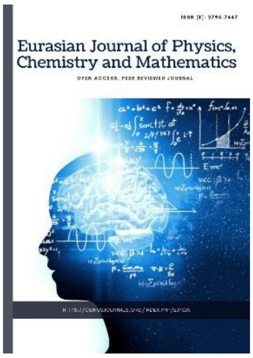

G -structure on Kähler manifold

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ABSTRACT

Purpose Of the Thesis: The main goal is to study some questions of the geometry of Kähler manifolds

The Importance of Thesis: To achieve this goal, the following tasks are solved:

1. Construction of the associated G -structure.
2. Get the first group of structural equations of this structure.
3. Describe the structural tensors of this structure.
4. Obtain the first group of structural equations of the Kähler structure.
5. Obtain the second group of structural equations of the Kähler structure

Material And Method: The presentation of the material is carried out by the systematic use of Cartan's method of external forms in combination with the method of invariant Koschul calculus. Structural equations are written in a specialized frame, ie on the space of the adjoint G -structure.

Keywords: G-structure, structure equations, Kähler manifold, Riemannian metric, almost complex structure.

1. Introduction

Kähler manifolds are one of the most remarkable mathematical objects studied intensively both in differential geometry and in algebraic geometry, the theory of Lie groups and homogeneous spaces, topology, the theory of differential operators, and mathematical physics. Kähler manifolds were first defined in 1933 by E. Kähler in [1]. Their significance for algebraic geometry became clear after the publication of Hodge's work, the results of which were subsequently combined in the book [2]. These works basically determined the direction of research on Kahlerian manifolds for many years. The flow of research into the geometry and topology of Kahlerian manifolds continues unabated even in our time.

Our work is devoted to the study of some aspects of the geometry of Kählerian manifolds and is structured as follows. In Section 2, we consider complex structures, the complexification of a linear space, and introduce the concept of linearity extension.

2. Complex structures

2.1. Tensor products of modules

Let A and B be modules over a commutative associative ring $\boldsymbol{K}$ with identity. Consider a free abelian group $A \circ B$whose set of generators is the set of all symbols of the form $a \circ b; a \in A, b \in B$. Its elements are all formal (finite) sums of such symbols, i.e. type records $a_1 \circ b_1 + \dots + a_N \circ b_N; N \in \boldsymbol{N}$. Consider its subgroup $S \subset A \circ B$generated by elements of the form $(a' + a'') \circ b - a' \circ b - a'' \circ b, a \circ (b' + b'') - a \circ b' - a \circ b'', (\alpha a) \circ b - a \circ (\alpha b)$; $\alpha \in \boldsymbol{K}$. Let's consider an abelian group $A \otimes B = A \circ B / S$. Its elements are finite formal sums of symbols of the form $a \otimes b = (a \circ b) + S$. It naturally introduces the structure of a $\boldsymbol{K}$ - module with an external composition operation $\alpha(\sum_{i=1}^{n} a_i \otimes b_i) = \sum_{i=1}^{n} (\alpha a_i) \otimes b_i \left(= \sum_{i=1}^{n} b_i \otimes (\alpha a_i)\right)$. This $\boldsymbol{K}$ -module is called *the tensor product* $\boldsymbol{K}$ *-* models A and B [3].

Remark 2.1. If A and B have the structure of a module over some ring $\boldsymbol{K}_1$, then $A \otimes B$, obviously, also has a natural $\boldsymbol{K}_1$-module structure. In particular, if A is an algebra, then it $A \otimes B$has the natural structure of an A - module.

2.2. Complexification of linear space

Let, in particular, $A = \boldsymbol{C}$be the field of complex numbers and $B = V$be an $\boldsymbol{R}$ -linear space. Then the $\boldsymbol{C}$ -linear space ($\boldsymbol{C}$ -module) $\boldsymbol{C} \otimes V$is denoted $V^{\boldsymbol{C}}$and called ***the complexification of the*** linear space V. Its elements are records of the form $\sum_{k=1}^{N} z_k \otimes X_k$; $z_k \in \boldsymbol{C}, X_k \in V, N$- an arbitrary natural number. *The sum* of two such elements $\sum_{k=1}^{N_1} z_k \otimes X_k$will $\sum_{p=1}^{N_2} z_p \otimes X_p$be a record of the form $\sum_{k=1}^{N_1} z_k \otimes X_k + \sum_{p=1}^{N_2} z_p \otimes X_p$, and *the product* element $\sum_{k=1}^{N_1} z_k \otimes X_k$to a complex number $z \in \boldsymbol{C}$- a record of the form $\sum_{k=1}^{N_1} (z z_k) \otimes X_k$.

In a $\boldsymbol{C}$ -linear space , $V^{\boldsymbol{C}}$the mapping is canonically defined $\tau: V^{\boldsymbol{C}} \to V^{\boldsymbol{C}}$, acting according to the formula $\tau\left(\sum_{k=1}^{N_1} z_k \otimes X_k\right) = \sum_{k=1}^{N_1} \overline{z_k} \otimes X_k$, where $z \to \bar{z}$is the usual complex conjugation in the field of complex numbers. It is directly verified that τ is an involutive anti-automorphism of a $\boldsymbol{C}$ -linear space $V^{\boldsymbol{C}}$, i.e., bijection with the properties
1) $\tau^2 = id$; 2) $\tau(X + Y) = \tau(X) + \tau(Y)$; 3) $\tau(zX) = \bar{z}\tau(X); z \in \boldsymbol{C}, X, Y \in V$.(2.1)

It is called *the complex conjugation operator* .

