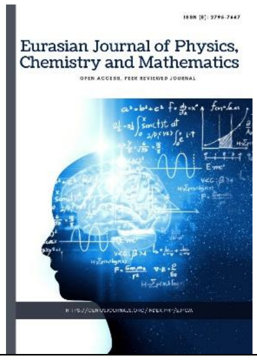

Energy- Momentum Relation in a Curved Space– Time Using General Relativity and Potential Dependent Special Relativity

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ABSTRACT

A new energy– momentum relation has been derived using the proper space – time expression in a curved space time. This relation link general and special relativity together. It describes how gravity affect particle energy. It also promotes the special relativity energy relation by recognizing the effect of fields through the potential term. This new expression reduces to the special relativistic one in a Minkoskian space. It also reduces to the Newtonian one for low speeds compared to the speed of light. This expression shows also that the energy is strongly affected by the space- time deformation through the time and space metrices. This new energy- momentum relation which link general with special relativity indicates that Einstein relativity theory is still one of the big achievements in physics.

Keywords: Curved Space Time, General Relativity, Special Relativity, Potential, Energy, Momentum

1. Introduction

The recent history of science shows that the nature of the gravitation starts from the Newton gravitational inverse square law which assumes that the gravity force is attractive [1]. The Newton gravitational law describes a large number of astronomical observations [2]. However many new astronomical phenomena can not be explained within the frame work of Newton gravitational law. These include the deflection of light by the sun, the observed red shift of light coming from remote stars, beside the discovered relic microwave background [3]. This motivates Einstein to generalize his special relativistic (SR) Minkowskian 4-dimensional space- time to be capable of describing a curved space- time using Reimman geometry. This leads to the born of the so called general relativity (GR), which describes gravity using geometrical language. According to his version, any gravitating mass influence the near by mass by deforming the space- time instead of spreading field lines. Einstein GR theory successfully describes the afore noted phenomena that can not be described by Newton theory of gravitation [4]. The so called big bang model is the cosmological model based on GR to describe the nature and evolution of the universe [5]. Despite these successes of SR and GR but them both suffer from noticeable setbacks namely that related to the notion of energy. In SR the expression of energy does not recognize the effect of fields through the potential term. For example if one have three particles like electrons, one move in vacuum and the others are move in a gravity field and nuclear field with the same speed. The three particles have the same SR energy, which is in direct conflict with gravitational laws and the quantum laws, which were confirmed experimentally [6, 7]. Moreover SR energy expression does not reduced to that of Newton for small speeds compared to the speed of light when neglecting the rest mass energy. This reduced SR energy gives only kinetic energy. The energy-momentum relation in a curved space time and in other fields does not intensively investigated. Only few attempts made like the models proposed by Savickas and M. Dirar [8, 9]. Savickas model needs to be promoted to satisfy Newtonian limit and describe the energy for all fields, although he found energy in a curved space- time [10],[11]. Mubarak Dirar model and ltilo model derives the energy-momentum relation for all fields and satisfies SR and Newtonian limit, but nothing is done for a curved space [12]. Another approach to promote the energy concept for the gravitational field is done Arbab by suggesting magnetic gravity counterpart of the electromagnetic field. A though his approach is promising, but unfortunately its only restricted to gravity without recognizing other fields [13]. The effect on space and time is not mentioned. This motivates to use in this work, the curved space proper time to construct a useful expression for energy. This is done with the discussion in section (2). Section (3) is devoted for conclusion.

2. Energy- Momentum Relation in a Curved Space- Time

In GR the proper space- time length which is assumed to be invariant takes the form

$$ds^2 = -ds_o^2 = -g_{\mu\nu} dx^\mu \, dx^\nu$$

$$ds_o^2 = g_{\mu\nu} dx^\mu \, dx^\nu \qquad (1)$$

For one spatial dimension say x- axis the proper space time length is given by:

$$ds_o^2 = g_{oo} dx^o + g_{11} dx^{r^2} = g_{oo} c^2 \, dt^2 + g_{xx} dx^2$$

Thus

$$ds^2 = -c^2 \, dt^2 = g_{oo} \, c^2 \, dt^2 \, g_{xx} \, dx^2$$

There fore

$$ds_o^2 = c^2 \, dt^2 = g_{oo} \, c^2 \, dt^2 + g_{xx} \, dx^2 \qquad (2)$$

For Minkoskian limit and free space

$$g_{oo} = 1 \quad , \quad g_{xx} = -1 \qquad (3)$$

Thus equation (2) reduces to

$$c^2 \, dt^2 = c^2 \, dt^2 - dx^2 \qquad (4)$$

Which is the ordinary SR expression for the proper length. Rearranging equation (4) yields.

