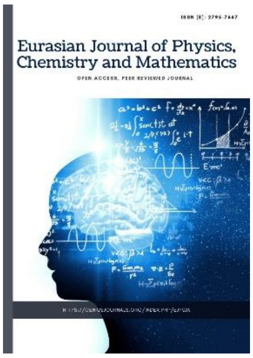

Nieuwenhuistensor of almost complex structure

Ahmed Thamer Abd Ali[1]

[1]Department of Mathematics, Faculty of Sciences, Çankırı Karatekin University, 18100 Çankırı, Turkey e-mail: at3486217@gmail.com

Ufuk Öztürk[2],

[2]Assoc. Prof. UFUK ÖZTÜRK, Department of Mathematics, Faculty of Sciences, Çankırı Karatekin University, 18100 Çankırı, Turkeye-mail: ozturkufuk06@gmail.com, uuzturk@asu.edu, ORCID: 0000-0002-8800-7869

A. A. Shihab[3]

[3]Dr. prof. Ali Abdul Al Majeed. Shihab, e-mail: draliabd@tu.edu.iq , ali.abd82@yahoo.com
Department of Mathematics, College of Education for Pure Sciences, University of Tikrit, Iraq- Salahaddin Tikrit

ABSTRACT

In this work, complex structures are considered, we build two bases: RA -basis and A -basis. We also define an almost complex structure and its adjoint$G$ -structure. We define locally the Nijenhuis tensor of an almost complex structure, prove that this tensor is defined globally, and obtain an explicit expression for this tensor.

Keywords: Adjoint$g$ -structure, complex structure, almost complex structure, Nijenhuis tensor.

1. Introduction

A complex manifold is a manifold with an atlas of charts although an almost complex manifold is a smooth manifold equipped with a smooth linear complex structure on each tangent space([6]). Therefore, every complex manifold is an almost complex manifold, but there are almost complex manifolds that are not complex manifolds. Since aKähler manifoldis a Riemannian manifold, a complex manifold, and a symplectic manifold([1]), a complex manifold and an almost complex manifold are introductory to Kahlerian manifolds. Hence, they are one of the most remarkable mathematical objects studied intensively in differential geometry and algebraic geometry, the theory of Lie groups and homogeneous spaces, topology, the theory of differential operators, and mathematical physics. Moreover, their significance for algebraic geometry became clear after the publication of Hodge's work ([2]), the results of which were subsequently combined. These works determined the direction of research on Kahlerian manifolds for many years. The flow of research into the geometry and topology of Kahlerian manifolds continues unabated even in our time.

Our work is devoted to the study of the Nijenhuis tensor of almost complex manifolds and is structured as follows.

In Section 2, we consider complex structures, the complexification of a linear space, and introduce the concept of linearity extension. We give the definition of a complex structure and build two bases: RA -basis and A-basis. In Section 3, we define an almost complex structure and its adjoint$G$-structure. In Section 4, we define the Nijenhuis tensor of an almost complex structure, prove that this tensor is defined globally, and obtain an explicit expression for this tensor.

2. Complex structures

2.1 Tensor products of modules

Let A and B be modules over a commutative associative ring $\boldsymbol{K}$ with identity. Consider a free abelian group $A \circ B$whose set of generators is the set of all symbols of the form $a \circ b; a \in A, b \in B$. Its elements are all formal (finite) sums of such symbols, i.e. type records $a_1 \circ b_1 + \cdots + a_N \circ b_N; N \in \boldsymbol{N}$. Consider its subgroup $S \subset A \circ B$generated by elements of the form $(a' + a'') \circ b - a' \circ b - a'' \circ b, a \circ (b' + b'') - a \circ b' - a \circ b'', (\alpha a) \circ b - a \circ (\alpha b);\ \alpha \in \boldsymbol{K}$. Let's consider an abelian group $A \otimes B = A \circ B / S$. Its elements are finite formal sums of symbols of the form $a \otimes b = (a \circ b) + S$. It naturally introduces the structure of a $\boldsymbol{K}$ - module with an external composition operation $\alpha(\sum_{i=1}^{n} a_i \otimes b_i) = \sum_{i=1}^{n} (\alpha a_i) \otimes b_i \big(= \sum_{i=1}^{n} b_i \otimes (\alpha a_i)\big)$. This $\boldsymbol{K}$ - module is called *the tensor product*$\boldsymbol{K}$ **-** models A and B ([4]).

Remark 2.1. If A and B have the structure of a module over some ring $\boldsymbol{K}_1$, then $A \otimes B$, obviously, also has a natural $\boldsymbol{K}_1$-module structure. In particular, if A is an algebra, then it $A \otimes B$has the natural structure of an A - module.

2.2 Complexification of linear space

Let, in particular, $A = \boldsymbol{C}$be the field of complex numbers and $B = V$be an $\boldsymbol{R}$ -linear space. Then the $\boldsymbol{C}$ -linear space ($\boldsymbol{C}$-module) $\boldsymbol{C} \otimes V$is denoted $V^{\boldsymbol{C}}$and called *the complexification of the*linear space V. Its elements are records of the form $\sum_{k=1}^{N} z_k \otimes X_k$; $z_k \in \boldsymbol{C}, X_k \in V, N$- an arbitrary natural number.*The sum* of two such elements $\sum_{k=1}^{N_1} z_k \otimes X_k$will $\sum_{p=1}^{N_2} z_p \otimes X_p$be a record of the form $\sum_{k=1}^{N_1} z_k \otimes X_k + \sum_{p=1}^{N_2} z_p \otimes X_p$, and *the product* element $\sum_{k=1}^{N_1} z_k \otimes X_k$to a complex number $z \in \boldsymbol{C}$- a record of the form $\sum_{k=1}^{N_1} (z z_k) \otimes X_k$.

In a $\boldsymbol{C}$ -linear space,$V^{\boldsymbol{C}}$the mapping is canonically defined $\tau: V^{\boldsymbol{C}} \to V^{\boldsymbol{C}}$, acting according to the formula $\tau\big(\sum_{k=1}^{N_1} z_k \otimes X_k\big) = \sum_{k=1}^{N_1} \overline{z_k} \otimes X_k$, where $z \to \bar{z}$is the usual complex conjugation in the field of complex numbers. It is directly verified that τ is an involutive antiautomorphism of a $\boldsymbol{C}$ -linear space $V^{\boldsymbol{C}}$, i.e., bijection with the properties

1) $\tau^2 = id$; 2) $\tau(X + Y) = \tau(X) + \tau(Y)$; 3) $\tau(zX) = \bar{z}\tau(X); z \in \boldsymbol{C}, X, Y \in V$,(2.1)

is called *the complex conjugation operator*.

Note that V naturally admits an embedding j in $V^{\boldsymbol{C}}$by identifying $X \equiv j(X) = 1 \otimes X; X \in V$. At the same time $\tau(X) \equiv \tau(1 \otimes X) = 1 \otimes X \equiv X$. Moreover, $\tau\big(\sum_{k=1}^{N_1} z_k \otimes X_k\big) = \sum_{k=1}^{N_1} z_k \otimes X_k \Leftrightarrow z_k = \overline{z_k} (k = 1, \dots, N) \Leftrightarrow z_k = x_k \in \boldsymbol{R}$, which means $\sum_{k=1}^{N_1} z_k \otimes X_k = \sum_{k=1}^{N_1} x_k (1 \otimes X_k) \equiv \sum_{k=1}^{N_1} x_k X_k \in V$. This proves

Proposition 2.1. Let $Y = \sum_{k=1}^{N_1} z_k \otimes X_k \in V^{\boldsymbol{C}}$. Then, taking into account the accepted identification, $Y \in V \Leftrightarrow \tau(Y) = Y$.

