation

$$\langle X, JX\rangle = 0;\ X \in V. \quad (3.3)$$

Recall that a Hermitian form on a complex linear space $W$is a mapping $h: W \times W \to \boldsymbol{C}$such that:
1) $h(X + Y, Z) = h(X, Z) + h(Y, Z)$;
2) $h(X, Y + Z) = h(X, Y) + h(X, Z)$;
3) $h(zX, Y) = zh(X, Y);\ h(X, zY) = \bar{z}h(X, Y)$;
4) $h(X, Y) = \overline{h(Y, X)};\ X, Y, Z \in W, z \in \boldsymbol{C}$.

The first two properties are, as usual, called *additivity*, the third, *sesquilinearity*, and the fourth, *hermitian*. The notions of non-degeneracy and positive definiteness of a Hermitian form are defined in the usual way. The non-degenerate Hermitian form will often be called the *Hermitian metric*, and the $\boldsymbol{C}$-linear space in which the Hermitian metric is fixed will be called the *Hermitian space*.

Theorem 3.1. *Specifying a Hermitian structure $(J, \langle\cdot,\cdot\rangle)$in a linear spaceV is equivalent to specifying a non-degenerate Hermitian form $h = \langle\langle\cdot,\cdot\rangle\rangle$in V, considered as a $\boldsymbol{C}$ - linear with respect to Jspace. The positive definiteness of a form is $\langle\langle\cdot,\cdot\rangle\rangle$equivalent to the positive definiteness of a bilinear form $\langle\cdot,\cdot\rangle$.*

Proof. Let be $(J, \langle\cdot,\cdot\rangle)$a Hermitian structure on V . Let $\langle\langle X, Y\rangle\rangle = \langle X, Y\rangle + \sqrt{-1}\langle X, JY\rangle; X, Y \in V$. Taking into account (4.1) and (4.2), it is obvious that $\langle\langle JX, Y\rangle\rangle = \langle JX, Y\rangle + \sqrt{-1}\langle JX, Y\rangle = \sqrt{-1}\langle X, Y\rangle - \langle X, JY\rangle = \sqrt{-1}(\langle X, Y\rangle + \sqrt{-1}\langle X, JY\rangle) = \sqrt{-1}\langle\langle X, Y\rangle\rangle$. Similarly, $\langle\langle X, JY\rangle\rangle = -\sqrt{-1}\langle\langle X, Y\rangle\rangle$, whence, taking into account the definition of a $\boldsymbol{C}$-module in V, it follows that the form $\langle\langle\cdot,\cdot\rangle\rangle$is linear in the first and antilinear in the second argument. In addition, $\langle\langle Y, X\rangle\rangle = \langle Y, X\rangle + \sqrt{-1}\Omega(Y, X) = \langle X, Y\rangle - \sqrt{-1}\Omega(X, Y) = \overline{\langle\langle X, Y\rangle\rangle}$. Thus, $\langle\langle\cdot,\cdot\rangle\rangle$is a Hermitian form on V. Obviously, it is non-degenerate.

Conversely, let h be a non-degenerate Hermitian form in V . Consider the bilinear forms $g = \Re h$and $\Omega = \Im h$- the real and imaginary parts of the form h, respectively. Thus, $h(X, Y) = g(X, Y) + \sqrt{-1}\Omega(X, Y); X, Y \in V$. Since $g(X, Y) + \sqrt{-1}\Omega(X, Y) = h(X, Y) = \overline{h(Y, X)} = g(Y, X) - \sqrt{-1}\Omega(Y, X)$, then, comparing the real and imaginary parts, we have:

$$1)\ g(X, Y) = g(Y, X);\ \ 2)\ \Omega(X, Y) = -\Omega(Y, X). \quad (4.4)$$

Next, $\sqrt{-1}g(X, Y) - \Omega(X, Y) = \sqrt{-1}h(X, Y) = -h(X, JY) = -g(X, JY) - \sqrt{-1}\Omega(X, JY)$.
Comparing the real and imaginary parts, we get that
1) $\Omega(X, Y) = g(X, JY)$; 2) $g(X, Y) = -\Omega(X, JY)$.
In particular,
$g(JX, JY) = \Omega(JX, Y) = -\Omega(Y, JX) = g(X, Y)$.
Wherein
$h(X, Y) = g(X, Y) + \sqrt{-1}g(X, JY) = \langle\langle X, Y\rangle\rangle$.
Taking into account the last relation, it is obvious that the bilinear form is non-$g$degenerate, and the positive definiteness of the form is $g$equivalent to the positive definiteness of the form h. Thus, $(J, g)$is a Hermitian structure, and $\Omega$is its fundamental form. □

Remark 3.1. *In what follows, unless otherwise stated, we will always assume that $g = (\cdot,\cdot)$is a Euclidean structure, which means that the form is $\langle\langle\cdot,\cdot\rangle\rangle$positive definite.*

Let be $(J, g = \langle\cdot,\cdot\rangle)$a Hermitian structure on a linear space V. Then the $V^{\boldsymbol{C}} = \boldsymbol{C} \otimes V\boldsymbol{C}$-bilinear form is naturally defined in space

$$g^{\boldsymbol{C}}(\textstyle\sum_k z_k X_k, \sum_m w_m Y_m) = \sum_{k,m} z_k w_k \langle X_k, Y_m\rangle,$$

or, alternatively,
$g^{C}(X+\sqrt{-1}Y, Z+\sqrt{-1}W) = (\langle X,Z\rangle - \langle Y,W\rangle) + \sqrt{-1}(\langle X,W\rangle + \langle Y,Z\rangle)$.
Obviously, this form is non-degenerate (which is easier to see from its alternative definition). It is called the *linearity extension* of theform g. Allowing the liberty of speech, we will further designate it in the same way as the form itself g.

Theorem 3.2. *The proper subspaces of the endomorphism $J$are completely isotropic with respect to the form g.*

Proof. Let $\tilde{X}, \tilde{Y} \in D_J^{\sqrt{-1}}$. Since the mappings $\sigma|_V : V \to D_J^{\sqrt{-1}}$and $\bar{\sigma}|_V : V \to D_J^{-\sqrt{-1}}$are, respectively, an isomorphism and an antiisomorphism of $\boldsymbol{C}$ -linear spaces, $\tilde{X} = \sigma X, \tilde{Y} = \sigma Y$for some $X, Y \in V$. So

$$\langle \tilde{X}, \tilde{Y}\rangle = \langle \sigma X, \sigma Y\rangle = \frac{1}{4}\left(\langle X - \sqrt{-1}JX, Y - \sqrt{-1}JY\rangle\right) = \frac{1}{4}\left(\langle X,Y\rangle - \langle JX,JY\rangle + \sqrt{-1}\langle X,JY\rangle + \sqrt{-1}\langle JX,Y\rangle\right) = 0.$$

Quite similarly, $\langle \tilde{X}, \tilde{Y}\rangle = 0;\ \tilde{X}, \tilde{Y} \in D_J^{\sqrt{-1}}$.