Note that V naturally admits an embedding j in $V^{\boldsymbol{C}}$by identifying $X \equiv j(X) = 1 \otimes X; X \in V$. At the same time $\tau(X) \equiv \tau(1 \otimes X) = 1 \otimes X \equiv X$. Moreover, $\tau\left(\sum_{k=1}^{N_1} z_k \otimes X_k\right) = \sum_{k=1}^{N_1} z_k \otimes X_k \Leftrightarrow z_k = \overline{z_k}\ (k = 1, \dots, N) \Leftrightarrow z_k = x_k \in \boldsymbol{R}$, which means $\sum_{k=1}^{N_1} z_k \otimes X_k = \sum_{k=1}^{N_1} x_k (1 \otimes X_k) \equiv \sum_{k=1}^{N_1} x_k X_k \in V$. This proves

Proposition 2.1. Let $Y = \sum_{k=1}^{N_1} z_k \otimes X_k \in V^{\boldsymbol{C}}$. Then, taking into account the accepted identification, $Y \in V \Leftrightarrow \tau(Y) = Y$.

□

Note that if V is an n -dimensional $\boldsymbol{R}$ - linear space, then V $V^{\boldsymbol{C}}$is an n -dimensional $\boldsymbol{C}$ - linear space. Moreover, if $b = \{e_1, \dots, e_n\}$is a basis of $\boldsymbol{R}$ -linear space V, then under the above canonical identification of elements $e_k \in V$with elements, the $1 \otimes e_k \in V^{\boldsymbol{C}}$set b is also a basis of $\boldsymbol{C}$ -linear space $V^{\boldsymbol{C}}$. This easily follows from a more general fact of independent interest ([4, p. 171]):

Proposition 2.2. Let be $\{e_1, \dots, e_n\}$a basis of a real linear space V, $\{\varepsilon_1, \dots, \varepsilon_n\}$be a basis of an $\boldsymbol{R}$ -linear space W. _ Then $\{e_i \otimes \varepsilon_a | i = 1, \dots, n; a = 1, \dots, m\}$is a basis of the $\boldsymbol{R}$ -linear space $V \otimes W$. □

Indeed, from this Proposition it follows that the elements $\left\{1 \otimes e_k, \sqrt{-1} \otimes e_k | k = 1, \dots, n\right\}$form a basis of an $\boldsymbol{R}$ -linear space $V^{\boldsymbol{C}}$, from which it already easily follows that the first n elements of this basis form a basis $V^{\boldsymbol{C}}$as a $\boldsymbol{C}$ - linear space.

Any operator $f: V \to V$canonically defines a $\boldsymbol{C}$ -linear mapping $f^{\boldsymbol{C}} = id \otimes f: V^{\boldsymbol{C}} \to V^{\boldsymbol{C}}$by the formula $f^{\boldsymbol{C}}(\sum_{k=1}^{N} z_k \otimes X_k) = \sum_{k=1}^{N} z_k \otimes f(X_k)$.
Obviously, taking into account the indicated identification, $f^{\boldsymbol{C}}\big|_N = f$, in view of which the mapping $f^{\boldsymbol{C}}$is called **the extension in linearity** of the operator f.

Proposition 2.3. A $\boldsymbol{C}$ -linear operator $F: V^{\boldsymbol{C}} \to V^{\boldsymbol{C}}$is a linear extension of some $\boldsymbol{R}$ -

linear operator $f: V \to V$ if and only if $\tau \circ F = F \circ \tau$.

Proof. Indeed, if $F = f^{\boldsymbol{C}}$, then $\tau \circ F(\sum_{k=1}^{N} z_k \otimes X_k) = \sum_{k=1}^{N} \overline{z_k} \otimes f(X_k) = F \circ \tau(\sum_{k=1}^{N} z_k \otimes X_k)$, whence it follows that $\tau \circ F = F \circ \tau$. Conversely, if this relation holds, then $\tau \circ F(1 \otimes X) = F \circ \tau(1 \otimes X) = F(1 \otimes X), X \in V$, and, by virtue of Proposition 2.1, the restriction $f = F|_V$ of the operator F to V *is defined by the* formula $1 \otimes f(X) = F(1 \otimes X)$. Obviously, in this case $F(\sum_{k=1}^{N} z_k \otimes X_k) = \sum_{k=1}^{N} F(z_k \otimes X_k) = \sum_{k=1}^{N} z_k F(1 \otimes X_k) = \sum_{k=1}^{N} z_k\big(1 \otimes f(X_k)\big) = \sum_{k=1}^{N} z_k \otimes f(X_k) = f^{\boldsymbol{C}}(\sum_{k=1}^{N} z_k \otimes X_k)$, and hence $F = f^{\boldsymbol{C}}$. □

The following assertion is proved in exactly the same way:

Proposition 2.4. An r -ary $\boldsymbol{C}$ -linear mapping $T: V^{\boldsymbol{C}} \times \ldots \times V^{\boldsymbol{C}} \to V^{\boldsymbol{C}}$ is a linear extension of an r -ary $\boldsymbol{R}$ -linear mapping $T: V \times \ldots \times V \to V$ if and only if $\tau \circ T(X_1, \ldots, X_r) = T(\tau X_1, \ldots, \tau X_r)$; $X_1, \ldots, X_r \in V^{\boldsymbol{C}}$. □

One can give another definition of complexification that is equivalent to the above one. Let V be an R -linear space. Let us introduce the following operations in the set :$V \times V$

1) ***Addition.*** If $X_1 = (A_1, B_1), X_2 = (A_2, B_2)$ are elements from $V \times V$, then the pair $(A_1 + A_2, B_1 + B_2)$ is called their *sum* and denoted by $X_1 + X_2$.