$$1 = g_{oo} \left(\frac{dt}{d\tau}\right)^2 + \frac{g_{xx}}{c^2} \left(\frac{dx}{d\tau}\right)^2$$

$$= \left(\frac{dt}{d\tau}\right)^2 \left[g_{oo} + \frac{g_{xx}}{c^2} \left(\frac{dx}{d\tau}\right)^2\right] = \gamma^2 \left[g_{oo} + \frac{g_{xx}}{c^2} \, u^2\right] \qquad (5)$$

Where γ satisfies

$$\gamma = \left(\frac{dt}{d\tau}\right) = \frac{1}{\sqrt{g_{oo} + g_{xx} \frac{u^2}{c^2}}} \qquad (6)$$

To see how the energy in a curved space looks like, multiply both sides of equation (5) by $m_o^2 \ c^4$, to get

$$m_o^2 \ c^4 = \gamma^2 \, m_o^2 \ c^4 \left[g_{oo} + \frac{g_{xx}}{c^2} \, u^2 / c^2\right] = m^2 c^4 \ \left[g_{oo} + g_{xx} \ u^2 / c^2\right]$$

$$= g_{oo} \, m^2 c^4 + g_{xx} \, m^2 u^2 c^4 = g_{oo} \, E^2 + g_{xx} \, p^2 c^2 \qquad (7)$$

Where one follows SR defining

$$m = \gamma \, m_o$$

$$E = m \, c^2 = m_o c^2 \qquad (8)$$

$$\sqrt{g_{oo} + g_{xx} \frac{u^2}{c^2}} \qquad (9)$$

For free space, one can use equation (3), in equation (6), (7) and (9) to get

$$\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$E^2 = m_o c^2 + p^2 c^2$$

$$E = \frac{m_o\, c^2}{\sqrt{1 - \frac{u^2}{c^2}}} \qquad (10)$$

Which are the ordinary SR relativistic relations. One can also use the weak field approximation used in GR, i.e

$$g_{oo} = \left(1 + {}^{2\varphi}/_{c^2}\right) , \quad g_{xx} = -1 \qquad (11)$$

Where φ is a potential per unit mass for any field in equation (9) to get

$$E = m_o c^2 \left(g_{oo} - {}^{v^2}/_{c^2}\right)^{-\frac{1}{2}} \qquad (12)$$

$$E = m_o c^2 \left(1 + {}^{2\varphi}/_{c^2} - {}^{v^2}/_{c^2}\right)^{-\frac{1}{2}} \qquad (13)$$

Using the identity

$$(1 + x)^n = 1 + nx \qquad (14)$$

For small x, where

$$x = \frac{2\varphi}{c^2} - {}^{v^2}/_{c^2}$$

$$n = -\frac{1}{2} \qquad (15)$$

One thus gets equation (13) in the form

$$E = m_o c^2 \left(1 - \frac{1}{2}\, x\right) = m_o c^2 \left[1 - \frac{\varphi}{c^2} + \frac{1}{2}\ \frac{v^2}{c^2}\right]$$

$$E = m_o c^2 - m_o \varphi + \frac{1}{2}\, m_o v^2 \qquad (16)$$

Since the potential energy$V$, and kinetic T, are defined by

$$V = m_o \varphi \quad , \quad T = \frac{1}{2}\, m_o v^2 \qquad (17)$$

It follows that

$$E = m_o c^2 + T - V \qquad (18)$$

Which, when neglect rest mass term does not satisfy the Newtonian limit, which has energy given by

$$E_N = T + V \qquad (19)$$

But fortunately it resembles the lagrangian function L when rest mass term is neglected, i.e

$$L_N = E = T - V \qquad (20)$$

This set back may be related to the use of vacuum time instead of curved space one in the velocity definition in equation (5). However one needs a more precise, proper energy expression in a curved space, which reduces to SR in free space, and ordinary Newtonian one a weak field. Despite this short coming expression of (18) for not satisfying the Newtonian limit, but this form shows the flexibility of the curved space energy form. This stems from the fact that the curved space energy relation (18) represents the lagrangian function which is widely used in all physical branches including Newton laws, Maxwell's equations, quantum mechanics, quantum field theories and unification theories.