□

The form $g$naturally introduces the Hermitian form

$$H(X,Y) = 2\langle X, \tau Y\rangle;\ X, Y \in V^{C}$$

in space V^{C}. From the non-degeneracy of the form g(and of the operator τ) it follows that the form *H is non-degenerate*. Further, it is fair

Proposition 3.1. *The proper subspaces of the endomorphism $J$are orthogonal with respect to the Hermitian metric H.*

Proof. This immediately follows from Theorem 3.1 and the definition of the metric H, because if $\tilde{X} \in D_J^{\sqrt{-1}}, \tilde{Y} \in D_J^{-\sqrt{-1}}$, then
$H(\tilde{X}, \tilde{Y}) = 2\langle \tilde{X}, \tau\tilde{Y}\rangle = 2\langle \sigma X, \tau\bar{\sigma} Y\rangle = 2\langle \sigma X, \sigma Y\rangle = 0;\ X, Y \in V$.
□

Since the linear spaceV^{C}decomposes into a direct sum of eigenspaces of the endomorphism Jcorresponding to the eigenvalues $\sqrt{-1}$and $-\sqrt{-1}$, i.e., $V^{C} = D_J^{\sqrt{-1}} \oplus D_J^{-\sqrt{-1}}$, and the endomorphisms $\sigma$and $\bar{\sigma}$are projections onto the subspaces $D_J^{\sqrt{-1}}$and $D_J^{-\sqrt{-1}}$, respectively, we obtain:

Theorem 3.3. *The linear space V^{C}decomposes into an orthogonal direct sum of the eigenspaces of the endomorphism Jcorresponding to the eigenvalues $\sqrt{-1}$and $-\sqrt{-1}$, i.e.,$V^{C} = D_J^{\sqrt{-1}} \oplus D_J^{-\sqrt{-1}}$.*

Theorem 3.4. *Themappings $\sigma : V \to D_J^{\sqrt{-1}}$and $\bar{\sigma} : V \to D_J^{-\sqrt{-1}}$are, respectively, an isometry and an anti-isometry of C -linear spaces with respect to the Hermitian metrics $\langle\langle \cdot,\cdot \rangle\rangle$on V and H on $D_J^{\pm\sqrt{-1}}$.*

Proof. Let $X, Y \in V$. Then

$$H(\sigma X, \sigma Y) = \frac{1}{4}H\left(X - \sqrt{-1}JX, Y - \sqrt{-1}JY\right) = \frac{1}{2}\langle X - \sqrt{-1}JX, Y + \sqrt{-1}JY\rangle = \frac{1}{2}\left(\langle X,Y\rangle + \langle JX,JY\rangle + \sqrt{-1}\langle X,JY\rangle - \sqrt{-1}\langle JX,Y\rangle\right) = \left(\langle X,Y\rangle + \sqrt{-1}\langle X,JY\rangle\right) = \langle\langle X,Y\rangle\rangle.$$

It is proved similarly that $H(\bar{\sigma}X, \bar{\sigma}Y) = \overline{\langle\langle X,Y\rangle\rangle}$.

□

Theorems 3.1, 3.3, and 3.4 immediately imply
Proposition 3.2.*A Hermitian metric H is positive definite if and only if gis a Euclidean structure.*

Proof. By virtue of Theorem 3.3, it suffices to prove the assertion for the restrictions of the metric H to proper subspaces of the endomorphism J. But for them, it is true by virtue of Theorems 3.4 and 3.1.

□

Let, in particular, V be a finite-dimensional $\boldsymbol{R}$ -linear space, $\dim M = 2n$, and let $b = \{e_1, \dots, e_n\}$be its basis as a $\boldsymbol{C}$ -module. Applying, if necessary, the Gram-Schmidt orthogonalization procedure, we can assume without loss of generality that $b = \{e_1, \dots, e_n\}$is a basis orthonormal with respect to the Hermitian metric $\langle\langle\cdot,\cdot\rangle\rangle$. Note the following useful

Proposition 3.3.*The RA -basis corresponding to the orthonormal basis $b = \{e_1, \dots, e_n\}$, is orthonormal with respect to the metric g.*

Proof. Due to the orthonormality of the basis $b = \{e_1, \dots, e_n\}$with respect to $\langle\langle\cdot,\cdot\rangle\rangle$, $\langle\langle e_a, e_b\rangle\rangle = \langle e_a, e_b\rangle + \sqrt{-1}\langle e_a, Je_b\rangle = \delta_{ab}$. From here $\langle e_a, e_b\rangle = \delta_{ab}$; $\langle e_a, Je_b\rangle = 0$; $\langle Je_a, Je_b\rangle = \langle e_a, e_b\rangle = \delta_{ab}$; $\langle Je_a, e_b\rangle = -\langle e_a, Je_b\rangle = 0$.

□

Consider a system of vectors $b_A = \{\varepsilon_1, \dots, \varepsilon_n, \varepsilon_{\hat{1}}, \dots, \varepsilon_{\hat{n}}\}$, where $\varepsilon_a = \sqrt{2}\sigma(e_a), \varepsilon_{\hat{a}} = \sqrt{2}\bar{\sigma}(e_a); a = 1, \dots, n$.By Theorem 3.4, the vectors $\{\varepsilon_1, \dots, \varepsilon_n\}$form orthogonal with respect to the Hermitian metric $H$the basis of the space $D_J^{\sqrt{-1}}$, and the vectors are the basis of the space $\{\varepsilon_{\hat{1}}, \dots, \varepsilon_{\hat{n}}\}$orthogonal with respect to the same metric $D_J^{-\sqrt{-1}}$, and, as in the case of almost complex structures, $\tau\varepsilon_a = \varepsilon_{\hat{a}}$. Moreover, by Theorem 3.3, the system of vectors $b_A$forms a Hermitian space basis orthogonal with respect to the same metric $(V^{\boldsymbol{C}}, H)$(the norm of basis vectors in such a metric is obviously $\sqrt{2}$). Let's call a basis of this kind *modified A-basis*. The modified A -basis differs from the usual *A -basis* attached to an almost complex structure J, firstly, by the obligatory orthogonality, and secondly, by the presence of a factor $\sqrt{2}$in the definition of its elements. However, taking the liberty of speech, by A -bases of a Hermitian space we will always understand modified A -bases.