2) ***Multiplication by a complex number.*** If $X = (A, B) \in V \times V, z = \alpha + \sqrt{-1}\beta \in \boldsymbol{C}$, then put $zX = (\alpha a - \beta b, \alpha b + \beta a)$. Let's call the element$zX$ the *product of* a complex number z and an element X.

is directly verified that this $V \times V$ introduces in the set the structure of a $\boldsymbol{C}$ -linear space $\tilde{V}^{\boldsymbol{C}}$, which is naturally isomorphic to the $\boldsymbol{C}$ -linear space $V^{\boldsymbol{C}}$. The natural isomorphism $\varphi: V^{\boldsymbol{C}} \to \tilde{V}^{\boldsymbol{C}}$ associates an element with $\sum_{k=1}^{N}\big(\alpha_k + \sqrt{-1}\beta_k\big)X_k \in V^{\boldsymbol{C}}$ a pair $(A, B) \in \tilde{V}^{\boldsymbol{C}}$, where $A = \sum_{k=1}^{N} \alpha_k X_k, B = \sum_{k=1}^{N} \beta_k X_k$. Under this isomorphism, the embedding described above $j: V \subset V^{\boldsymbol{C}}$ corresponds to the embedding $\tilde{j}: V \subset \tilde{V}^{\boldsymbol{C}}$ defined by the formula $\tilde{j}(X) = (X, 0); X \in V$, the τcomplex conjugation operator corresponds to the operator $\tilde{\tau}: \tilde{V}^{\boldsymbol{C}} \to \tilde{V}^{\boldsymbol{C}}$ defined by the formula $\tilde{\tau}(X, Y) = (X, -Y); X, Y \in V$, and the $\boldsymbol{C}$ -linear operator corresponds to $f^{\boldsymbol{C}} = id \otimes f$ the ***C -linear*** operator $\tilde{j}^{\boldsymbol{C}}: \tilde{V}^{\boldsymbol{C}} \subset \tilde{V}^{\boldsymbol{C}}$ defined by the formula $\tilde{f}^{\boldsymbol{C}}(X, Y) = (fX, fY); X, Y \in V$.

2.3. Complex structures

Let V be a complex linear space. It can, in particular, be viewed as a real linear space $V^{\boldsymbol{R}}$ (called the *reification* of the space V) in which an $\boldsymbol{R}$ -linear endomorphism $J_0: V^{\boldsymbol{R}} \to V^{\boldsymbol{R}}$ is given, defined by $J_0(X) = \sqrt{-1}X; X \in V^{\boldsymbol{R}}$. This endomorphism allows us to completely restore the structure of a complex linear space to *V.* Namely, if $z = \alpha + \sqrt{-1}\beta \in \boldsymbol{C}, X \in V$, then $zX = \alpha X + \beta J_0(X)$. Moreover, it is obvious that the endomorphism J_0 is anti-involutive, i.e. $J_0^2 = -id$.

Let V be a real linear space.

Definition 2.1. A ***complex structure*** in V is an endomorphism $J: V \to V$ such that $J^2 = -id$. In other words, a complex structure is an anti-involutive automorphism of a real linear space.

Fixing a complex structure in V canonically determines in V the structure of a complex linear space (that is, a $\boldsymbol{C}$ - module). Indeed, if $X \in V, z = \alpha + \sqrt{-1}\beta \in \boldsymbol{C}$, then we set

$$zX = \alpha X + \beta(JX). \tag{2.2}$$

It is directly verified that in this case all 8 axioms of a $\boldsymbol{C}$ -linear space are satisfied, which we will denote by the same symbol V. Obviously, V as an $\boldsymbol{R}$ -linear space is its reification $V^{\boldsymbol{R}}$.

Let the dimension $dim_{\boldsymbol{C}}V$ of the linear space V as a complex space be equal to n , and let $\{e_1, \ldots, e_n\}$ be the basis of this space. Let $X \in V$. Then $X = z^k e_k$, where $z^k = \alpha^k + \sqrt{-1}\beta^k \in \boldsymbol{C}; k = 1, \ldots, n$. Taking into account (1.2), $X = \alpha^k e_k + \beta^k(Je_k)$, i.e., every vector $X \in V^{\boldsymbol{R}}$ is represented as a linear combination of vectors $e_1, \ldots, e_n, Je_1, \ldots, Je_n$. On the other hand , let $\lambda^k e_k + \mu^k Je_k = 0, \lambda^k, \mu^k \in \boldsymbol{R}$Then, due to (1.2), $\big(\lambda^k + \sqrt{-1}\mu^k\big)e_k = 0$, and due to the $\boldsymbol{C}$ -linear independence of the vectors $e_1, \ldots, e_n, \lambda^k + \sqrt{-1}\mu^k = 0$, and hence $\lambda^k = \mu^k = 0; k = 1, \ldots, n$. Therefore, the vectors $\{e_1, \ldots, e_n, Je_1, \ldots, Je_n\}$ form a basis for the space V as an $\boldsymbol{R}$ -linear space (i.e. a basis for the space $V^{\boldsymbol{R}}$). Such a basis is called *a real-adapted complex structure* , in short, an *RA - basis* .

Remark 2.2. Obviously, any complex structure is defined in the *RA* basis by a matrix of the form

$$(J_j^i) = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}. \tag{2.3}$$

As a simple but important corollary, we get the following

Proposition 2.5. A finite-dimensional real linear space admits a complex structure if and only if it is even-dimensional.

Proof. Since the RA -basis contains an even number of vectors, a space that admits a complex structure is necessarily even-dimensional. Conversely, let V be a $2n$-dimensional real linear space. We fix an arbitrary basis in it $\{e_1, \dots, e_{2n}\}$. Then the endomorphism of the Jspace V given by the matrix (2.3) in this basis is obviously a complex structure. □

Let be Ja complex structure in an $\boldsymbol{R}$ -linear space V . Consider the endomorphism $\sigma: V^{\boldsymbol{C}} \to V^{\boldsymbol{C}}$defined by the formula $\sigma = \frac{1}{2}\left(id - \sqrt{-1}J^{\boldsymbol{C}}\right)$. Obviously, $\sigma^2 = \sigma$, i.e. σ- projector. The projector complementary to it $\bar{\sigma}$is determined by the formula $\bar{\sigma} = \frac{1}{2}\left(id + \sqrt{-1}J^{\boldsymbol{C}}\right)$. In the future, allowing freedom of speech, $J^{\boldsymbol{C}}$we will simply denote endomorphism J. Note that $J \circ \sigma = \frac{1}{2}\left(J + \sqrt{-1}id\right) = \frac{\sqrt{-1}}{2}\left(id - \sqrt{-1}J\right) = \sqrt{-1}\sigma$, which means $Im\,\sigma \subset D_J^{\sqrt{-1}}$. (Here and in what follows, the symbol D_F^{λ}denotes the proper subspace of the endomorphism F corresponding to the eigenvalue λ).