Note that if V is an n-dimensional $\boldsymbol{R}$ -linear space, then $V^{\boldsymbol{C}}$is an n-dimensional $\boldsymbol{C}$ -linear space. Moreover, if $b = \{e_1, \dots, e_n\}$is a basis of $\boldsymbol{R}$ -linear space V, then under the above canonical identification of elements $e_k \in V$with elements, the $1 \otimes e_k \in V^{\boldsymbol{C}}$set b is also a basis of $\boldsymbol{C}$ -linear space $V^{\boldsymbol{C}}$. This easily follows from a more general fact of independent interest ([4, p. 171]):

Proposition 2.2. Let be $\{e_1, \dots, e_n\}$a basis of a real linear space V,$\{\varepsilon_1, \dots, \varepsilon_n\}$be a basis of an $\boldsymbol{R}$ -linear space$W$. Then $\{e_i \otimes \varepsilon_a | i = 1, \dots, n; a = 1, \dots, m\}$is a basis of the $\boldsymbol{R}$ -linear space $V \otimes W$.

Indeed, from this Proposition, it follows that the elements $\{1 \otimes e_k, \sqrt{-1} \otimes e_k | k = 1, \dots, n\}$form a basis of an $\boldsymbol{R}$ -linear space $V^{\boldsymbol{C}}$, from which it already easily follows that the first n elements of this are basis form a basis $V^{\boldsymbol{C}}$as a $\boldsymbol{C}$ -linear space.

Any operator $f: V \to V$canonically defines a $\boldsymbol{C}$ -linear mapping $f^{\boldsymbol{C}} = id \otimes f: V^{\boldsymbol{C}} \to V^{\boldsymbol{C}}$by the formula $f^{\boldsymbol{C}}(\sum_{k=1}^{N} z_k \otimes X_k) = \sum_{k=1}^{N} z_k \otimes f(X_k)$. Obviously, taking into account the indicated identification, $f^{\boldsymbol{C}}\big|_N = f$, because of which the mapping $f^{\boldsymbol{C}}$is called *the extension in the linearity*of the operator *f*.

Proposition 2.3. A$\boldsymbol{C}$ -linear operator $F: V^{\boldsymbol{C}} \to V^{\boldsymbol{C}}$is a linear extension of some $\boldsymbol{R}$ -linear operator $f: V \to V$if and only if $\tau \circ F = F \circ \tau$.

Proof. Indeed, if $F = f^{\boldsymbol{C}}$, then $\tau \circ F(\sum_{k=1}^{N} z_k \otimes X_k) = \sum_{k=1}^{N} \overline{z_k} \otimes f(X_k) = F \circ \tau(\sum_{k=1}^{N} z_k \otimes X_k)$, whence it follows that $\tau \circ F = F \circ \tau$. Conversely, if this relation holds, then $\tau \circ F(1 \otimes X) = F \circ \tau(1 \otimes X) = F(1 \otimes X), X \in V$, and, by Proposition 2.1, the restriction $f = F|_V$of the operator F to V *is* defined by theformula $1 \otimes f(X) = F(1 \otimes X)$. Obviously, in this case,$F(\sum_{k=1}^{N} z_k \otimes X_k) = \sum_{k=1}^{N} F(z_k \otimes X_k) = \sum_{k=1}^{N} z_k F(1 \otimes X_k) = \sum_{k=1}^{N} z_k\big(1 \otimes f(X_k)\big) = \sum_{k=1}^{N} z_k \otimes f(X_k) = f^{\boldsymbol{C}}(\sum_{k=1}^{N} z_k \otimes X_k)$, and hence $F = f^{\boldsymbol{C}}$. □

The following assertion is proved in the same way:

Proposition 2.4. An r -ary$\boldsymbol{C}$ -linear mapping $T: V^{\boldsymbol{C}} \times \ldots \times V^{\boldsymbol{C}} \to V^{\boldsymbol{C}}$is a linear extension of an r -ary$\boldsymbol{R}$-linear mapping $T: V \times \ldots \times V \to V$if and only if $\tau \circ T(X_1, \ldots, X_r) = T(\tau X_1, \ldots, \tau X_r)$; $X_1, \ldots, X_r \in V^{\boldsymbol{C}}$.

One can give another definition of complexification that is equivalent to the above one. Let Vbe an R -linear space. Let us introduce the following operations in the set:$V \times V$

1) ***Addition.*** If $X_1 = (A_1, B_1), X_2 = (A_2, B_2)$are elements from $V \times V$, then the pair $(A_1 + A_2, B_1 + B_2)$is called their *sum* and denoted by $X_1 + X_2$.

2) ***Multiplication by a complex number.*** If $X = (A, B) \in V \times V, z = \alpha + \sqrt{-1}\beta \in \boldsymbol{C}$, then put $zX = (\alpha a - \beta b, \alpha b + \beta a)$.

Let's call the element$X$ the *product of* a complex number z and an element $X$is directly verified that this $V \times V$introduces in the set the structure of a $\boldsymbol{C}$ -linear space $\tilde{V}^{\boldsymbol{C}}$, which is naturally isomorphic to the $\boldsymbol{C}$ -linear space$V^{\boldsymbol{C}}$. The natural isomorphism $\varphi: V^{\boldsymbol{C}} \to \tilde{V}^{\boldsymbol{C}}$associates an element with $\sum_{k=1}^{N}\big(\alpha_k + \sqrt{-1}\beta_k\big)X_k \in V^{\boldsymbol{C}}$a pair $(A, B) \in \tilde{V}^{\boldsymbol{C}}$, where $A = \sum_{k=1}^{N} \alpha_k X_k, B = \sum_{k=1}^{N} \beta_k X_k$. Under this isomorphism, the embedding described above $j: V \subset V^{\boldsymbol{C}}$corresponds to the embedding $\tilde{j}: V \subset \tilde{V}^{\boldsymbol{C}}$defined by the formula $\tilde{j}(X) = (X, 0); X \in V$, the τcomplex conjugation operator corresponds to the operator $\tilde{\tau}: \tilde{V}^{\boldsymbol{C}} \to \tilde{V}^{\boldsymbol{C}}$defined by the formula $\tilde{\tau}(X, Y) = (X, -Y); X, Y \in V$, and the $\boldsymbol{C}$ -linear operator corresponds to $f^{\boldsymbol{C}} = id \otimes f$the ***C -linear*** operator $\tilde{j}^{\boldsymbol{C}}: \tilde{V}^{\boldsymbol{C}} \subset \tilde{V}^{\boldsymbol{C}}$defined by the formula $\tilde{f}^{\boldsymbol{C}}(X, Y) = (fX, fY); X, Y \in V$.

2.3. Complex structures

Let V be a complex linear space. It can, in particular, be viewed as a real linear space $V^{\boldsymbol{R}}$(called the *reification* of the space V) in which an $\boldsymbol{R}$ -linear endomorphism $J_0: V^{\boldsymbol{R}} \to V^{\boldsymbol{R}}$is given, defined by $J_0(X) = \sqrt{-1}X; X \in V^{\boldsymbol{R}}$. This endomorphism allows us to completely restore the structure of a complex linear space to *V*. Namely, if $z = \alpha + \sqrt{-1}\beta \in \boldsymbol{C}, X \in V$, then $zX = \alpha X + \beta J_0(X)$. Moreover, it is obvious that the endomorphism $J_0$is anti-involutive, i.e. $J_0^2 = -id$.

Definition 2.1. A *complex structure*in V is an endomorphism $J: V \to V$such that $J^2 = -id$. In other words, a complex structure is an anti-involutive automorphism of a real linear space.

Fixing a complex structure in V canonically determines in V the structure of a complex linear space (that is, a $\boldsymbol{C}$ - module). Indeed, if $X \in V, z = \alpha + \sqrt{-1}\beta \in \boldsymbol{C}$, then we set

$$zX = \alpha X + \beta(JX). \quad (2.2)$$

It is directly verified that in this case all 8 axioms of a $\boldsymbol{C}$ -linear space are satisfied, which we will denote by the same symbol V. Obviously, Vas an $\boldsymbol{R}$ -linear space is its reification $V^{\boldsymbol{R}}$.