Proposition 3.4.*The modified A -basis of the Hermitian space is characterized by the fact that the matrices of the components of the tensors Jand ghave the form in it, respectively:*

1) $\left(J_j^i\right) = \begin{pmatrix} \sqrt{-1}I_n & 0 \\ 0 & -\sqrt{-1}I_n \end{pmatrix}$; 2) $\left(g_{ij}\right) = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$. (3.5)

Proof. The first of these relations is defined by Theorem 3.3, the system of vectors $b_A = \{\varepsilon_1, \dots, \varepsilon_n, \varepsilon_{\hat{1}}, \dots, \varepsilon_{\hat{n}}\}$forms a basis of the space $V^{\boldsymbol{C}}$, characterized by the fact that the endomorphism matrix Jin this basis has the form (3.5:1). As for the second relation, then, taking into account Theorem 3.2, $g_{ab} = \langle\varepsilon_a, \varepsilon_b\rangle = 0$; $g_{\hat{a}\hat{b}} = \langle\varepsilon_{\hat{a}}, \varepsilon_{\hat{b}}\rangle = 0$. Moreover, taking into account Theorem 3.4, $g_{a\hat{b}} = g_{\hat{a}b} = \langle\varepsilon_a, \varepsilon_{\hat{b}}\rangle = \langle\varepsilon_a, \tau\varepsilon_b\rangle = \frac{1}{2}H(\varepsilon_a, \varepsilon_b) = H(\sigma e_a, \sigma e_b) = \langle\langle e_a, e_b\rangle\rangle = \delta_a^b$. □

Remark 3.2. *Taking into account the formula (3.5:2), the operation of lowering the index $X^i \to X_i = g_{ij}X^j$in the modified A -basis will be written as follows: $X_a = g_{ab}X^b + g_{a\hat{b}}X^{\hat{b}} = X^{\hat{a}}$; $X_{\hat{a}} = g_{\hat{a}b}X^b + g_{\hat{a}\hat{b}}X^{\hat{b}} = X^a$and, thus,*
$X_a = X^{\hat{a}}$; $X_{\hat{a}} = X^a$.
Similarly for tensors of arbitrary type.

Now let $b = \{e_1, \dots, e_n\}$and $\tilde{b} = \{\tilde{e}_1, \dots, \tilde{e}_n\}$be two orthonormal bases of the space V , $C = C_{b\tilde{b}} = (c_b^a)$be the transition matrix from basis $b$to basis $\tilde{b}$. Obviously, $C \in U(n)$, and the formula

$$C_{b_A\tilde{b}_A} = \begin{pmatrix} C & 0 \\ 0 & \bar{C} \end{pmatrix}, C \in U(n), \tag{3.6}$$

defines an embedding of Lie groups $U(n) \subset GL(2n, \boldsymbol{C})$and hence a right action of a Lie group $U(n)$on the set of orthonormal bases of a given Hermitian structure.

4. Almost Hermitian structure and its associated *G* -structure

Definition 4.1. *Almost Hermitian (in short, $\mathcal{AH}$-) structure onan n-dimensional manifold$M^{2n}$is called a pair (J, g), where $J$is an almost complex structure on M, $g = \langle\cdot,\cdot\rangle$is a Riemannian metric on M. Wherein*
$\langle JX, JY\rangle = \langle X, Y\rangle; X, Y \in \mathcal{X}(M)$,
where $\mathcal{X}(M)$is $C^\infty(M)$the modulus of smooth vector fields on M^{2n}. An endomorphism $J$is called a structural endomorphism. A manifold on which

an almost Hermitian structure is fixed is called an almost Hermitian (in short, $\mathcal{AH}$-) manifold.

Proposition 4.1. *Every almost complex manifold has an almost Hermitian structure.*

Proof. Let be $\tilde{g}$an arbitrary Riemannian metric on an almost complex manifold (M, J). Let's build a bilinear form $g(X,Y) = \tilde{g}(X,Y) + \tilde{g}(JX,JY); X,Y \in \mathcal{X}(M)$. Obviously, the form is $g$positive definite and hence is a Riemannian structure. It is also obvious that $g(JX,JY) = g(X,Y)$, which means that the pair $(J,g)$is an almost Hermitian structure on M^{2n}.

□

Obviously, an almost Hermitian structure can be considered as a Hermitian structure of a module $\mathcal{X}(M)$considered as an $\boldsymbol{R}$ -linear space. Setting an almost Hermitian structure is (J,g)equivalent to setting a Hermitian structure $\langle\langle X,Y\rangle\rangle = \langle X,Y\rangle + \sqrt{-1}\langle X,JY\rangle$in this space, considered as a $\boldsymbol{C}$ -module with respect to the complex structure J.

Let be (J,g)an almost Hermitian structure on the manifold M. It induces almost Hermitian structures (J_m, g_m)at every point $m \in M$.

Theorem 4.1. *Specifying an almost Hermitian structure on a smooth manifold $M^{2n}$is equivalent to specifying a G -structure on this manifold with the structure group $G = U(n)$.*

Proof. Let be Jan almost Hermitian structure on M. Then, at each point $m \in M$, a family of $\mathcal{R}_m$orthonormal frames of the space is defined $T_m(M)$, which is considered as an n-dimensional $\boldsymbol{C}$ -linear space. It follows from the definition of a frame that a group $U(n)$acts in each such family freely and transitively.