Conversely, if $X \in D_J^{\sqrt{-1}}$, then $\sigma X = \frac{1}{2}\left(X - \sqrt{-1}JX\right) = \frac{1}{2}(2X) = X$, in particular, $X \in Im\,\sigma$. Thus, $Im\,\sigma = D_J^{\sqrt{-1}}$. Likewise, $Im\,\bar{\sigma} = D_J^{-\sqrt{-1}}$. Since $\mathcal{X}^{\boldsymbol{C}}(M) = D_J^{\sqrt{-1}} \oplus D_J^{-\sqrt{-1}}$we get:

Theorem 2.1. linear space$V^{\boldsymbol{C}}$ decomposes into a direct sum of eigenspaces of the endomorphism Jcorresponding to the eigenvalues $\sqrt{-1}$and $-\sqrt{-1}$, i.e., $V^{\boldsymbol{C}} = D_J^{\sqrt{-1}} \oplus D_J^{-\sqrt{-1}}$, and the endomorphisms $\sigma$and $\bar{\sigma}$are projections onto the subspaces $D_J^{\sqrt{-1}}$and $D_J^{-\sqrt{-1}}$, respectively. □

Moreover, fair

Theorem 2.2. Defining a complex structure on an $\boldsymbol{R}$ -linear space V is equivalent to splitting $V^{\boldsymbol{C}}$into a direct sum of two complex conjugate subspaces serving as proper subspaces of this complex structure.

Proof. Necessity follows from Theorem 2.1. Let now $V^{\boldsymbol{C}} = D \oplus \tau D$. Then $\forall X \in V^{\boldsymbol{C}} \Longrightarrow X = X_1 + X_2;\ X_1 \in D, X_2 \in \tau D$. We construct an endomorphism $\mathcal{J}: V^{\boldsymbol{C}} \longrightarrow V^{\boldsymbol{C}}$by setting $\mathcal{J}(X) = \sqrt{-1}(X_1 - X_2)$. Obviously, $\tau(X) = \tau(X_1) + \tau(X_2)$, and $\tau(X_1) \in \tau D, \tau(X_2) \in D$. Therefore $(\mathcal{J} \circ \tau)(X) = \sqrt{-1}\big(\tau(X_2) - \tau(X_1)\big)$. On the other hand, due to the antilinearity of the operator τ, $(\tau \circ \mathcal{J})(X) = -\sqrt{-1}\big(\tau(X_1) - \tau(X_2)\big) = \sqrt{-1}(\tau X_2 - \tau X_1)$. Thus, $\mathcal{J} \circ \tau = \tau \circ \mathcal{J}$. By Proposition 2.3 $\mathcal{J} = J^{\boldsymbol{C}}$, for some R -linear endomorphism $J: V \to V$. Obviously, $\mathcal{J}^2 = -id$, in particular, $J^2 = -id$, i.e. $J$is the complex structure on V. If $X \in D$, then $X = X_1$, which means $\mathcal{J}(X) = \sqrt{-1}X_1 = \sqrt{-1}X$. Therefore, $D \subset D_J^{\sqrt{-1}}$. Conversely, if $X \in D_J^{\sqrt{-1}}$, then $\sqrt{-1}(X_1 - X_2) = \mathcal{J}(X) = \sqrt{-1}X = \sqrt{-1}(X_1 + X_2)$, whence $X_2 = 0$, and hence $X \in D$. Therefore, $D_J^{\sqrt{-1}} \subset D$, i.e. $D_J^{\sqrt{-1}} = D$. Likewise, $D_J^{-\sqrt{-1}} = \tau D$. □

Lemma 2.1. In the introduced notation, 1) $\tau \circ \sigma = \bar{\sigma} \circ \tau$, 2) $\tau \circ \bar{\sigma} = \sigma \circ \tau$.

Proof. Taking into account the antilinearity of the mapping τand using Proposition 2.3, we have: $\tau \circ \sigma(X) = \frac{1}{2}\tau\big(X - \sqrt{-1}JX\big) = \frac{1}{2}\big(\tau X + \sqrt{-1}\tau \circ JX\big) = \frac{1}{2}\big(\tau X + \sqrt{-1}J \circ \tau X\big) = \sigma \circ \tau(X); X \in V^{\boldsymbol{C}}$.

The second relation is proved similarly. □

Theorem 2.3. The mappings $\sigma|_V: V \to D_J^{\sqrt{-1}}$and $\bar{\sigma}|_V: V \to D_J^{-\sqrt{-1}}$are, respectively, an isomorphism and an anti-isomorphism of $\boldsymbol{C}$ -linear spaces.

Proof. The additivity of the mappings $\sigma|_V$and $\bar{\sigma}|_V$is obvious. Let now $z = \alpha + \sqrt{-1}\beta \in \boldsymbol{C}, X \in V$. As already seen, $\sigma \circ J = J \circ \sigma = \sqrt{-1}\sigma, \bar{\sigma} \circ J = J \circ \bar{\sigma} = -\sqrt{-1}\sigma$. Therefore $\sigma(zX) = \sigma(\alpha X + \beta JX) = \alpha\sigma X + \beta\sqrt{-1}\sigma X = z(\sigma X)$. Similarly, $\bar{\sigma}(zX) = \bar{z}(\bar{\sigma}X)$, and thus the maps $\sigma|_V$and $\bar{\sigma}|_V$are, respectively, a homomorphism and an antihomomorphism of $\boldsymbol{C}$ -linear spaces.