To do this, one needs to return back to the original definition of the proper space- time interval (ds) which is defined in SR to be

$$\underline{ds} = i\ c\ dt\ \hat{e_o} + dx\ \hat{e_1} \qquad (21)$$

Thus the square of the proper interval (21) is given by

$$ds^2 = \underline{ds} \,.\, \underline{ds} = i^2\ c^2\ dt^2 + dx^2 = -c^2\ dt^2 + dx^2 \qquad (22)$$

Since one needs use of proper time τ it is suitable to use equation (2) to get

$$ds_o^2 = -ds^2 = c^2\, dt^2 - dx^2 \qquad (23)$$

Which represents Minkowskian space form as pointed out by equation (3). For a curved space- time the proper time τ can be written according to equation (2), to take the form

$$ds_o^2 = g_{oo}\, c^2\, dt^2 + g_{xx}\, dx^2 \qquad (24)$$

The proper time τ is defined by

$$dt = d\tau\,, \quad dx = 0 \qquad (25)$$

Thus equation (24) gives

$$ds_o^2 = g_{oo}\, c^2\, d\tau^2 \qquad (26)$$

One can follow Hilo and Savickas [8, 9] to define the time in a curved space to be given by

$$dt_o = \sqrt{g_{oo}}\, dt \qquad (27)$$

Inserting equation (26) in equation (24) and using equation (27) gives

$$g_{oo}\, c^2\, d\tau^2 = g_{oo}\, c^2\, dt^2 + g_{xx}\, dx^2 = c^2\, dt_o^2 + g_{xx}\, dx^2$$

Rearranging gives

$$g_{oo} = \left(\frac{dt}{d\tau}\right)^2 g_{oo} + \frac{g_{xx}}{c^2}\left(\frac{dx}{d\tau}\right)^2 \qquad (28)$$

Further rearrangement in which the curved time is used to define velocity yields

$$g_{oo} = \left(\frac{dt}{d\tau}\right)^2 \left[g_{oo} + \frac{g_{xx}}{c^2}\left(\frac{dx}{d\tau}\right)^2\right] \qquad (29)$$

$$g_{oo} = \frac{1}{g_{oo}}\left(\frac{\sqrt{g_{oo}}\, dt}{d\tau}\right)^2 \left[g_{oo} + \frac{g_{xx} g_{oo}}{c^2}\left(\frac{dx}{\sqrt{g_{oo}}\, dt}\right)^2\right] \qquad (30)$$

$$g_{oo} = \frac{1}{g_{oo}}\left(\frac{dt_o}{d\tau}\right)^2 \left[g_{oo} + \frac{g_{oo} g_{xx}}{c^2}\left(\frac{dx}{dt_o}\right)^2\right] \qquad (31)$$

$${g_{oo}}^2 = \gamma^2 \left[g_{oo} + g_{oo} g_{xx}\frac{v^2}{c^2}\right] \qquad (32)$$

Where γ is defined by

$$\gamma = \frac{g_{oo}}{\sqrt{g_{oo} + g_{oo} g_{xx}\frac{v^2}{c^2}}} = \frac{g_{oo}}{\sqrt{g_{oo}}\sqrt{1 + g_{xx}\frac{v^2}{c^2}}} \qquad (33)$$

$$\gamma = \frac{g_{oo}}{\sqrt{1 + g_{xx}\frac{v^2}{c^2}}} \qquad (34)$$

Now multiply both sides of equation (32) by $m_o^2\ c^4$ to get

$$g_{oo}^2\, m_o^2\ c^4 = g_{oo}\, \gamma^2\, m_o^2\ c^4 + g_{oo} g_{xx}\, \gamma^2 \frac{m_o^2\ c^4 v^2}{c^2} \qquad (35)$$

Now define the mass m, energy E, and momentum P to be

$$m = \gamma\, m_o \qquad (36)$$

$$E = mc^2 = \gamma\, m_o c^2 \qquad (37)$$

$$P = mv = \gamma\, m_o v \qquad (38)$$

Using equations (36), (37) and (38), in equation (35) yields

$$g_{oo}^2\, m_o^2\ c^4 = g_{oo}\, E^2 + g_{oo} g_{xx}\, p^2 c^2 \qquad (39)$$

Which is the energy- momentum relation in a curved space- time.

For Minkowskian free space [see equation (3)]

$g_{oo} = 1 \quad , \quad g_{xx} = -1$

Thus equations (34), (37), (38) and (39) gives

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$E = \gamma\, m_o\, c^2$$

$$P = \gamma\, m_o\, v \qquad (40)$$

Which are the ordinary SR relations.