□

Lemma 4.1. *In some neighborhood U of an arbitrary point $m \in M$, one can construct a family of vector fields $\{e_1^0, \dots, e_n^0\}$on U that form an orthonormal basis of a module $\mathcal{X}(U)$as a $\boldsymbol{C} \otimes C^\infty(U)$-module.*

Proof. We fix $m \in M$some basis at a point $p = \{\xi_1, \dots, \xi_n, J_m\xi_1, \dots, J_m\xi_n\}$. The system of vectors $\xi_k$can be extended to a system of vector fields $e_k^0 (k = 1, \dots, n)$on M . In this case, the system of vectors $J_m\xi_k$will continue to the system of vector fields Je_k^0. Since the linear independence of the vectors of the frame $p$is equivalent to the inequality zero of the determinant of the transition matrix from the natural basis at the point $m$to the basic part of the frame p, this property is preserved in some neighborhood U of the point $m$for vector fields as well $\{e_1^0, \dots, e_n^0, Je_1^0, \dots, Je_n^0\}$. But then, obviously, the system $\{e_1^0, \dots, e_n^0\}$of vector fields on U will be $\boldsymbol{C} \otimes C^\infty(U)$-linearly independent, and hence forms a basis of the $\boldsymbol{C} \otimes C^\infty(U)$-module $\mathcal{X}(U)$. Applying the Gram-Schmidt orthogonalization procedure to this basis, we obtain the desired orthonormal basis. □

Let's continue the proof of Theorem 4.1. The basis of the view $\{e_1, \dots, e_n, Je_1, \dots, Je_n\}$is called the RA -basis. Let us denote $B_J M = \bigcup_{m \in M} \mathcal{R}_m$, and introduce the natural projection $\pi: \mathcal{R} \to M$that assigns the vertex to the frame $p \in \mathcal{R}$. Now we can construct the mapping $F_U: \pi^{-1}(U) \to GL(n, \boldsymbol{C})$by setting $F_U(p) = g$, where $g$is the transition matrix from the frame $(m, e_1^0|_m, \dots, e_n^0|_m)$to the frame p. Further, it is easy to verify that the quadruple $B_J(M) = \big(\mathcal{R}, M, \pi, G = GL(n, \boldsymbol{C})\big)$forms a principal bundle. This principal bundle can be considered as a G - structure with respect to the monomorphism $(f, \rho)$of the principal bundle $B_J(M)$into the principal bundle $B(M)$, where $f: \mathcal{R} \to BM$is the map that associates the $(m, e_1, \dots, e_n)$space frame $T_m(M)$as a $\boldsymbol{C}$ - module with the corresponding RA -frame, and $\rho: GL(n, \boldsymbol{C}) \to GL(2n, \boldsymbol{R})$is the canonical Lie group monomorphism that associates the matrix with $C = A + \sqrt{-1}B \in GL(n, \boldsymbol{C})$the matrix $\rho(C) = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$whose image is the Lie group $GL^R(n, \boldsymbol{C})$.

Conversely, let be $\big(\mathcal{R}, M, \pi, GL^R(n, \boldsymbol{C})\big)$a G -structure of this type on M . Let be J_0a standard complex structure in space $\boldsymbol{R}^{2n}$given by a matrix of the form $(J_j^i) = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$. Let be $m \in M$an arbitrary point. We define an endomorphism $J_m$in space by the $T_m(M)$formula $J_m = p \circ J_0 \circ p^{-1}; p \in \pi^{-1}(m)$. Obviously, $J_m^2 = -id$, i.e. $J_m$is the complex structure on $T_m(M)$. Let us show that it is well defined in the sense of being independent of the choice of the element $p \in \pi^{-1}(m)$. Indeed, if $\tilde{p} \in \pi^{-1}(m)$is another such element, then $\exists h \in GL^R(n, \boldsymbol{C})$and $\tilde{p} = ph$. Therefore, $\tilde{p} \circ J_0 \circ \tilde{p}^{-1} = (ph) \circ J_0 \circ (ph)^{-1} = p \circ (h \circ J_0 \circ h^{-1}) \circ p^{-1} = p \circ J_0 \circ p^{-1} = J_m$since the group $GL^R(n, \boldsymbol{C})$is obviously an endomorphism invariance group J_0, i.e., $hJ_0 = J_0 h; h \in GL^R(n, \boldsymbol{C})$, which is checked directly.

Let us show that the family of tensors $J = \{J_m | m \in M\}$defines a smooth tensor field on the manifold M . To do this, it suffices to prove that any admissible map (U, φ)on M of functions $m \to J_j^i(m) = dx^i\left(J_m\left(\frac{\partial}{\partial x^j}\Big|_m\right)\right), m \in M,$ are smooth on U . Let us fix a local section $s: U \to \mathcal{R}$of the space of the G -structure. Then, by construction, in the RA -frame $\sigma(m)$(and dual-coreframe) we have:

$$\left(e^i\left(J_m(e_j)\right)\right) = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} = \left((J_0)_j^i\right).$$

The smoothness of the section is expressed in the fact that the components of the matrix C of the transition from the natural basis of the module $\mathcal{X}(U)$to the RA -basis $\sigma(U)$of this module, and hence the components of the inverse matrix $\tilde{C}$, are smooth functions. Therefore, $m \to J_j^i(m) = dx^i\left(J_m\left(\frac{\partial}{\partial x^j}\Big|_m\right)\right) = C_k^i(m)e^k\left(J_m(\tilde{C}_j^r(m)e_r)\right) = C_k^i(m)\tilde{C}_j^r(m)e^k\left(J_m(e_r)\right) = C_k^i(m)\tilde{C}_j^r(m)(J_0)_r^k$are smooth functions on U . Thus, $J$is an almost complex structure. Obviously, the family of RA -frames generated by it coincides with the space of the G -structure.

□

Definition 4.2. *The$G$ - structure $B_J M$constructed above is called the G -structure attached to an almost Hermitian structure (J, g).*

5. Approximately Kahlerian structures

Definition 5.1. *An almost Hermitian structure on a manifold M is called an approximately (nearly) Kahlerian (in short, $\mathcal{NK}$-) structure if the identity*

$$\nabla_X(J)Y + \nabla_Y(J)X = 0; X, Y \in \mathcal{X}(M). \quad (5.1)$$

Theorem 5.1. An almost Hermitian structure (J, g)on a manifold M *is approximately* Kählerian if and only if the identities hold on M

$$1)\ B(X, Y) = 0;\ \ 2)\ C(X, Y) + C(Y, X) = 0. \quad (4.2)$$

Proof. First of all, we note that for approximately Kählerian manifolds we have the identities

$$\nabla_{JX}(J)Y = \nabla_X(J)JY = -J \circ \nabla_X(J)Y. \quad (5.3)$$

Indeed, in view of the identity $J \circ \nabla_X(J)Y + \nabla_X(J)JY = 0$, which is valid for any almost Hermitian manifold,

$\nabla_{JX}(J)Y = -\nabla_Y(J)(JX) = J \circ \nabla_Y(J)X = -J \circ \nabla_X(J)Y = \nabla_X(J)(JY)$.