Let the dimension $dim_{\boldsymbol{C}}V$of the linear space Vas a complex space be equal to n, and let

$\{e_1, \dots, e_n\}$be the basis of this space. Let $X \in V$. Then $X = z^k e_k$, where $z^k = \alpha^k + \sqrt{-1}\beta^k \in \boldsymbol{C}; k = 1, \dots, n$. Taking into account (1.2), $X = \alpha^k e_k + \beta^k (Je_k)$, i.e., every vector $X \in V^{\boldsymbol{R}}$is represented as a linear combination of vectors $e_1, \dots, e_n, Je_1, \dots, Je_n$. On the other hand, let $\lambda^k e_k + \mu^k Je_k = 0, \lambda^k, \mu^k \in \boldsymbol{R}$.Then, due to (1.2), $(\lambda^k + \sqrt{-1}\mu^k)e_k = 0$, and due to the $\boldsymbol{C}$ -linear independence of the vectors $e_1, \dots, e_n, \lambda^k + \sqrt{-1}\mu^k = 0$, and hence $\lambda^k = \mu^k = 0; k = 1, \dots, n$. Therefore, the vectors $\{e_1, \dots, e_n, Je_1, \dots, Je_n\}$form a basis for the space V as an $\boldsymbol{R}$ -linear space (i.e. a basis for the space $V^{\boldsymbol{R}}$). Such a basis is called *a real-adapted complex structure*, in short, an *RA - basis*.

Remark 2.2. Obviously, any complex structure is defined on the *RA* basis by a matrix of the form

$$(J_j^i) = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}. \qquad (2.3)$$

As a simple but important corollary, we get the following

Proposition 2.5. A finite-dimensional real linear space admits a complex structure if and only if it is even-dimensional.

Proof. Since the *RA* -basis contains an even number of vectors, a space that admits a complex structure is necessarily even-dimensional. Conversely, let V be a $2n$-dimensional real linear space. We fix an arbitrary basis in it $\{e_1, \dots, e_{2n}\}$. Then the endomorphism of the Jspace V given by the matrix (2.3) on this basis is obviously a complex structure.

□

Let Jbe a complex structure in an $\boldsymbol{R}$ -linear space V . Consider the endomorphism $\sigma: V^{\boldsymbol{C}} \to V^{\boldsymbol{C}}$defined by the formula $\sigma = \frac{1}{2}(id - \sqrt{-1}J^{\boldsymbol{C}})$. Obviously, $\sigma^2 = \sigma$, i.e. σ- projector. The projector complementary to it $\bar{\sigma}$is determined by the formula $\bar{\sigma} = \frac{1}{2}(id + \sqrt{-1}J^{\boldsymbol{C}})$. In the future, allowing freedom of speech, $J^{\boldsymbol{C}}$we will simply denote endomorphism J. Note that $J \circ \sigma = \frac{1}{2}(J + \sqrt{-1}id) = \frac{\sqrt{-1}}{2}(id - \sqrt{-1}J) = \sqrt{-1}\sigma$, which means $Im\, \sigma \subset D_J^{\sqrt{-1}}$. (Here and in what follows, the symbol D_F^{λ}denotes the proper subspace of the endomorphism F corresponding to the eigenvalue λ).

Conversely, if $X \in D_J^{\sqrt{-1}}$, then $\sigma X = \frac{1}{2}(X - \sqrt{-1}JX) = \frac{1}{2}(2X) = X$, in particular, $X \in Im\, \sigma$. Thus, $Im\, \sigma = D_J^{\sqrt{-1}}$. Likewise, $Im\, \bar{\sigma} = D_J^{-\sqrt{-1}}$. Since $\mathcal{X}^{\boldsymbol{C}}(M) = D_J^{\sqrt{-1}} \oplus D_J^{-\sqrt{-1}}$we get:

Theorem 2.1.$V^{\boldsymbol{C}}$linear spacedecomposes into a direct sum of eigenspaces of the endomorphism Jcorresponding to the eigenvalues $\sqrt{-1}$and $-\sqrt{-1}$, i.e., $V^{\boldsymbol{C}} = D_J^{\sqrt{-1}} \oplus D_J^{-\sqrt{-1}}$, and the endomorphisms $\sigma$and $\bar{\sigma}$are projections onto the subspaces $D_J^{\sqrt{-1}}$and $D_J^{-\sqrt{-1}}$, respectively.

Moreover, fair

Theorem 2.2. Defining a complex structure on an $\boldsymbol{R}$ -linear space V is equivalent to splitting $V^{\boldsymbol{C}}$into a direct sum of two complex conjugate subspaces serving as proper subspaces of this complex structure.

Proof. Necessity follows from Theorem 2.1. Let now $V^{\boldsymbol{C}} = D \oplus \tau D$. Then $\forall X \in V^{\boldsymbol{C}} \Longrightarrow X = X_1 + X_2$; $X_1 \in D, X_2 \in \tau D$. We construct an endomorphism $\mathcal{J}: V^{\boldsymbol{C}} \longrightarrow V^{\boldsymbol{C}}$by setting $\mathcal{J}(X) = \sqrt{-1}(X_1 - X_2)$. Obviously, $\tau(X) = \tau(X_1) + \tau(X_2)$, and $\tau(X_1) \in \tau D, \tau(X_2) \in D$. Therefore $(\mathcal{J} \circ \tau)(X) = \sqrt{-1}(\tau(X_2) - \tau(X_1))$. On the other hand, due to the antilinearity of the operator τ, $(\tau \circ \mathcal{J})(X) = -\sqrt{-1}(\tau(X_1) - \tau(X_2)) = \sqrt{-1}(\tau X_2 - \tau X_1)$. Thus, $\mathcal{J} \circ \tau = \tau \circ \mathcal{J}$. By Proposition 2.3 $\mathcal{J} = J^{\boldsymbol{C}}$, for some R -linear endomorphism $J: V \to V$. Obviously, $\mathcal{J}^2 = -id$, in particular, $J^2 = -id$, i.e. $J$is the complex structure on V. If $X \in D$, then $X = X_1$, which means $\mathcal{J}(X) = \sqrt{-1}X_1 = \sqrt{-1}X$. Therefore, $D \subset D_J^{\sqrt{-1}}$. Conversely, if $X \in D_J^{\sqrt{-1}}$, then $\sqrt{-1}(X_1 - X_2) = \mathcal{J}(X) = \sqrt{-1}X = \sqrt{-1}(X_1 + X_2)$, whence $X_2 = 0$, and hence $X \in D$. Therefore, $D_J^{\sqrt{-1}} \subset D$, i.e. $D_J^{\sqrt{-1}} = D$. Likewise, $D_J^{-\sqrt{-1}} = \tau D$.

□

Lemma 2.1. In the introduced notation, 1) $\tau \circ \sigma = \bar{\sigma} \circ \tau$, 2) $\tau \circ \bar{\sigma} = \sigma \circ \tau$.

Proof. Taking into account the antilinearity of the mapping $\tau$and using Proposition 2.3, we have: $\tau \circ \sigma(X) = \frac{1}{2}\tau\big(X - \sqrt{-1}JX\big) = \frac{1}{2}\big(\tau X + \sqrt{-1}\tau \circ JX\big) = \frac{1}{2}\big(\tau X + \sqrt{-1}J \circ \tau X\big) = \sigma \circ \tau(X); X \in V^{\boldsymbol{C}}$.
The second relation is proved similarly.

□

Theorem 2.3. The mappings $\sigma|_V: V \to D_J^{\sqrt{-1}}$and $\bar{\sigma}|_V: V \to D_J^{-\sqrt{-1}}$are, respectively, an isomorphism and an anti-isomorphism of $\boldsymbol{C}$ -linear spaces.

Proof. The additivity of the mappings $\sigma|_V$and $\bar{\sigma}|_V$is obvious. Let now $z = \alpha + \sqrt{-1}\beta \in \boldsymbol{C}, X \in V$. As already seen, $\sigma \circ J = J \circ \sigma = \sqrt{-1}\sigma, \bar{\sigma} \circ J = J \circ \bar{\sigma} = -\sqrt{-1}\sigma$. Therefore $\sigma(zX) = \sigma(\alpha X + \beta JX) = \alpha\sigma X + \beta\sqrt{-1}\sigma X = z(\sigma X)$. Similarly, $\bar{\sigma}(zX) = \bar{z}(\bar{\sigma}X)$, and thus the maps $\sigma|_V$and $\bar{\sigma}|_V$are, respectively, a homomorphism and an antihomomorphism of $\boldsymbol{C}$ -linear spaces.