Now consider a weak field limit in equation (11), then substitute it in (34) to get

$$\gamma = \left(1+\frac{2\varphi}{c^2}\right)^{\frac{1}{2}} \left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} \qquad (41)$$

Using the identity (14) with $\left(x = \frac{2\varphi}{c^2}, -\frac{v^2}{c^2}\right)$ and $\left(n = \frac{1}{2}, -\frac{1}{2}\right)$, yields

$$\gamma = \left(1+\frac{\varphi}{c^2}\right)\left(1+\frac{1}{2}\frac{v^2}{c^2}\right) = 1+\frac{1}{2}\frac{v^2}{c^2}+\frac{\varphi}{c^2}+\left(\frac{\varphi}{c^2}\right)\left(\frac{v^2}{2c^2}\right) \quad (42)$$

Neglect higher order terms to get

$$\gamma = 1+\frac{1}{2}\frac{v^2}{c^2}+\frac{\varphi}{c^2} \qquad (43)$$

A direct substitution of equation (43) in (37) gives

$$E = m_o\, c^2\, \gamma = m_o\, c^2 \left(1+\frac{1}{2}\frac{v^2}{c^2}+\frac{\varphi}{c^2}\right)$$

$$E = m_o\, c^2 + \frac{1}{2} m_o\, c^2 + m_o \varphi$$

Neglect the rest mass term and use equation (17) to get, [see equation (19)]

$$E = T + V = E_N$$

This means that the relativistic expression of energy (37), with γ defined according to equation (34) satisfies the Newtonian limit.

3. Conclusion

The curved space energy- momentum relation shows that strong dependence on the space-time curvature via the metric terms. This means that the energy is considerably affected by the space- time deformation. Fortunately, this relation reduces to the SR one in vacuum. It is also satisfying the Newtonian limit for small speeds compared to the speed of light. This relation also recognizes the effect of the field through the potential term.

References

1. Raymond A. Serway, physics for Scientists and Engineers with modern physics, Saunders college publishing, USA, 2004.
2. M. Carmeli, Classical fields, general relativity and guage theory , John Wiley and sons, New York, 1992.
3. S. Weinbery, Gravitation and cosmology, John Wiley and sons, New York, 1972.
4. Derek F. Lowden, An introduction to tensor calculus and relativity, John Wiley and sons, New York, 1982.
5. Narlikar, J. V, introduction to cosmology 2nd Edition, Cambridge university press, Cambridge, 1993.
6. M. D. Abd alla, A. EL-Tahir, M. H. Eisa, Abdulaziz S. ALaamer, M. ELnabhani, K. G.Elgaylani, Gravitational self energy mass and gravitational radiation quantization within the framework of generalized general relativity, International Journal of Astronomy and Astrophysics, 3, 131- 136. 2013.
7. M. H. M. Hilo, R. Abd Elhai, M. D. Abd Allah, Amel A. A. Elfaki, Deriving the generalized special relativity using mirror Clcck and Lorentz transformations, Natural Science, 6,1275- 1281. 2014.
8. D. Savickas, Relation Between Newtonian Mechanics, General relativity and Quantum Mechanics, Am. J. Phy. 70 (8), 2002.
9. Hilo, M. H. M, Using of the Generalized Special Relativity in Deriving the Equation of the Gravitational Red Shift, Journal of Modern Physics, 2,370- 373. 2011.
10. Mahmoud, H. A. M., Abdalgadir, L. M., Rashed, M. M., Widaa, E. M. A., & Alshekh, M. A. (2022). Energy Conservation for Fluids and Radiation Beams Using Pressure Notion to Account for Thermal, Mechanical, and Frictional Collisional Energy. NeuroQuantology, 20(4), 389-392.
11. Mahmoud, H. M., & Abdalgadir, L. M. (2021). Relativistic Quantum Equation For Electromagnetic Field And Bio

Photons. NVEO-NATURAL VOLATILES & ESSENTIAL OILS Journal| NVEO, 3020-3030.

12. Abdelgadir, H. M. (2016). Mass and Charge Generation Mechanisms Within Framework of Generalized Lagrangian and Generalized Special Relativity (Doctoral dissertation, Sudan University of Science and Technology).
13. Arbab Ibrahim Arbab. The Generalized Newton's Law of Gravitation Versus the General Theory of Relativity, Journal of Modern Physics, 2012,3,1231- 1235. 2012.