Let $\exists X \in V$and $\sigma X = 0$. Applying the operator to both parts of this identity τ, taking into account Lemma 2.1 and Proposition 2.1, we obtain that $\bar{\sigma}X = 0$, and hence $X = \sigma X + \bar{\sigma}X = 0$. Therefore, $\ker \sigma|_V = \{0\}$. Similarly, $\ker \bar{\sigma}|_V = \{0\}$, i.e. $\sigma$and $\bar{\sigma}$are monomorphism and antimonomorphism, respectively.

Let, finally $X \in D_J^{\sqrt{-1}}$. Consider the vector $Y = X + \tau X$. By Proposition 2.1, $Y \in V$. On the other hand, since, $X \in Im\,\sigma = \ker \bar{\sigma}$taking into account Lemma 2.1, we have: $\sigma Y = \sigma X + (\tau \circ \sigma)X = X + (\tau \circ \bar{\sigma})X = X$. Similarly, if $X \in D_J^{-\sqrt{-1}}$, then $\bar{\sigma}Y = X$, and, thus, $\sigma|_V$and $\bar{\sigma}|_V$are an epimorphism and an anti-epimorphism, respectively. □

Let, in particular, V be a real $\boldsymbol{R}$ -linear space, $\dim V = 2n$, and let $b = \{e_1, \dots, e_{2n}\}$be its basis as a $\boldsymbol{C}$ -module. Consider a system of vectors $b_A = \{\varepsilon_1, \dots, \varepsilon_n, \varepsilon_{\hat{1}}, \dots, \varepsilon_{\hat{n}}\}$, where $\varepsilon_a = \sigma(e_a), \varepsilon_{\hat{a}} = \bar{\sigma}(e_a); a = 1, \dots, n$. By virtue of Theorem 2.3, the vectors $\{\varepsilon_1, \dots, \varepsilon_n\}$form the basis of the $\boldsymbol{C}$-linear space $D_J^{\sqrt{-1}}$, and the vectors $\{\varepsilon_{\hat{1}}, \dots, \varepsilon_{\hat{n}}\}$form the basis of the $\boldsymbol{C}$ -linear space $D_J^{-\sqrt{-1}}$, and, by virtue of Lemma 2.1 and Proposition 2.1, $\tau\varepsilon_a = (\tau \circ \sigma)e_a = (\bar{\sigma} \circ \tau)e_a = \bar{\sigma}e_a = \varepsilon_{\hat{a}}$. Moreover, by virtue of Theorem 2.1, the system of vectors $b_A$forms a basis of the space $V^{\boldsymbol{C}}$, which is characterized by the fact that the endomorphism matrix Jin this basis has the form

$$(J_j^i) = \begin{pmatrix} \sqrt{-1}I_n & 0 \\ 0 & -\sqrt{-1}I_n \end{pmatrix}, \qquad (2.4)$$

Such a basis is called ***an adapted complex structure*** , in short ***A-basis*** .

Conclusion. Fixing a complex structure Jin a $2n$-dimensional real linear space $V^{\boldsymbol{R}}$induces the assignment of $V^{\boldsymbol{R}}$an n -dimensional complex linear space V *to the structures* . Each basis $b = \{e_1, \dots, e_{2n}\}$of the space V canonically induces two bases:

1) *RA is the* basis $b_{RA} = \{e_1, \dots, e_n, Je_1, \dots, Je_n\}$of the space $V^{\boldsymbol{R}}$. Of course, taking into account the canonical identification, $X \equiv 1 \otimes X$this basis can also be considered as the basis of the $\boldsymbol{C}$-linear space $(V^{\boldsymbol{R}})^{\boldsymbol{C}}$.

2) A -basis$b_A = \{\varepsilon_1, \dots, \varepsilon_n, \varepsilon_{\hat{1}}, \dots, \varepsilon_{\hat{n}}\}$ $\boldsymbol{C}$ -linear space $(V^{\boldsymbol{R}})^{\boldsymbol{C}}$.

Now let $b = \{e_1, \dots, e_n\}$and $\tilde{b} = \{\tilde{e}_1, \dots, \hat{e}_n\}$be two bases of the space V , $C = C_{b\tilde{b}} = (c_b^a)$be the transition matrix from basis $b$to basis $\tilde{b}$, $C = A + \sqrt{-1}B$, where $A = (\alpha_b^a)$and $B = (\beta_b^a)$are the real and imaginary parts of the matrix C , respectively. Since, taking into account (1.2), $\tilde{e}_a = c_a^b e_b = \alpha_a^b e_b + \beta_a^b (Je_b)$, we have: $J\tilde{e}_a = \alpha_a^b Je_b + \beta_a^b J^2 e_b = -\beta_a^b e_b + \alpha_a^b Je_b$, which means that

$$C_{b_{RA}\tilde{b}_{RA}} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}. \qquad (2.5)$$

Next, $\tilde{\varepsilon}_a = \sigma(\tilde{e}_a) = \sigma(c_a^b e_b) = c_a^b \sigma(e_b) = c_a^b \varepsilon_b$; $\tilde{\varepsilon}_{\hat{a}} = \bar{\sigma}(\tilde{e}_a) = \bar{\sigma}(c_a^b e_b) = \bar{c}_a^b \bar{\sigma}(e_b) = \bar{c}_a^b \varepsilon_{\hat{b}}$, which means

$$C_{b_a\tilde{b}_A} = \begin{pmatrix} C & 0 \\ 0 & \bar{C} \end{pmatrix}. \qquad (2.6)$$

Obviously, both matrices, $C_{b_{RA}\tilde{b}_{RA}}$and $C_{b_a\tilde{b}_A}$, can be treated as matrices of the same linear space endomorphism , $(V^{\boldsymbol{R}})^{\boldsymbol{C}}$namely, the endomorphism $f^{\boldsymbol{C}}$, where $f$is the space endomorphism $V^{\boldsymbol{R}}$, considered as an endomorphism of the space V, transforming the basis b into the basis $\tilde{b}$. In particular,

$$\det\begin{pmatrix} A & -B \\ B & A \end{pmatrix} = \det\begin{pmatrix} C & 0 \\ 0 & \bar{C} \end{pmatrix} = |\det C|^2. \qquad (2.7)$$