Let $\exists X \in V$and $\sigma X = 0$. Applying the operator to both parts of this identity τ, taking into account Lemma 2.1 and Proposition 2.1, we obtain that $\bar{\sigma}X = 0$, and hence $X = \sigma X + \bar{\sigma}X = 0$. Therefore, $\ker\sigma|_V = \{0\}$. Similarly, $\ker\bar{\sigma}|_V = \{0\}$, i.e. $\sigma$and $\bar{\sigma}$are monomorphism and antimonomorphism, respectively.

Let, finally $X \in D_J^{\sqrt{-1}}$. Consider the vector $Y = X + \tau X$. By Proposition 2.1, $Y \in V$. On the other hand, since, $X \in Im\,\sigma = \ker\bar{\sigma}$taking into account Lemma 2.1, we have: $\sigma Y = \sigma X + (\tau \circ \sigma)X = X + (\tau \circ \bar{\sigma})X = X$. Similarly, if $X \in D_J^{-\sqrt{-1}}$, then $\bar{\sigma}Y = X$, and, thus, $\sigma|_V$and $\bar{\sigma}|_V$are an epimorphism and an anti-epimorphism, respectively.

□

Let, in particular, V be a real $\boldsymbol{R}$ -linear space, $\dim V = 2n$, and let $b = \{e_1, \dots, e_{2n}\}$be its basis as a $\boldsymbol{C}$ -module. Consider a system of vectors $b_A = \{\varepsilon_1, \dots, \varepsilon_n, \varepsilon_{\hat{1}}, \dots, \varepsilon_{\hat{n}}\}$, where $\varepsilon_a = \sigma(e_a), \varepsilon_{\hat{a}} = \bar{\sigma}(e_a); a = 1, \dots, n$. By Theorem 2.3, the vectors $\{\varepsilon_1, \dots, \varepsilon_n\}$form the basis of the $\boldsymbol{C}$ -linear space $D_J^{\sqrt{-1}}$, and the vectors $\{\varepsilon_{\hat{1}}, \dots, \varepsilon_{\hat{n}}\}$form the basis of the $\boldsymbol{C}$ -linear space $D_J^{-\sqrt{-1}}$, and, by Lemma 2.1 and Proposition 2.1, $\tau\varepsilon_a = (\tau \circ \sigma)e_a = (\bar{\sigma} \circ \tau)e_a = \bar{\sigma}e_a = \varepsilon_{\hat{a}}$.
Moreover, by Theorem 2.1, the system of vectors $b_A$forms a basis of the space $V^{\boldsymbol{C}}$, which is characterized by the fact that the endomorphism matrix Jin this basis has the form

$$(J_j^i) = \begin{pmatrix} \sqrt{-1}I_n & 0 \\ 0 & -\sqrt{-1}I_n \end{pmatrix}, \quad (2.4)$$

Such a basis is called *an adapted complex structure*, in short ***A-basis***.

Conclusion. Fixing a complex structure Jin a $2n$-dimensional real linear space $V^{\boldsymbol{R}}$induces the assignment of $V^{\boldsymbol{R}}$an n-dimensional complex linear space V *to the structures*. Each basis $b = \{e_1, \dots, e_{2n}\}$of the space V canonically induces two bases:
1) *RA is the* basis $b_{RA} = \{e_1, \dots, e_n, Je_1, \dots, Je_n\}$of the space $V^{\boldsymbol{R}}$. Of course, taking into account the canonical identification, $X \equiv 1 \otimes X$this basis can also be considered as the basis of the $\boldsymbol{C}$ -linear space $(V^{\boldsymbol{R}})^{\boldsymbol{C}}$.
2) A -basis$b_A = \{\varepsilon_1, \dots, \varepsilon_n, \varepsilon_{\hat{1}}, \dots, \varepsilon_{\hat{n}}\}\boldsymbol{C}$ -linear space $(V^{\boldsymbol{R}})^{\boldsymbol{C}}$.

Now let $b = \{e_1, \dots, e_n\}$and $\tilde{b} = \{\tilde{e}_1, \dots, \hat{e}_n\}$be two bases of the space V,$C = C_{b\tilde{b}} = (c_b^a)$be the transition matrix from basis $b$to basis $\tilde{b}$, $C = A + \sqrt{-1}B$, where $A = (\alpha_b^a)$and $B = (\beta_b^a)$are the real and imaginary parts of matrix C, respectively. Since, taking into account (1.2), $\tilde{e}_a = c_a^b e_b = \alpha_a^b e_b + \beta_a^b(Je_b)$, we have: $J\tilde{e}_a = \alpha_a^b Je_b + \beta_a^b J^2 e_b = -\beta_a^b e_b + \alpha_a^b Je_b$, which means that

$$C_{b_{RA}\tilde{b}_{RA}} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}. \quad (2.5)$$

Next, $\tilde{\varepsilon}_a = \sigma(\tilde{e}_a) = \sigma(c_a^b e_b) = c_a^b \sigma(e_b) = c_a^b \varepsilon_b$; $\tilde{\varepsilon}_{\hat{a}} = \bar{\sigma}(\tilde{e}_a) = \bar{\sigma}(c_a^b e_b) = \bar{c}_a^b \bar{\sigma}(e_b) = \bar{c}_a^b \varepsilon_{\hat{b}}$, which means

$$C_{b_a\tilde{b}_A} = \begin{pmatrix} C & 0 \\ 0 & \bar{C} \end{pmatrix}. \quad (2.6)$$

Obviously, both matrices, $C_{b_{RA}\tilde{b}_{RA}}$and $C_{b_a\tilde{b}_A}$, can be treated as matrices of the same linear space endomorphism, $(V^{\boldsymbol{R}})^{\boldsymbol{C}}$namely, the endomorphism $f^{\boldsymbol{C}}$, where $f$is the space endomorphism $V^{\boldsymbol{R}}$, considered as an endomorphism of the space V, transforming the basis b into the basis $\tilde{b}$. In particular,

$$\det\begin{pmatrix} A & -B \\ B & A \end{pmatrix} = \det\begin{pmatrix} C & 0 \\ 0 & \bar{C} \end{pmatrix} = |\det C|^2. \quad (2.7)$$

An important consequence of this relation is

Proposition 2.6. Fixing a complex structure in an $\boldsymbol{R}$ -linear space $V^{\boldsymbol{R}}$canonically determines the orientation of this space. It consists of bases oriented in the same way as any RA -basis. □

3. Almost complex structures and the associated $\boldsymbol{G}$ - structure

Definition 3.1. An *almost complex structure*on a manifold M is an anti-involutive endomorphism of a module $\mathcal{X}(M)$, i.e. $C^\infty(M)$-linear mapping $J: \mathcal{X}(M) \to \mathcal{X}(M)$such that $J^2 = -id$. An endomorphism $J$is also called a *structural endomorphism.* A manifold on which an almost complex structure is fixed is called an almost complex manifold. A diffeomorphism of an $f: M_1 \to M_2$almost complex manifold (M_1, J_1)onto an almost complex manifold $(M_2, J_2)$is called a holomorphic diffeomorphism if $f_* \circ J_1 = J_2 \circ f_*$.

It is obvious that an almost complex structure can be considered as a complex structure of the module $\mathcal{X}(M)$, considered as an $\boldsymbol{R}$ -linear space. As we have seen, the structure of a ***C* -linear** space is naturally induced on this linear space, and hence the structure of a $\boldsymbol{C} \otimes C^\infty(M)$-module, i.e., module over the ring of smooth complex-valued functions on the manifold M . The smoothness of such a function is understood as the smoothness of its real and imaginary parts. For a better understanding of this structure, it is convenient to use the alternative definition $\boldsymbol{C} \otimes C^\infty(M)$as a complexification of an $\boldsymbol{R}$ -linear space $C^\infty(M)$, according to $\boldsymbol{C} \otimes C^\infty(M) = C^\infty(M) \times C^\infty(M)$. If $(f, g) = f + \sqrt{-1}g \in \boldsymbol{C} \otimes C^\infty(M), X \in \mathcal{X}(M)$, then by definition $\left(f + \sqrt{-1}g\right)X = fX + g(JX)$.