From this, it immediately follows that

$B(X, Y) = \frac{1}{2}\{\nabla_{JX}(J)Y - \nabla_X(J)(JY)\} = 0;$

$C(X, Y) = \frac{1}{2}\{\nabla_{JX}(J)Y + \nabla_X(J)(JY)\} = -J \circ \nabla_X(J)Y = J \circ \nabla_Y(J)X = -C(Y, X)$.

Conversely, if these relations hold, then

$\nabla_X(J)Y = J \circ C(X, Y) = -J \circ C(Y, X) = \nabla_Y(J)X; X, Y \in \mathcal{X}(M)$.

Therefore, M is an approximately Kählerian manifold.

□

Since an almost Hermitian structure is quasi-Kählerian if and only if its virtual tensor is equal to zero, we obtain

Corollary 5.1. *Any approximately Kählerian manifold is a quasi-Kählerian manifold.*

Almost Hermitian structure (J, g)on the manifold M *is* approximately Kählerian if and only if therelations

$$1)\ B^{ab}{}_c = 0;\ \ 2)\ B^{abc} = B^{[abc]}; \quad (5.4)$$

and complex conjugate formulas (in short, f.c.s.). By virtue of these relations, the first group of structural equations of the NK - structure has the form:

1) $d\omega^a = -\theta_b^a \wedge \omega^b + B^{abc}\omega_b \wedge \omega_c$;
2) $d\omega_a = \theta_a^b \wedge \omega_b + B_{abc}\omega^b \wedge \omega^c$. (5.5)

6. Complete group of structural equations

The second group of structural equations can be found using the procedure of differential continuation of relations (5.5). To do this, we differentiate (5.5:1) externally:

$$-d\theta_b^a \wedge \omega^b + \theta_b^a \wedge d\omega^b + dB^{abc} \wedge \omega_b \wedge \omega_c + B^{abc} d\omega_b \wedge \omega_c - B^{abc}\omega_b \wedge d\omega_c = 0.$$

We substitute the values from (4.5) into the resulting equality, then we get:

$$-d\theta_b^a \wedge \omega^b + \theta_b^a \wedge (-\theta_c^b \wedge \omega^c + B^{bcd}\omega_c \wedge \omega_d) + dB^{abc} \wedge \omega_b \wedge \omega_c + B^{abc}\left(\theta_b^d \wedge \omega_d + B_{bdh}\omega^d \wedge \omega^h\right) \wedge \omega_c - B^{abc}\omega_b \wedge (\theta_c^d \wedge \omega_d + B_{cdh}\omega^d \wedge \omega^h) = 0.$$

We rewrite the resulting equality in the form:

$$-\Delta\theta_b^a \wedge \omega^b + \Delta B^{abc} \wedge \omega_b \wedge \omega_c = 0, \quad (6.1)$$

where

1) $\Delta\theta_b^a = d\theta_b^a + \theta_c^a \wedge \theta_b^c + 2B^{adh}B_{hbc}\omega^c \wedge \omega_d$;
2) $\Delta B^{abc} = dB^{abc} + B^{dbc}\theta_d^a + B^{adc}\theta_d^b + B^{abd}\theta_d^c$. (6.2)

Restricting, as usual, to the area $U \subset M$of some map on M, and setting $W = \pi^{-1}(U)$, we expand the restriction of these forms in standard bases of modules $\Lambda_2(W)$and $\Lambda_1(W)$, respectively:

1) $\Delta\theta_b^a = A_{bcd}^{afh}\theta_f^c \wedge \theta_h^d + A_{bcd}^{af}\theta_f^c \wedge \omega^d + A_{bf}^{acd}\theta_c^f \wedge \omega_d + A_{bcd}^{a}\omega^c \wedge \omega^d + A_{bc}^{ad}\omega^c \wedge \omega_d + A_b^{acd}\omega_c \wedge \omega_d$; (6.3)
2) $\Delta B^{abc} = B^{abcd}{}_f\theta_d^f + B^{abc}{}_d\omega^d + B^{abcd}\omega_d$.

Substituting these relations into (6.1), we obtain

$$-A_{bcd}^{afh}\theta_f^c \wedge \theta_h^d \wedge \omega^b - A_{[b|c|d]}^{af}\theta_f^c \wedge \omega^d \wedge \omega^b - A_{bf}^{acd}\theta_c^f \wedge \omega_d \wedge \omega^b - A_{[bcd]}^{a}\omega^c \wedge \omega^d \wedge \omega^b - A_{[bc]}^{ad}\omega^c \wedge \omega_d \wedge \omega^b - A_b^{acd}\omega_c \wedge \omega_d \wedge \omega^b + B^{abcd}{}_f\theta_d^f \wedge \omega_b \wedge \omega_c + B^{abc}{}_d\omega^d \wedge \omega_b \wedge \omega_c + B^{a[bcd]}\omega_d \wedge \omega_b \wedge \omega_c = 0.$$

Hence, taking into account the linear independence of the basic forms, we obtain that

1) $A_{bcd}^{afh} = 0$; 2) $A_{[b|c|d]}^{af} = 0$; 3) $A_{bf}^{acd} = 0$; 4) $A_{[bcd]}^{a} = 0$; 5) $A_{[bc]}^{ad} = 0$; 6) $A_b^{acd} + B^{abc}{}_d = 0$; 7) $B^{abcd}{}_f = 0$; 8) $B^{a[bcd]} = 0$. (6.4)

Similarly, we differentiate (4.5:2) externally:

$$d\theta_a^b \wedge \omega_b - \theta_a^b \wedge d\omega_b + dB_{abc} \wedge \omega^b \wedge \omega^c + B_{abc}d\omega^b \wedge \omega^c - B_{abc}\omega^b \wedge d\omega^c = 0.$$

We substitute the values from (4.5) into the resulting equality, then we get:

$$d\theta_a^b \wedge \omega_b - \theta_a^b \wedge (\theta_b^c \wedge \omega_c + B_{bcd}\omega^c \wedge \omega^d) + dB_{abc} \wedge \omega^b \wedge \omega^c + B_{abc}\left(-\theta_d^b \wedge \omega^d + B^{bdh}\omega_d \wedge \omega_h\right) \wedge \omega^c - B_{abc}\omega^b \wedge (-\theta_d^c \wedge \omega^d + B^{cdh}\omega_d \wedge \omega_h) = 0.$$

We rewrite the resulting equality in the form:

$$\Delta\theta_a^b \wedge \omega_b + \Delta B_{abc} \wedge \omega^b \wedge \omega^c = 0, \quad (6.5)$$

where

$$\Delta B_{abc} = dB_{abc} \wedge \omega^b - B_{dbc}\theta_a^d - B_{adc}\theta_b^d - B_{abd}\theta_c^d. \quad (6.6)$$