An important consequence of this relation is

Proposition 2.6. Fixing a complex structure in an $\boldsymbol{R}$ -linear space $V^{\boldsymbol{R}}$canonically determines the orientation of this space. It consists of bases oriented in the same way as any RA -basis.□

3. Almost complex structures and the associated *G* - structure

Definition 3.1. An ***almost complex structure*** on a manifold M is an anti-involutive endomorphism of a module $\mathcal{X}(M)$, i.e. $C^\infty(M)$-linear mapping $J: \mathcal{X}(M) \to \mathcal{X}(M)$such that $J^2 = -id$. An endomorphism $J$is also called a ***structural endomorphism*** . A manifold on which an almost complex structure is fixed is called an almost complex manifold. A diffeomorphism of an $f: M_1 \to M_2$almost complex manifold (M_1, J_1)onto an almost complex manifold $(M_2, J_2)$is called a holomorphic diffeomorphism if $f_* \circ J_1 = J_2 \circ f_*$.

It is obvious that an almost complex structure can be considered as a complex structure of the module $\mathcal{X}(M)$, considered as an $\boldsymbol{R}$ -linear space. As we have seen, the structure of a ***C -linear*** space is naturally induced on this linear space, and hence the structure of a $\boldsymbol{C} \otimes C^{\infty}(M)$-module, i.e., module over the ring of smooth complex-valued functions on the manifold M. The smoothness of such a function is understood as the smoothness of its real and imaginary parts. For a better understanding of this structure, it is convenient to use the alternative definition $\boldsymbol{C} \otimes C^{\infty}(M)$as a complexification of an $\boldsymbol{R}$ -linear space $C^{\infty}(M)$, according to $\boldsymbol{C} \otimes C^{\infty}(M) = C^{\infty}(M) \times C^{\infty}(M)$. If $(f,g) = f + \sqrt{-1}g \in \boldsymbol{C} \otimes C^{\infty}(M), X \in \mathcal{X}(M)$, then by definition $\left(f + \sqrt{-1}g\right)X = fX + g(JX)$.

Let be Jan almost complex structure on the manifold M. It induces complex structures $J_m: T_m(M) \to T_m(M)$at every point $m \in M$. In view of Proposition 2.5, the space $T_m(M)$, and hence the manifold M *itself* , are even-dimensional. Let $\dim M = 2n$. The number n is called ***the complex dimension of*** the manifold M.

Theorem 3.1. Specifying an almost complex structure on a smooth manifold $M^{2n}$is equivalent to specifying a G *-structure on this manifold* with the structure group $G = GL^R(n, \boldsymbol{C})$.

Proof. Let be Jan almost complex structure on the manifold M. Then, at each point $m \in M$, a family of $\mathcal{R}_m$frames of the space is defined $T_m(M)$, which is considered as an n -dimensional $\boldsymbol{C}$ -linear space. It follows from the definition of a frame that a group $GL(n, \boldsymbol{C})$acts in each such family freely and transitively.

Lemma 3.1. In some neighborhood U of an arbitrary point $m \in M$, one can construct a family of vector fields $\{e_1^0, \dots, e_n^0\}$on U that form the basis of a module $\mathcal{X}(U)$as a $\boldsymbol{C} \otimes C^{\infty}(U)$-module.

Proof. We fix some RA - frame $p = \{\xi_1, \dots, \xi_n, J_m\xi_1, \dots, J_m\xi_n\}$at the point m. As we know, a system of vectors $\xi_k$can be extended to a system of vector fields e_k^0 $(k = 1, \dots, n)$on M. In this case, the system of vectors $J_m\xi_k$will continue to the system of vector fields Je_k^0. Since the linear independence of the vectors of the frame p is equivalent to the inequality zero of the determinant of the transition matrix from the natural basis at the point m to the basic part of the frame p , this property is preserved in some neighborhood U of the point m and for some vector fields $\{e_1^0, \dots, e_n^0, Je_1^0, \dots, Je_n^0\}$. But then, obviously, the system $\{e_1^0, \dots, e_n^0\}$of vector fields on U will be $\boldsymbol{C} \otimes C^{\infty}(U)$-linearly independent, and hence forms a basis of the $\boldsymbol{C} \otimes C^{\infty}(U)$-module $\mathcal{X}(U)$.

□

We continue the proof of the theorem. Let us denote $\mathcal{R} = \bigcup_{m \in M} \mathcal{R}_m$, and introduce the natural projection $\pi: \mathcal{R} \to M$that assigns the vertex to the frame $p \in \mathcal{R}$. Now we can construct the mapping $F_U: \pi^{-1}(U) \to GL(n, \boldsymbol{C})$by setting $F_U(p) = g$, where $g$is the transition matrix from the frame $(m, e_1^0|_m, \dots, e_n^0|_m)$to the frame p. It is easy to verify that the quadruple $B_J(M) = \left(\mathcal{R}, M, \pi, G = GL(n, \boldsymbol{C})\right)$forms a principal bundle. This principal bundle can be considered as a G - structure with respect to the monomorphism $\left(\tilde{f}, \tilde{\rho}\right)$of the principal bundle $B_J(M)$into the principal bundle $B(M)$, where $\tilde{f}: \mathcal{R} \to BM$is the map that associates the $(m, e_1, \dots, e_n)$space frame $T_m(M)$as a $\boldsymbol{C}$ - module with the corresponding RA -frame, and $\tilde{\rho}: GL(n, \boldsymbol{C}) \to GL(2n, \boldsymbol{R})$is the canonical Lie group monomorphism that associates the matrix with $C = A + \sqrt{-1}B \in GL(n, \boldsymbol{C})$the matrix $\tilde{\rho}(C) = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$whose image is the Lie group $GL^{\boldsymbol{R}}(n, \boldsymbol{C})$.