Let be Jan almost complex structure on the manifold M . It induces complex structures $J_m: T_m(M) \to T_m(M)$at every point $m \in M$. In view of Proposition 2.5, the space $T_m(M)$, and hence the manifold M *itself*, are even-dimensional. Let $\dim M = 2n$. The number n is called *the complex dimension* ofthe manifold *M.*

Theorem 3.1. Specifying an almost complex structure on a smooth manifold $M^{2n}$is equivalent to specifying a G *-structure on this manifold* with the structure group $G = GL^R(n, \boldsymbol{C})$.

Proof. Let be Jan almost complex structure on the manifold M. Then, at each point $m \in M$, a family of $\mathcal{R}_m$frames of the space is defined $T_m(M)$, which is considered as an n-dimensional $\boldsymbol{C}$ -linear space. It follows from the definition of a frame that a group $GL(n, \boldsymbol{C})$acts in each such family freely and transitively.

Lemma 3.1. In some neighborhood U of an arbitrary point $m \in M$, one can construct a family of vector fields $\{e_1^0, \dots, e_n^0\}$on U that form the basis of a module $\mathcal{X}(U)$as a $\boldsymbol{C} \otimes C^\infty(U)$-module.

Proof. We fixsome RA - frame $p = \{\xi_1, \dots, \xi_n, J_m\xi_1, \dots, J_m\xi_n\}$at the point m. As we know, a system of vectors $\xi_k$can be extended to a system of vector fields $e_k^0 (k = 1, \dots, n)$on M. In this case, the system of vectors $J_m\xi_k$will continue to the system of vector fields Je_k^0. Since the linear independence of the vectors of the frame p is equivalent to the inequality zero of the determinant of the transition matrix from the natural basis at the point m to the basic part of the frame p, this property is preserved in some neighborhood U of the point m and for some vector fields $\{e_1^0, \dots, e_n^0, Je_1^0, \dots, Je_n^0\}$. But then, obviously, the system $\{e_1^0, \dots, e_n^0\}$of vector fields on U will be $\boldsymbol{C} \otimes C^\infty(U)$-linearly independent, and hence forms a basis of the $\boldsymbol{C} \otimes C^\infty(U)$-module $\mathcal{X}(U)$. □

We continue the proof of Theorem 3.1. Let us denote $\mathcal{R} = \bigcup_{m \in M} \mathcal{R}_m$, and introduce the natural projection $\pi: \mathcal{R} \to M$that assigns the vertex to the frame $p \in \mathcal{R}$. Now we can construct the mapping $F_U: \pi^{-1}(U) \to GL(n, \boldsymbol{C})$by setting $F_U(p) = g$, where $g$is the transition matrix from the frame $(m, e_1^0|_m, \dots, e_n^0|_m)$to the frame p. It is easy to verify that the quadruple $B_J(M) = (\mathcal{R}, M, \pi, G = GL(n, \boldsymbol{C}))$forms a principal bundle. This principal bundle can be considered as a G - structure with respect to the monomorphism$(\tilde{f}, \tilde{\rho})$of the principal bundle $B_J(M)$into the principal bundle $B(M)$, where $\tilde{f}: \mathcal{R} \to BM$is the map that associates the $(m, e_1, \dots, e_n)$space frame $T_m(M)$as a $\boldsymbol{C}$ - module with the corresponding RA -frame, and $\tilde{\rho}: GL(n, \boldsymbol{C}) \to GL(2n, \boldsymbol{R})$is the canonical Lie group monomorphism that associates the matrix with $C = A + \sqrt{-1}B \in GL(n, \boldsymbol{C})$the matrix $\tilde{\rho}(C) = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$whose image is the Lie group $GL^{\boldsymbol{R}}(n, \boldsymbol{C})$.

Conversely, let be $(\mathcal{R}, M, \pi, GL^R(n, \boldsymbol{C}))$a G -structure of this type on M . Let be J_0a standard complex structure in the space $\boldsymbol{R}^{2n}$given by a matrix of the form (2.3). Let be $m \in M$an arbitrary point. We define an endomorphism $J_m$in space by the $T_m(M)$formula $J_m = p \circ J_0 \circ p^{-1}; p \in \pi^{-1}(m)$. Obviously, $J_m^2 = -id$, i.e. $J_m$is the complex structure on $T_m(M)$. Let us show that it is well defined in the sense of being independent of the choice of the element $p \in \pi^{-1}(m)$. Indeed, if $\tilde{p} \in \pi^{-1}(m)$is another such element, then $\exists h \in GL^R(n, \boldsymbol{C})$and $\tilde{p} = ph$. Therefore, $\tilde{p} \circ J_0 \circ \tilde{p}^{-1} = (ph) \circ J_0 \circ (ph)^{-1} = p \circ (h \circ J_0 \circ h^{-1}) \circ p^{-1} = p \circ J_0 \circ p^{-1} = J_m$since the group $GL^R(n, \boldsymbol{C})$is obviously an endomorphism invariance group J_0, i.e., $hJ_0 = J_0 h; h \in GL^{\boldsymbol{R}}(n, \boldsymbol{C})$, which is checked directly.

Let us show that the family of tensors $J = \{J_m | m \in M\}$defines a tensor field on the manifold M . To do this, it suffices to prove that for any admissible map (U, φ)on M the functions $m \to J_j^i(m) = dx^i\left(J_m\left(\frac{\partial}{\partial x^j}\Big|_m\right)\right), m \in M,$ are smooth on U . Let us fix a local section $s: U \to \mathcal{R}$of the space of the G -structure. Then, by construction, in the RA -frame $\sigma(m)$(and dual-coreframe) we have: $\left(e^i\left(J_m(e_j)\right)\right) = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} = \left((J_0)_j^i\right)$. The smoothness of the section is expressed in the fact that the components of the matrix C of the transition from the natural basis of the module $\mathcal{X}(U)$to the RA -basis $\sigma(U)$of this module, and hence the components of the inverse matrix $\tilde{C}$, are smooth functions. Hence,$m \to J_j^i(m) = dx^i\left(J_m\left(\frac{\partial}{\partial x^j}\Big|_m\right)\right) =$ $= C_k^i(m) e^k\left(J_m\left(\tilde{C}_j^r(m) e_r\right)\right) = C_k^i(m)\tilde{C}_j^r(m) e^k\left(J_m(e_r)\right) = C_k^i(m)\tilde{C}_j^r(m)(J_0)_r^k$are smooth functions on U . Thus, $J$is an almost complex structure. Obviously, the family of RA -frames generated by it coincides with the space of the G -structure.

□

Theorem 3.2. Every almost complex manifold is even-dimensional and orientable.

Proof. The even-dimensionality of an almost complex manifold (M^{2n}, J)follows from Proposition 2.5 applied to any linear space of the form $T_m(M); m \in M$. The orientability of this manifold, by definition, means the existence on it of a differential -form τthat does not vanish anywhere $2n$. But its existence is obvious for any neighborhood of local triviality of the bundle $B_J(M)$. Indeed, if U is such a neighborhood, $s: U \to \mathcal{R}$is a section of this bundle over it, given by vector fields $\{e_1, \dots, e_n, Je_1, \dots, Je_n\}$, then, by Proposition 2.6, it suffices to set $\tau_U = e_1 \wedge \dots \wedge e_n \wedge J^* e_1 \wedge \dots \wedge J^* e_n$, where $J^*(u)(X) = u(JX); X \in \mathcal{X}(U), u \in \mathcal{X}^*(U)$. Let now $\{\psi_\alpha\}_{\alpha \in A}$be a partition of unity subject to the covering of the $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$manifold M *by* the local triviality domains of the bundle $B_J(M)$. Since the manifold is paracompact, this cover can be considered locally finite without loss of generality. Let be $\tau_\alpha$the $2n$-form constructed for the domain U_α; $\alpha \in A$. Then $\tau = \sum_{\alpha \in A} \psi_\alpha \tau_\alpha$is a well-defined $2n$-form on M . Indeed, due to the local finiteness of the cover $\mathfrak{U}$in some neighborhood U of each point, the $m \in M$form $\tau|_U$is the sum of at most a finite number of smooth forms $(\tau_\alpha)|_U$, and hence is τa nowhere vanishing $2n$form on *M.*

Remark 3.1. The even-dimensionality and orientability of a manifold are thus necessary conditions for the existence of an almost complex structure on this manifold. However, these conditions are not sufficient. For example, a well-known deep result of a topological nature is the assertion that an $2n$-dimensional sphere S^{2n}admits an almost complex structure if and only if $n = 1$either $n = 3$(see [5]). Therefore, for example, a 4-dimensional sphere, being, as is well known, an even-dimensional orientable manifold, does not admit an almost complex structure. The question of finding necessary and sufficient conditions for the existence of an almost complex structure on a smooth manifold is still open.