Let there be equality

$$\Delta B_{abc} = B_{abcd}{}^f\theta_f^d + B_{abcd}\omega^d + B_{abc}{}^d\omega_d. \quad (6.7)$$

Substituting these ratios and ratios (6.3:1) into (6.5), we get:

$$A_{acd}^{bf}\theta_f^c \wedge \omega^d \wedge \omega_b + A_{acd}^{b}\omega^c \wedge \omega^d \wedge \omega_b + A_{ac}^{[bd]}\omega^c \wedge \omega_d \wedge \omega_b + A_a^{[bcd]}\omega_c \wedge \omega_d \wedge \omega_b + B_{abcd}{}^f\theta_f^d \wedge \omega^b \wedge \omega^c + B_{a[bcd]}\omega^d \wedge \omega^b \wedge \omega^c + B_{abc}{}^d\omega_d \wedge \omega^b \wedge \omega^c = 0.$$

Hence, taking into account the linear independence of the basic forms, we obtain that

1) $A_{acd}^{bf} = 0$; 2) $A_{acd}^{b} + B_{abc}{}^d = 0$; 3) $A_{ac}^{[bd]} = 0$; 4) $A_a^{[bcd]} = 0$; 5) $B_{abcd}{}^f = 0$;
6) $B_{a[bcd]} = 0$. (6.8)

Taking into account the obtained equalities (6.4) and (6.8), expansions (6.3) and (6.7) take the form:

1) $d\theta_b^a = -\theta_c^a \wedge \theta_b^c + A_{bcd}^{a}\omega^c \wedge \omega^d + \left(A_{bc}^{ad} - 2B^{adh}B_{hbc}\right)\omega^c \wedge \omega_d + A_b^{acd}\omega_c \wedge \omega_d$; (6.9)
2) $dB^{abc} + B^{dbc}\theta_d^a + B^{adc}\theta_d^b + B^{abd}\theta_d^c = B^{abc}{}_d\omega^d + B^{abcd}\omega_d$.
3) $dB_{abc} \wedge \omega^b - B_{dbc}\theta_a^d - B_{adc}\theta_b^d - B_{abd}\theta_c^d = B_{abcd}\omega^d + B_{abc}{}^d\omega_d$.

From equalities (6.4) and (6.8) we obtain:

$$A_{bcd}^{a} = A_b^{acd} = B_{abc}{}^d = B^{abc}{}_d = 0.$$

Thus, relations (6.9) take the form:

1) $d\theta_b^a = -\theta_c^a \wedge \theta_b^c + \left(A_{bc}^{ad} - 2B^{adh}B_{hbc}\right)\omega^c \wedge \omega_d$; (6.10)

2) $dB^{abc} + B^{dbc}\theta_d^a + B^{adc}\theta_d^b + B^{abd}\theta_d^c = B^{abcd}\omega_d$;
3) $dB_{abc} - B_{dbc}\theta_a^d - B_{adc}\theta_b^d - B_{abd}\theta_c^d = B_{abcd}\omega^d$,
where
$A_{[bc]}^{ad} = A_{bc}^{[ad]} = 0, B^{a[bcd]} = B_{a[bcd]} = 0.$ (6.11)

Remark 6.1. The components of the form of $\zeta$the first canonical connection of $\tilde{\nabla}$an arbitrary almost Hermitian structure on the space of the adjoint G -structure has the form:
1) $\zeta_b^a = \theta_b^a + B_b^{ac}\omega_c$; 2) $\zeta_{\hat{b}}^a = 0$; 3) $\zeta_{\hat{b}}^{\hat{a}} = -\theta_a^b + B_{ac}^b\omega^c$; 4) $\zeta_b^{\hat{a}} = 0$. (6.12)
In particular, in the case of approximately Kahlerian structures

$$(\zeta_j^j) = \begin{pmatrix} \theta_b^a & 0 \\ 0 & \theta_{\hat{b}}^{\tilde{a}} = -\theta_a^b \end{pmatrix}. \quad (6.13)$$

From these relations it follows that
$\tilde{\nabla}_{\hat{d}}B^{abc} = B^{abcd}$; $\tilde{\nabla}_d B_{abc} = B_{abcd}$; (6.14)
(here and below $\tilde{\nabla}_k t_{i_1\ldots i_r}^{j_1\ldots j_s} = t_{i_1\ldots i_r;k}^{j_1\ldots j_s}$, are the corresponding components of the covariant differential of the tensor t in the first canonical connection). The remaining components of the tensor $\tilde{\nabla}(Alt\,\tilde{C})$on the space of the attached G -structure are equal to zero. It follows from what has been said that the functions $B^{abcd}$and $B_{abcd}$are defined globally on the space of the associated G -structure. Moreover, the differential continuation of relations (6.10) (taking into account the second fundamental identity) leads to the relations

1) $dB^{abcd} + B^{hbcd}\theta_h^a + B^{ahcd}\theta_h^b + B^{abhd}\theta_h^c + B^{abch}\theta_h^d = B^{abcdh}\omega_h$;
2) $dB_{abcd} - B_{hbcd}\theta_a^h - B_{ahcd}\theta_b^h - B_{abhd}\theta_c^h - B_{abch}\theta_d^h = B_{abcdh}\omega^h$

where $\{B^{abcdh}, B_{abcdh}\}$is a globally defined system of functions on the space of the adjoint G -structure that serve as components of the second covariant differential of the tensor $Alt\,\tilde{C}$in the first canonical connection. The nonzero components of the $\Phi = D\zeta = d\zeta + \frac{1}{2}[\zeta,\zeta]$curvature form of the connection $\tilde{\nabla}$have the form $\Phi_b^a = (A_{bc}^{ad} - 2B^{adh}B_{hbc})\omega^c \wedge \omega_d$and f.c.s.
In particular, since the functions $B^{abc}$and B_{abc}on the space of the adjoint G -structure are defined globally, the functions are A_{bc}^{ad}also defined globally.Thus. Proven

Theorem 6.1. *The complete group of structural equations of an approximately Kahlerian structure has the form:*

1) $d\omega^a = -\theta_b^a \wedge \omega^b + B^{abc}\omega_b \wedge \omega_c$;
2) $d\omega_a = \theta_a^b \wedge \omega_b + B_{abc}\omega^b \wedge \omega^c$;
3) $d\theta_b^a = -\theta_c^a \wedge \theta_b^c + (A_{bc}^{ad} - 2B^{adh}B_{hbc})\omega^c \wedge \omega_d$;
4) $dB^{abc} + B^{dbc}\theta_d^a + B^{adc}\theta_d^b + B^{abd}\theta_d^c = B^{abcd}\omega_d$;
5) $dB_{abc} \wedge \omega^b - B_{dbc}\theta_a^d - B_{adc}\theta_b^d - B_{abd}\theta_c^d = B_{abcd}\omega^d$,

where $\{A_{bc}^{ad}\}$is a globally defined system of functions on the space of the adjoint G -structure, which is symmetric in superscripts and subscripts.