Conversely, let be $\left(\mathcal{R}, M, \pi, GL^R(n, \boldsymbol{C})\right)$a G -structure of this type on M. Let be J_0a standard complex structure in the space $\boldsymbol{R}^{2n}$given by a matrix of the form (2.3). Let be $m \in M$an arbitrary point. We define an endomorphism $J_m$in space by the $T_m(M)$formula $J_m = p \circ J_0 \circ p^{-1}; p \in \pi^{-1}(m)$. Obviously, $J_m^2 = -id$, i.e. $J_m$is the complex structure on $T_m(M)$. Let us show that it is well defined in the sense of being independent of the choice of the element $p \in \pi^{-1}(m)$. Indeed, if $\tilde{p} \in \pi^{-1}(m)$is another such element, then $\exists h \in GL^R(n, \boldsymbol{C})$and $\tilde{p} = ph$. Therefore, $\tilde{p} \circ J_0 \circ \tilde{p}^{-1} = (ph) \circ J_0 \circ (ph)^{-1} = p \circ (h \circ J_0 \circ h^{-1}) \circ p^{-1} = p \circ J_0 \circ p^{-1} = J_m$since the group $GL^R(n, \boldsymbol{C})$is obviously an endomorphism invariance group J_0, i.e., $hJ_0 = J_0h; h \in GL^{\boldsymbol{R}}(n, \boldsymbol{C})$, which is checked directly.

Let us show that the family of tensors $J = \{J_m | m \in M\}$defines a tensor field on the manifold M . To do this, it suffices to prove that for any admissible map (U, φ)on M the functions $m \to J_j^i(m) = dx^i\left(J_m\left(\frac{\partial}{\partial x^j}\Big|_m\right)\right), m \in M,$ are smooth on U . Let us fix a local section $s: U \to \mathcal{R}$of the space of the G -structure. Then, by construction, in the RA -frame $\sigma(m)$(and dual coreframe) we have: $\left(e^i\left(J_m(e_j)\right)\right) = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} = \left((J_0)_j^i\right)$. The smoothness of the section is expressed in the fact that the components of the matrix C of the transition from the natural basis of the module $\mathcal{X}(U)$to the RA -basis $\sigma(U)$of this module, and hence the components of the inverse matrix $\tilde{C}$, are smooth functions. Hence,$m \to J_j^i(m) = dx^i\left(J_m\left(\frac{\partial}{\partial x^j}\Big|_m\right)\right) =$
$= C_k^i(m)e^k\left(J_m\left(\tilde{C}_j^r(m)e_r\right)\right) = C_k^i(m)\tilde{C}_j^r(m)e^k\left(J_m(e_r)\right) = C_k^i(m)\tilde{C}_j^r(m)(J_0)_r^k$are smooth functions on U . Thus, $J$is an almost complex structure. Obviously, the family of RA -frames generated by it coincides with the space of the G -structure.

□

Theorem 3.2. Every almost complex manifold is even-dimensional and orientable.

Proof. The even-dimensionality of an almost complex manifold (M^{2n}, J)follows from Proposition 2.5 applied to any linear space of the form $T_m(M); m \in M$. The orientability of this manifold, by definition, means the existence on it of a differential -form τthat does not vanish anywhere $2n$. But its existence is obvious for any neighborhood of local triviality of the bundle $B_J(M)$. Indeed, if U is such a neighborhood, $s: U \to \mathcal{R}$is a section of this bundle over it, given by vector fields $\{e_1, \ldots, e_n, Je_1, \ldots, Je_n\}$, then, by Proposition 2.6, it suffices to set $\tau_U = e_1 \wedge \ldots \wedge e_n \wedge J^*e_1 \wedge \ldots \wedge J^*e_n$, where $J^*(u)(X) = u(JX); X \in \mathcal{X}(U), u \in \mathcal{X}^*(U)$. Let now $\{\psi_\alpha\}_{\alpha \in A}$be a partition of unity subject to the covering of the $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$manifold M *by* the local triviality domains of the bundle $B_J(M)$. Since the manifold is paracompact, this cover can be considered locally finite without loss of generality. Let be $\tau_\alpha$the $2n$-form constructed for the domain U_α; $\alpha \in A$. Then $\tau = \sum_{\alpha \in A} \psi_\alpha \tau_\alpha$is a well-defined $2n$-form on M . Indeed, due to the local finiteness of the cover $\mathfrak{U}$in some neighborhood U of each point , the $m \in M$form $\tau|_U$is the sum of at most a finite number of smooth forms $(\tau_\alpha)|_U$, and hence is τa □nowhere vanishing $2n$form on M.

Remark 3.1. The even-dimensionality and orientability of a manifold are thus necessary conditions for the existence of an almost complex structure on this manifold. However, these conditions are not sufficient. For example, a well-known deep result of a topological nature is the assertion that an $2n$-dimensional sphere S^{2n}admits an almost complex structure if and only if $n = 1$either $n = 3$(see [5]). Therefore, for example, a 4-dimensional sphere, being, as is well known, an even-dimensional orientable manifold, does not admit an almost complex structure. The question of finding necessary and sufficient conditions for the existence of an almost complex structure on a smooth manifold is still open.