Remark 3.2. Along with the principal bundle of VM frames over a smooth manifold M^n, we can consider a more extensive principal bundle of complex frames over M, which we denote $B^{\boldsymbol{C}}(M) = \left(B^{\boldsymbol{C}}M, M, \pi, GL(n, \boldsymbol{C})\right)$by, where $B^{\boldsymbol{C}}M$is the union of all frames of the spaces $\left(T_m(M)\right)^{\boldsymbol{C}}; m \in M$. The corresponding justifications do not differ in any way from the corresponding justifications for the main bundle of the WM. This principal bundle plays a particularly important role for almost complex manifolds (M^{2n}, J), since it allows, along with the G -structure constructed above, ***to*** *consider* another $B_J(M)$defining G -structure $(m, e_1, \dots, e_n)$defined by the monomorphism $(f, \rho)$of $T_m(M)$the principal bundle $B_J(M)$into the principal bundle $B^{\boldsymbol{C}}M$, where $f: \mathcal{R} \to B^{\boldsymbol{C}}MA$ is a frame and $\rho: GL(n, \boldsymbol{C}) \to GL(2n, \boldsymbol{C})$is a canonical monomorphism of Lie groups that associates a matrix with $C \in GL(n, \boldsymbol{C})$a matrix $\rho(C) = \begin{pmatrix} C & 0 \\ 0 & \bar{C} \end{pmatrix} \in GL(2n, \boldsymbol{C})$. As above, it is proved that specifying such a G -structure is equivalent to specifying the original almost complex structure. This G-structure will be especially important for our subsequent considerations; we call it a G -structure attached to an almost complex structure.

4. Nijenhuis tensor

4.1 Local definition

Let be (M, J)an almost complex manifold, $dim\, M = 2n$. Let us agree that in what follows unless otherwise stated, the indices $i, j, k, \dots$range over the values from 1 to $2n$, the indices $a, b, c. d, \dots$range over the values from 1 to n. and denote $\hat{a} = a + n$. Let be (U, φ)a local chart on the manifold M . According to the Main Theorem of tensor analysis, the assignment of an almost complex structure Jon the manifold M induces the assignment on the total space BM of a bundle of frames over M of a system of functions $\{J^i_j\}$satisfying in a coordinate neighborhood $W = \pi^{-1}(U) \subset BM$a system of differential equations of the form

$$\triangle J^i_j \equiv dJ^i_j - J^i_k \omega^k_j + J^k_j \omega^i_k = J^i_{jk} \omega^k. \qquad (4.1)$$

We differentiate these relations externally:

$$\left(\triangle J^i_{jk} + J^i_r \omega^r_{jk} - J^r_j \omega^i_{rk}\right) \wedge \omega^k = 0.$$

According to Cartan's lemma,

$$\triangle J^i_{jk} + J^i_r \omega^r_{jk} - J^r_j \omega^i_{rk} = J^i_{jkr} \omega^r \qquad (4.2)$$

for suitable functions J^i_{jkr}. Since, by virtue of the Corollary to the Generalized Cartan Lemma, $\omega^i_{jk} \equiv \omega^i_{kj} mod(\omega^1, \dots, \omega^n)$, then alternating both parts of (4.2)

by indices *j* and *k*, we get:

$$\triangle \left(J^i_{jk} - J^i_{kj}\right) = -J^r_k \omega^i_{rj} + J^r_k \omega^i_{rj} + A^i_{jkr} \omega^r$$

for suitable functions A^i_{jkr}. Multiply both sides of this ratio by $J^j_s$and sum over *j* from 1 to $2n$:

$$J^j_s \triangle \left(J^i_{jk} - J^i_{kj}\right) = -J^j_s J^r_k \omega^i_{rj} + J^j_s J^r_k \omega^i_{rj} + J^j_s A^i_{jkr} \omega^r$$

or,taking into account (4.1),

$$\triangle \left\{J^j_s \left(J^i_{jk} - J^i_{kj}\right)\right\} = -\omega^i_{sk} - J^j_s J^r_k \omega^i_{jr} + B^i_{skr} \omega^r$$

for suitable functions B^i_{skr}. Alternating by indices *s* and *k*, we finally get:

$$\triangle \left\{J^j_s \left(J^i_{jk} - J^i_{kj}\right) - J^j_k \left(J^i_{js} - J^i_{sj}\right)\right\} = C^i_{skr} \omega^r$$

for suitable functions C^i_{skr}. By virtue of the Main Theorem of Tensor Analysis, the set of functions

$$N^i_{sk} = \frac{1}{4}\left\{J^j_s \left(J^i_{jk} - J^i_{kj}\right) - J^j_k \left(J^i_{js} - J^i_{sj}\right)\right\}. \qquad (4.3)$$

defines in some neighborhood U of an arbitrary point of the manifold M a tensor of type (2,1), called ***the Nijenhuis tensor*** of almost complex structure J. Our immediate task is to prove that this tensor is in fact defined globally on M and to find an explicit expression for it. To achieve this, first of all, we note that, taking into account (1.4), relations (4.1) on the space of the associated G -structure can be written in the form

1) $\omega_b^{\hat{a}} = -\frac{1}{2}J_{bk}^{\hat{a}}\omega^k$; 2) $\omega_{\hat{b}}^a = \frac{1}{2}J_{\hat{b}k}^a\omega^k$; 3) $J_{bk}^a = 0$; 4) $J_{\hat{b}k}^{\hat{a}} = 0$. (4.4)

Indeed, putting in (4.1) $i = a, j = b$, we obtain, taking into account (1.4):

$dJ_b^{\hat{a}} - J_{\hat{c}}^{\hat{a}}\omega_b^{\hat{c}} + J_b^c\omega_c^{\hat{a}} = J_{bk}^{\hat{a}}\omega^k$,

whence, taking into account (1.4), we obtain that $2\sqrt{-1}\omega_b^{\hat{a}} = J_{bk}^{\hat{a}}\omega^k$, and hence $\omega_b^{\hat{a}} = -\frac{\sqrt{-1}}{2}J_{bk}^{\hat{a}}\omega^k$. The rest of the relations are proved similarly.

Taking into account (3.4 1-2), we obtain that the first group of structural equations of an almost complex manifold on the space of the associated G -structure can be written in the form

1) $d\omega^a = -\omega_b^a \wedge \omega^b + \frac{\sqrt{-1}}{2}J_{\hat{b}c}^a\omega^{\hat{b}} \wedge \omega^c + \frac{\sqrt{-1}}{2}J_{[\hat{b}\hat{c}]}^a\omega^{\hat{b}} \wedge \omega^{\hat{c}}$;
2) $d\omega^{\hat{a}} = -\omega_{\hat{b}}^{\hat{a}} \wedge \omega^{\hat{b}} - \frac{\sqrt{-1}}{2}J_{b\hat{c}}^{\hat{a}}\omega^b \wedge \omega^{\hat{c}} - \frac{\sqrt{-1}}{2}J_{[bc]}^{\hat{a}}\omega^b \wedge \omega^c$.

(4.5)

Further, taking into account (3.3), (1.4), and (3.4:3-4), we calculate the components of the Nijenhuis tensor in the A-frame (i.e., on the space of the associated G - structure):

$N_{\hat{b}\hat{c}}^a = \sqrt{-1}J_{[\hat{b}\hat{c}]}^a$; $N_{bc}^{\hat{a}} = -\sqrt{-1}J_{[bc]}^{\hat{a}}$. (4.6)

the other components of this tensor are zero.

Taking into account (4.6), structural equations (4.5) can be written in the form

1) $d\omega^a = -\omega_b^a \wedge \omega^b + \frac{1}{2}N_{\hat{b}\hat{c}}^a\omega^{\hat{b}} \wedge \omega^{\hat{c}} + \frac{\sqrt{-1}}{2}J_{\hat{b}c}^a\omega^{\hat{b}} \wedge \omega^c$;
2) $d\omega^{\hat{a}} = -\omega_{\hat{b}}^{\hat{a}} \wedge \omega^{\hat{b}} - \frac{\sqrt{-1}}{2}J_{b\hat{c}}^{\hat{a}}\omega^b \wedge \omega^{\hat{c}} - \frac{1}{2}N_{bc}^{\hat{a}}\omega^b \wedge \omega^c$.