Remark 6.2. *Taking into account that $\overline{\omega^a} = \omega_a$; $\overline{\theta_b^a} = -\theta_a^b$; $\overline{B^{abc}} = B_{abc}$, and performing complex conjugation of the relation (6.10:1), taking into account the linear independence of the basic forms of the module $\Lambda_2(M)$, we obtain that*
$\overline{A_{bc}^{ad}} = A_{ad}^{bc}$. (6.17)

7. Fundamental identities of approximately Kahlerian manifolds

Let's differentiate relations (6.10:1) externally:
$-d\theta_c^a \wedge \theta_b^c + \theta_c^a \wedge d\theta_b^c + d(A_{bc}^{ad} - 2B^{adh}B_{hbc}) \wedge \omega^c \wedge \omega_d + (A_{bc}^{ad} - 2B^{adh}B_{hbc})d\omega^c \wedge \omega_d - (A_{bc}^{ad} - 2B^{adh}B_{hbc})\omega^c \wedge d\omega_d = 0.$
We substitute the values from (5.5) and (6.10) into the resulting equality, then we get: $-\{\theta_d^a \wedge \theta_c^d + (A_{cd}^{ah} - 2B^{ahg}B_{gcd})\omega^d \wedge \omega_h\} \wedge \theta_b^c + \theta_c^a \wedge \{-\theta_d^c \wedge \theta_b^d + (A_{bd}^{ch} - 2B^{chg}B_{gbd})\omega^d \wedge \omega_h\}d\theta_b^c +$

$dA^{ad}_{bc} \wedge \omega^c \wedge \omega_d - 2B_{hbc} dB^{adh} \wedge \omega^c \wedge \omega_d + \left(A^{ad}_{bc} - 2B^{adh}B_{hbc}\right)\left(-\theta^c_g \wedge \omega^g + B^{cgf}\omega_g \wedge \omega_f\right) \wedge \omega_d - \left(A^{ad}_{bc} - 2B^{adh}B_{hbc}\right)\omega^c \wedge \left(\theta^g_d \wedge \omega_g + B_{dgf}\omega^g \wedge \omega^f\right) = 0.$

We open the brackets and make a ghost of similar terms, then we get

$$\left(dA^{ad}_{bc} + A^{hd}_{bc}\theta^a_h + A^{ah}_{bc}\theta^d_h - A^{ad}_{hc}\theta^h_b - A^{ad}_{bh}\theta^h_c\right) \wedge \omega^c \wedge \omega_d - \left(A^{af}_{b[c} - 2B^{afg}B_{gb[c}\right)B_{|f|dh]}\omega^c \wedge \omega^d \wedge \omega^h + 2B^{ahg}B_{gb[cd]}\omega^c \wedge \omega^d \wedge \omega_h - 2B^{a[d|g|h]}B_{gbc}\omega^c \wedge \omega_d \wedge \omega_h + \left(A^{a[d}_{bc} - 2B^{a[d|h}B_{hbc}\right)B^{c|gf]}\omega_d \wedge \omega_g \wedge \omega_f = 0. \quad (7.1)$$

We expand the restriction of forms $\Delta A^{ad}_{bc} = dA^{ad}_{bc} + A^{hd}_{bc}\theta^a_h + A^{ah}_{bc}\theta^d_h - A^{ad}_{hc}\theta^h_b - A^{ad}_{bh}\theta^h_c$to W in the standard modulus basis $\Lambda_1(M)$:

$$\Delta A^{ad}_{bc} = dA^{ad}_{bc} + A^{hd}_{bc}\theta^a_h + A^{ah}_{bc}\theta^d_h - A^{ad}_{hc}\theta^h_b - A^{ad}_{bh}\theta^h_c = A^{adh}_{bcg}\theta^g_h + A^{ad}_{bch}\omega^h + A^{adh}_{bc}\omega_h. \quad (7.2)$$

We substitute this relation in (7.1), then we get:

$$A^{adh}_{bcg}\theta^g_h \wedge \omega^c \wedge \omega_d + A^{ad}_{b[ch]}\omega^h \wedge \omega^c \wedge \omega_d + A^{a[dh]}_{bc}\omega_h \wedge \omega^c \wedge \omega_d - \left(A^{af}_{b[c} - 2B^{afg}B_{gb[c}\right)B_{|f|dh]}\omega^c \wedge \omega^d \wedge \omega^h + 2B^{ahg}B_{gb[cd]}\omega^c \wedge \omega^d \wedge \omega_h - 2B^{a[d|g|h]}B_{gbc}\omega^c \wedge \omega_d \wedge \omega_h + \left(A^{a[d}_{bc} - 2B^{a[d|h}B_{hbc}\right)B^{c|gf]}\omega_d \wedge \omega_g \wedge \omega_f = 0.$$

And, taking into account the linear independence of the basic forms, we get:

1) $A^{adh}_{bcg} = 0$; 2) $\left(A^{af}_{b[c} - 2B^{afg}B_{gb[c}\right)B_{|f|dh]} = 0$; 3) $\left(A^{a[d}_{bc} - 2B^{a[d|h}B_{hbc}\right)B^{c|gf]}$; 4) $A^{ad}_{b[ch]} = 2B^{adg}B_{gb[ch]}$; 5) $A^{a[dh]}_{bc} = 2B^{a[d|g|h]}B_{gbc}$. (7.3)