Remark 3.2. Along with the principal bundle of VM frames over a smooth manifold M^n, we can consider a more extensive principal bundle of complex frames over M , which we denote $B^{\boldsymbol{C}}(M) = \left(B^{\boldsymbol{C}}M, M, \pi, GL(n, \boldsymbol{C})\right)$by , where $B^{\boldsymbol{C}}M$is the union of all frames of the spaces $\left(T_m(M)\right)^{\boldsymbol{C}}; m \in M$. The corresponding justifications do not differ in any way from the corresponding justifications for the main bundle of the WM . This principal bundle plays a particularly important role for almost complex manifolds (M^{2n}, J), since it allows, along with the G -structure constructed above, ***to** consider* another $B_J(M)$defining G -structure $(m, e_1, \ldots, e_n)$defined by the monomorphism $(f, \rho)$of $T_m(M)$the principal bundle $B_J(M)$into the principal bundle $B^{\boldsymbol{C}}M$, where $f: \mathcal{R} \to B^{\boldsymbol{C}}MA$ is a frame, and $\rho: GL(n, \boldsymbol{C}) \to GL(2n, \boldsymbol{C})$is a canonical monomorphism of Lie groups that associates a matrix with $C \in GL(n, \boldsymbol{C})$a matrix $\rho(C) = \begin{pmatrix} C & 0 \\ 0 & \bar{C} \end{pmatrix} \in GL(2n, \boldsymbol{C})$. As above, it is proved that specifying such a G -structure is equivalent to specifying the original almost complex structure. This G -structure will be especially important for our subsequent considerations; we call it a G -structure attached to an almost complex structure.

4. Hermitian structures

Definition 4.1. Let V be a real linear space. A Hermitian structure on V is a pair $(J, g = \langle\cdot,\cdot\rangle)$, where $J$is a complex structure on V , $g = \langle\cdot,\cdot\rangle$is a (pseudo) Euclidean structure, and

$$\langle JX, JY\rangle = \langle X, Y\rangle, X, Y \in V. \quad (4.1)$$

Let be $(J, g = \langle\cdot,\cdot\rangle)$a Hermitian structure on V . Let us construct a mapping $\Omega: V \times V \to R$by setting $\Omega(X, Y) = \langle X, JY\rangle, X, Y \in V$. Obviously $\Omega(Y, X) = \langle Y, JX\rangle = \langle JY, J^2X\rangle = -\langle JY, X\rangle = -\langle X, JY\rangle = -\Omega(X, Y)$. Thus, $\Omega$is an outer 2-form on V. It is called **the fundamental form** of structure. Obviously, its skew-symmetry is equivalent to the identity

$$\langle JX, Y\rangle = -\langle X, JY\rangle;\ X, Y \in V, \quad (4.2)$$

which, in turn, is equivalent to (4.1). An obvious consequence of this identity is the important relation

$$\langle X, JX\rangle = 0;\ X \in V. \quad (4.3)$$

Recall that a Hermitian form on a complex linear space W is a mapping $h: W \times W \to \boldsymbol{C}$such that:

The first two properties are, as usual, called ***additivity*** , the third, ***sesquilinearity*** , and the fourth, ***hermitian*** . The notions of nondegeneracy and positive definiteness of a Hermitian form are defined in the usual way. The non-degenerate Hermitian form will often be called the ***Hermitian metric*** , and the ***C*** -linear space in which the Hermitian metric is fixed will be called the ***Hermitian space*** .

Theorem 4.1. Specifying a Hermitian structure $(J, \langle\cdot,\cdot\rangle)$in a linear space V *is* equivalent to specifying a non-degenerate Hermitian form $h = \langle\langle\cdot,\cdot\rangle\rangle$in V , considered as a ***C*** **-** linear with respect to Jspace. The positive definiteness of a form is $\langle\langle\cdot,\cdot\rangle\rangle$equivalent to the positive definiteness of a bilinear form $\langle\cdot,\cdot\rangle$.

Proof. Let be $(J, \langle\cdot,\cdot\rangle)$a Hermitian structure on V . Let $\langle\langle X, Y\rangle\rangle = \langle X, Y\rangle + \sqrt{-1}\langle X, JY\rangle; X, Y \in V$. Taking into account (4.1) and (4.2), it is obvious that $\langle\langle JX, Y\rangle\rangle = \langle JX, Y\rangle + \sqrt{-1}\langle JX, Y\rangle = \sqrt{-1}\langle X, Y\rangle - \langle X, JY\rangle =$
$\sqrt{-1}\big(\langle X, Y\rangle + \sqrt{-1}\langle X, JY\rangle\big) = \sqrt{-1}\langle\langle X, Y\rangle\rangle$.
Similarly, $\langle\langle X, JY\rangle\rangle = -\sqrt{-1}\langle\langle X, Y\rangle\rangle$, whence, taking into account the definition of a ***C*** -module in V, it follows that the form $\langle\langle\cdot,\cdot\rangle\rangle$is linear in the first and antilinear in the second arguments. In addition, $\langle\langle Y, X\rangle\rangle = \langle Y, X\rangle + \sqrt{-1}\Omega(Y, X) = \langle X, Y\rangle - \sqrt{-1}\Omega(X, Y) = \overline{\langle\langle X, Y\rangle\rangle}$. Thus, $\langle\langle\cdot,\cdot\rangle\rangle$is a Hermitian form on V . Obviously, it is non-degenerate.

Conversely, let h be a non-degenerate Hermitian form in V . Consider the bilinear forms $g = \Re h$and $\Omega = \Im h$- the real and imaginary parts of the form h , respectively. Thus, $h(X, Y) = g(X, Y) + \sqrt{-1}\Omega(X, Y); X, Y \in V$. Since $g(X, Y) + \sqrt{-1}\Omega(X, Y) = h(X, Y) = \overline{h(Y, X)} = g(Y, X) - \sqrt{-1}\Omega(Y, X)$, then, comparing the real and imaginary parts, we have:

$$1)\ g(X, Y) = g(Y, X);\ \ 2)\ \Omega(X, Y) = -\Omega(Y, X). \quad (4.4)$$

Next, $\sqrt{-1}g(X, Y) - \Omega(X, Y) = \sqrt{-1}h(X, Y) = -h(X, JY) = -g(X, JY) - \sqrt{-1}\Omega(X, JY)$.
Comparing the real and imaginary parts, we get that
$1)\ \Omega(X, Y) = g(X, JY);\ \ 2)\ g(X, Y) = -\Omega(X, JY)$.

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