(4.7)

From these relations follows the following important

Theorem 4.1. The eigendistributions of an almost complex structure J on a smooth manifold M^{2n} are involutive if and only if its Nijenhuis tensor vanishes identically.

Proof. This immediately follows from the fact that the eigendistributions $D_J^{\sqrt{-1}}$ and $D_J^{-\sqrt{-1}}$ are given, respectively, by the Pfaffian systems $s^*(\omega^{\hat{a}}) = 0$ and $s^*(\omega^a) = 0$ $(a = 1, \dots, n)$, (where s is the local section of the attached G - structure) by virtue of the definition of the A - frame, as well as relations (4.7).

4.2 . Almost complex connections

Definition 4.1. An *almost complex connection* on an almost complex manifold (M, J) is a connection in the principal bundle $B_J(M)$.

By virtue of the existence of a canonical monomorphism

$$(\tilde{f}, \tilde{\rho}): B_J(M) \to B(M)$$

an almost complex connection induces a path connection on M, and by virtue of the existence of a canonical monomorphism

$(f, \rho): B_J(M) \to B^C(M)$,

which is an isomorphism onto the image, the adjoint G -structure, the almost complex connection induces connections in these principal bundles. All these four connections can be identified. In particular, a connection in the principal bundle $B^C(M)$ as an almost complex connection is characterized by the fact that its connection form θ takes values in the Lie algebra of the structure group of the adjoint G - structure, and hence its components satisfy the relations

1) $\overline{\theta_b^a} = \theta_{\hat{b}}^{\hat{a}}$; 2) $\theta_b^{\hat{a}} = 0$; 3) $\theta_{\hat{b}}^a = 0$. (4.8)

Theorem 4.2. A linear connection ∇ on an almost complex manifold (M, J) is an almost complex connection if and only if $\nabla J = 0$.

Proof. Let be ∇ an arbitrary connection on M, θ be its form on $B^C(M)$. Then, as above, the identities

$\nabla J_j^i \equiv dJ_j^i - J_k^i\theta_j^k + J_j^k\theta_k^i = J_{j,k}^i\omega^k$.

As above, it is shown that in the space of the adjoint G -structure these identities take the form:

1) $\theta_b^{\hat{a}} = -\frac{1}{2}J_{b,k}^{\hat{a}}\omega^k$; 2) $\theta_{\hat{b}}^a = \frac{1}{2}J_{\hat{b},k}^a\omega^k$; 3) $J_{b,k}^a = 0$; 4) $J_{\hat{b},k}^{\hat{a}} = 0$. (4.9)

Relations (4.6) for the nonzero components of the Nijenhuis tensor then take the form:

$N_{\hat{b}\hat{c}}^a = \sqrt{-1}J_{[\hat{b},\hat{c}]}^a$; $N_{bc}^{\hat{a}} = -\sqrt{-1}J_{[b,c]}^{\hat{a}}$. (4.10)

Let now ∇ be an almost complex connection. Then, due to (4.8), (4.9:1-2), and due to the linear independence of the basic forms, we have: $\nabla J^{a}_{\hat{b},k}=0, \nabla J^{\hat{a}}_{b,k}=0$. In view of (3.9:3-4) $\nabla J^{i}_{j,k}=0$, i.e. $\nabla J=0$.

Back, let $\nabla J=0$. Then, taking into account (3.9:1-2), we obtain that $\theta^{a}_{\hat{b}}=0, \theta^{\hat{a}}_{b}=0$. On the other hand, since the connection is real ∇, it follows that $\overline{\theta^{a}_{b}}=\theta^{\hat{a}}_{\hat{b}}$. Therefore, $\theta$is a connection form with values in the Lie algebra of the structure group of the adjoint$G$ -structure, and, due to the isomorphism of (f,ρ)principal bundles can be identified with an almost complex connection.□

Theorem 4.3. Any torsion-free linear connection on an almost complex manifold induces an almost complex connection whose torsion tensor coincides with the Nijenhuis tensor.

Proof. Let be ∇an arbitrary torsion-free linear connection on an almost complex manifold (M,J), and let θbe the form of this connection. By virtue of relations (4.9) and (4.10), its first group of structural equations on the space of the adjoint$G$ -structure takes the form:

1) $d\omega^{a}=\left(-\theta^{a}_{b}\wedge\omega^{b}+\frac{\sqrt{-1}}{2}J^{a}_{\hat{c},b}\omega^{\hat{c}}\right)\wedge\omega^{b}+\frac{1}{2}N^{a}_{\hat{b}\hat{c}}\omega^{\hat{b}}\wedge\omega^{\hat{c}}$;

2) $d\omega^{\hat{a}}=-\left(\theta^{\hat{a}}_{\hat{b}}+-\frac{\sqrt{-1}}{2}J^{\hat{a}}_{c,\hat{b}}\omega^{c}\right)\wedge\omega^{\hat{b}}+\frac{1}{2}N^{\hat{a}}_{bc}\omega^{b}\wedge\omega^{c}$.

(4.11)

Let us introduce into consideration a 1-form $\zeta$with values in the complex complete matrix Lie algebra of order $2n$by its components

1) $\zeta^{a}_{b}=\theta^{a}_{b}-\frac{\sqrt{-1}}{2}J^{a}_{\hat{c},b}\omega^{\hat{c}}$; 2) $\zeta^{a}_{\hat{b}}=0$; 3) $\zeta^{\hat{a}}_{\hat{b}}=\theta^{\hat{a}}_{\hat{b}}+\frac{\sqrt{-1}}{2}J^{\hat{a}}_{c,\hat{b}}\omega^{c}$; 4) $\zeta^{\hat{a}}_{b}=0$. (4.12)

Obviously, this form takes values in the Lie algebra of the structure group of the adjoint$G$ -structure. In addition, externally differentiating (4.12:1) on the space of the associated G-structure, taking into account (4.9:3), we obtain:

$$d\zeta^{a}_{b}=d\theta^{a}_{b}-\frac{\sqrt{-1}}{2}dJ^{a}_{\hat{c},b}\wedge\omega^{\hat{c}}-\frac{\sqrt{-1}}{2}J^{a}_{\hat{c},b}d\omega^{\hat{c}}$$
$$=-\theta^{a}_{c}\wedge\theta^{c}_{b}-\theta^{a}_{\hat{c}}\wedge\theta^{\hat{c}}_{b}+\frac{1}{2}R^{a}_{bij}\omega^{i}\wedge\omega^{j}$$
$$+\frac{\sqrt{-1}}{2}\Big\{\Big(J^{a}_{\hat{h},b}\theta^{\hat{h}}_{\hat{c}}+J^{a}_{\hat{c},h}\theta^{h}_{b}+J^{a}_{\hat{c},\hat{h}}\theta^{\hat{h}}_{b}-J^{h}_{\hat{c},b}\theta^{a}_{h}+J^{a}_{\hat{c},\hat{b}}\omega^{k}\Big)\wedge\omega^{\hat{c}}$$
$$-J^{a}_{\hat{c},b}\left(\theta^{\hat{c}}_{h}\wedge\omega^{h}+\theta^{\hat{c}}_{\hat{h}}\wedge\omega^{\hat{h}}\right)\Big\}\equiv$$
$$\equiv\left(\zeta^{a}_{c}-\frac{\sqrt{-1}}{2}J^{a}_{\hat{h},c}\omega^{\hat{h}}\right)\wedge\left(\zeta^{c}_{b}-\frac{\sqrt{-1}}{2}J^{c}_{\hat{g},b}\omega^{\hat{g}}\right)+\frac{\sqrt{-1}}{2}\left(J^{a}_{\hat{h},b}\theta^{\hat{h}}_{\hat{c}}+J^{a}_{\hat{c},h}\theta^{h}_{b}-J^{h}_{\hat{c},b}\theta^{a}_{h}\right)\wedge\omega^{\hat{c}}-J^{a}_{\hat{c},b}\theta^{\hat{c}}_{\hat{h}}\wedge\omega^{\hat{h}}\equiv-\zeta^{a}_{c}\wedge\zeta^{c}_{b}=-\zeta^{a}_{k}\wedge\zeta^{k}_{b}\left(mod\ \omega^{i}\wedge\omega^{j}\right).$$