Performing complex conjugation of the equality $dA^{ad}_{bc} + A^{hd}_{bc}\theta^a_h + A^{ah}_{bc}\theta^d_h - A^{ad}_{hc}\theta^h_b - A^{ad}_{bh}\theta^h_c = A^{ad}_{bch}\omega^h + A^{adh}_{bc}\omega_h$, taking into account $\overline{\omega^a} = \omega_a$; $\overline{\theta^a_b} = -\theta^b_a$; $\overline{B^{abc}} = B_{abc}$both (6.17) and the linear independence of the basic forms, we obtain that

$$\overline{A^{adh}_{bc}} = A^{bc}_{adh}. \quad (7.4)$$

By the Main Theorem of tensor analysis and relation (6.13), the identity

$$dA^{ad}_{bc} + A^{hd}_{bc}\theta^a_h + A^{ah}_{bc}\theta^d_h - A^{ad}_{hc}\theta^h_b - A^{ad}_{bh}\theta^h_c = A^{ad}_{bch}\omega^h + A^{adh}_{bc}\omega_h, \quad (7.5)$$

shows that the system of functions $\{A^{ad}_{bc}\}$on the space of the associated G *-structure is* a system of components of some four-valent tensor A on the manifold M. Relation (6.17) shows that this is a real tensor. It is called *the structural tensor of the third kind or the holomorphic sectional*(in short, *HS* -) curvature tensor of an approximate Kähler manifold.

Wherein

$$\widetilde{\nabla}_h A^{ad}_{bc} = A^{ad}_{bch};\ \widetilde{\nabla}_{\hat{h}} A^{ad}_{bc} = A^{adh}_{bc}. \quad (7.6)$$

Due to the oblique symmetry of the system of functions $\{B^{ach}B_{hbdf}\}$with respect to the indices b and d and the symmetry with respect to the indices d and f, we obtain from this that the space of the associated G - structure

$$B^{ach}B_{hbdf} = 0;\ B_{ach}B^{hbdf} = \overline{B^{ach}}\overline{B_{hbdf}} = \overline{B^{ach}B_{hbdf}} = 0. \quad (7.7).$$

This identity is called *the first fundamental identity.*

Identity

$$\left(A^{ad}_{b[c} - 2B^{adh}B_{hb[c}\right)B_{gf]d} = 0, \quad (7.8)$$

is called *the second fundamental identity.*

Let's collapse (7.8) by indices a and b :

$$A^d_c B_{gfd} + A^d_g B_{fcd} + A^d_f B_{cgd} - 2B^d_c B_{gfd} - 2B^d_g B_{fcd} - 2B^d_f B_{cgd} = 0, \quad (7.9)$$

where $A^d_c = A^{hc}_{hc}, B^d_c = B^{ghd}B_{ghc}$. Now we fold (7.8) with respect to the indices a and c and rename $b$to c. Taking into account the symmetry properties of objects A and B, we get:

$$A^d_c B_{gfd} + 2B^d_c B_{gfd} - 2B^d_g B_{fcd} + 2B^d_f B_{cgd} = 0. \quad (7.10)$$

Subtracting (7.10) term by term from (7.9), taking into account the symmetry properties of the object B, we obtain the identity

$$A^d_{[g}B_{f]cd} - 2B^d_c B_{gfd} = 0. \quad (7.11).$$

The identity $A^d_{[g}B_{f]cd} - 2B^d_c B_{gfd} = 0$is called*the third fundamental identity.*

Conclusion

So, the complete group of structural equations of an approximately Kählerian manifold on the space of the associated G - structure has the form:

1) $d\omega^a = -\theta^a_b \wedge \omega^b + B^{abc}\omega_b \wedge \omega_c$;
2) $d\omega_a = \theta^b_a \wedge \omega_b + B_{abc}\omega^b \wedge \omega^c$;

3) $d\theta_b^a = -\theta_c^a \wedge \theta_b^c + \left(A_{bc}^{ad} - 2B^{adh}B_{hbc}\right)\omega^c \wedge \omega_d$;
4) $dB^{abc} + B^{dbc}\theta_d^a + B^{adc}\theta_d^b + B^{abd}\theta_d^c = B^{abcd}\omega_d$;
5) $dB_{abc} \wedge \omega^b - B_{dbc}\theta_a^d - B_{adc}\theta_b^d - B_{abd}\theta_c^d = B_{abcd}\omega^d$.

The fundamental identities of an approximately Kählerian manifold on the space of the associated G - structure have the form:

1) $B^{ach}B_{hbdf} = 0$; $B_{ach}B^{hbdf} = 0$;
2) $\left(A_{b[c}^{ad} - 2B^{adh}B_{hb[c}\right)B_{gf]d} = 0$;
3) $A_{[g}^{d}B_{f]cd} - 2B_c^dB_{gfd} = 0$.

References

[1] Frölicher, A. Zur Differentialgeometrie der komplexen Strukturen. (German) Math. Ann. 129 (1955), 50–95.

[2]Fukami, T., and Ishihara, S. Almost Hermitian structure on S^6. Tohoku Math. J. (2) 7(3) (1955), 151–156.

[3] Tachibana, S. On almost-analytic vectors in certain almost-Hermitian manifolds. Tohoku Math. J. (2) 11(3) (1959), 351–363.

[4] Gray, A. Nearly Kähler manifolds. J. Differential Geometry 4(3) (1970), 283–309.

[5] Gray, A. The structure of nearly Kähler manifolds. Math. Ann. 223(3) (1976), 233–248.

[6] Kiričenko, V. F. The differential geometry of K-spaces. (Russian) Problems in geometry, Vol. 8 (Russian), pp. 139–161, 279. Akad. Nauk SSSR Vsesojuz. Inst. Naučn. i Tehn. Informacii, Moscow, 1977.

[7] Kirichenko, V. F. Generalized quasi-Kaehlerian manifolds and axioms of CR-submanifolds in generalized Hermitian geometry. I. Geom. Dedicata 51(1) (1994), 75–104.

[8] Kirichenko, V. F. Generalized quasi-Kaehlerian manifolds and axioms of CR-submanifolds in generalized Hermitian geometry. II. Geom. Dedicata 52(1) (1994), 53–85.

[9] Takamatsu, K., and Watanabe, Y. Classification of a conformally flat K-space. Tohoku Math. J. (2) 24(3) (1972), 435–440

[10] Vanhecke, Lieven Some theorems for quasi- and nearly Kähler-manifolds. Collection in memory of Enrico Bompiani. Boll. Un. Mat. Ital. (4) 12(3) (1975), suppl., 74–188.

[11] Kobayashi, S. and Nomizu, K. Foundations of differential geometry. Vol. II. Reprint of the 1969 original. Wiley Classics Library. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1996. xvi+468 pp.

[12] Lichnerovich, A. Connection theory in general and holonomy groups. Moscow: Publishing House of Foreign Literature, 1960, 216 p.