Here the symbol " $\equiv$ "is understood as equality up to terms of the form $f\omega^{i}\wedge\omega^{j}; f\in C^{\infty}\left(B^{C}_{J}M\right)$. Thus, $d\zeta^{a}_{b}\equiv-\zeta^{a}_{k}\wedge\zeta^{k}_{b}\left(mod\omega^{i}\wedge\omega^{j}\right)$. Likewise, $d\zeta^{\hat{a}}_{\hat{b}}\equiv-\zeta^{\hat{a}}_{k}\wedge\zeta^{k}_{\hat{b}}\left(mod\omega^{i}\wedge\omega^{j}\right)$. Finally, taking into account (4.12),

$d\zeta^{a}_{\hat{b}}=0-\zeta^{a}_{c}\wedge\zeta^{c}_{\hat{b}}-\zeta^{a}_{\hat{c}}\wedge\zeta^{\hat{c}}_{\hat{b}}=-\zeta^{a}_{k}\wedge\zeta^{k}_{\hat{b}}$.

Similarly, $d\zeta^{\hat{a}}_{b}=0=-\zeta^{\hat{a}}_{k}\wedge\zeta^{k}_{b}$, and thus

$d\zeta^{i}_{j}\equiv-\zeta^{i}_{k}\wedge\zeta^{k}_{j}\left(mod\omega^{i}\wedge\omega^{j}\right)$.

According to the Kartina-Laptev theorem, $\zeta$is a connection form. What already. It was noted that this form takes values in the Lie algebra of the structure group of the adjoint G -structure, and hence is an almost complex connection. It follows from (4.11) and (4.12) that the torsion tensor S of this connection coincides with the Nijenhuis tensor. □

Definition 4.2. An almost complex connection whose torsion tensor coincides with the Nijenhuis tensor will be called semi-canonical.

4.3. Global Definition

We are now ready to prove that the Nijenhuis tensor of an almost complex manifold is defined globally and obtain an explicit formula for calculating it. First of all, we recall that in the space of the associated G -structure

1) $N^{c}_{ab}=0$; 2) $N^{c}_{\hat{a}b}=0$; 3) $N^{c}_{a\hat{b}}=0$;

(4.13)

and complex conjugate formulas (in short, f.c.s.).

Proposition 3.1. Relations (4.13) are equivalent to the following identities:

1) $\sigma N(\sigma X,\sigma Y)=0$; 2) $\sigma N(\bar{\sigma}X,\sigma Y)=0$ 3) $\sigma N(\sigma X,\bar{\sigma}Y)=0$;

andf.c.s., respectively.

Proof. Note that for $p=(m,\varepsilon_1,\dots,\varepsilon_{\hat{n}})\in B_J^C M$, $N_{ab}^c(p)=0 \iff N(\varepsilon_a,\varepsilon_b)^c\varepsilon_c=0 \iff$ $\sigma\left(N\big(\sigma(e_a),\sigma(e_b)\big)\right)=0$, and, since the vectors $\{e_1,\dots,e_n\}$form the basis of the space $T_m(M)$as a $\boldsymbol{C}$ - module, then $\sigma\left(N\big(\sigma(e_a),\sigma(e_b)\big)\right)=0 \iff$ $\sigma N(\sigma X,\sigma Y)=0$.
The rest of the assertions are proved similarly.

□

Proposition 4.2. Let be ∇an almost complex connection on an almost complex manifold (M,J). Then
1) $\nabla_X\circ\sigma=\sigma\circ\nabla_X$; 2) $\nabla_X\circ\bar{\sigma}=\bar{\sigma}\circ\nabla_X$; $X\in\mathcal{X}(M)$.
Proof. By condition. $\nabla_X J=0$. So
$2\nabla_X(\sigma Y)=\nabla_X Y-\sqrt{-1}\nabla_X(JY)=\nabla_X Y-\sqrt{-1}\{\nabla_X(J)Y+J\nabla_X Y\}=\big(id-\sqrt{-1}J\big)\nabla_X Y=2\sigma(\nabla_X Y)$.
The second relation is proved similarly.

□

Given these statements, if S is the torsion tensor of a semi-canonical connection ∇, then
$N(X,Y)=(\sigma+\bar{\sigma})N\big((\sigma+\bar{\sigma})X,(\sigma+\bar{\sigma})Y\big)=\sigma N(\bar{\sigma}X,\bar{\sigma}Y)+\bar{\sigma}N(\sigma X,\sigma Y)=\sigma S(\bar{\sigma}X,\bar{\sigma}Y)+\bar{\sigma}S(\sigma X,\sigma Y)=\sigma\{\nabla_{\bar{\sigma}X}(\bar{\sigma}Y)-\nabla_{\bar{\sigma}Y}(\bar{\sigma}X)-[\bar{\sigma}X,\bar{\sigma}Y]\}+\bar{\sigma}\{\nabla_{\sigma X}(\sigma Y)-\nabla_{\sigma Y}(\sigma X)-[\sigma X,\sigma Y]\}=-\sigma[\bar{\sigma}X,\bar{\sigma}Y]-\bar{\sigma}[\sigma X,\sigma Y]=-\frac{1}{8}\big(id-\sqrt{-1}J\big)\big([X-\sqrt{-1}JX,Y-\sqrt{-1}JY]\big)-\frac{1}{8}\big(id+\sqrt{-1}J\big)\big([X-\sqrt{-1}JX,Y-\sqrt{-1}JY]\big)=\frac{1}{4}\{-[X,Y]+[JX,JY]-J[JX,Y]-J[X,JY]\}$.
This proves
Theorem 4.4. The Nijenhuis tensor of an almost complex structure is Jcalculated by the formula $N(X,Y)=\frac{1}{4}\{-[X,Y]+[JX,JY]-J[JX,Y]-J[X,JY]\}$.

Conclusion

In this paper, we define a complex structure on a linear space and prove that a finite-dimensional real space admits a complex structure if and only if it is even-dimensional. It is shown that fixing a complex structure Jin a $2n$-dimensional real linear space $V^{\boldsymbol{R}}$induces the assignment of $V^{\boldsymbol{R}}$an n-dimensional complex linear space V *to the structure*. Each basis $b=\{e_1,\dots,e_{2n}\}$of the space V canonically induces two bases:

1) *RA is the* basis $b_{RA}=\{e_1,\dots,e_n,Je_1,\dots,Je_n\}$of the space $V^{\boldsymbol{R}}$.

2) A -basis$b_A=\{\varepsilon_1,\dots,\varepsilon_n,\varepsilon_{\hat{1}},\dots,\varepsilon_{\hat{n}}\}\boldsymbol{C}$ - linear space $(V^{\boldsymbol{R}})^{\boldsymbol{C}}$.

A G -structure attached to an almost complex structure is constructed.
We have given a local definition of the Nijenhuis tensor of an almost complex structure J. It is proved that in reality this tensor is defined globally on the manifold M and we have found an explicit expression for it.

References

1. Efimov, N. V.; Rozendorn, È. R. Linear algebra and multidimensional geometry. Mir Publishers, Moscow, 1975. 495 pp
2. Hodge, W. V. D. The theory and applications of harmonic integrals. Cambridge University Press, Cambridge, 1952.
3. Kähler, Erich. Über eine bemerkenswerte Hermitesche Metrik. Abh. Math. Sem. Univ. Hamburg 9(1) (1933), 173–186.
4. Kobayashi, S., & Nomizu, K. Fundamentals of Differential Geometry, Vol. 1. Interscience Publ (1963)
5. Samelson, Hans Über die Sphären, die als Gruppenräume auftreten. Comment. Math. Helv. 13 (1940), 144–155.
6. Van de Ven, A. (1966). On the Chern numbers of certain complex and almost complex manifolds. *Proceedings of the National Academy of Sciences of the United States of America*, *55*(6), 1624.