On Hesitant Fuzzy MA-Ideals

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ABSTRACT

In this research, we study the definition of Hesitant Fuzzy set, with some properties. We introduced the Hesitant fuzzy ideal of a ring with the result equivalent to the definition. We proved some of the characteristics for Hesitant fuzzy set and Hesitant fuzzy ideal. Also, we proved the image and the inverse image of the Hesitant fuzzy ideal with respect the homomorphism between two rings. We studied the level set and then proved that it is ideal of ring through Hesitant Fuzzy.

Keywords:

1 Introduction

The concept of fuzzy sets was first introduced by L. A. Zadeh [11] in 1965 as an extension of the classical notion of a set. Torra and Y.Narukawa [9] (2009) and Torra[10]in(2010) proposed a new generalized type of fuzzy set called Hesitant fuzzy set (HFS) and he defined the complement, union and intersection of HFSs. Xia and Xu (2011) gave some operational laws for HFSs, such as the addition and multiplication operations . Mohammad Abbasi , Aakif F. Talee , Sabahat A. Khan, and Kostaq Hila[1]in (2018) introduced the hesitant fuzzy ideal, hesitant fuzzy bi-ideal, and hesitant fuzzy interior ideal in Γ-semigroup. Kim, Lim and Lee [3]in (2019) defined the hesitant fuzzy subgroupoid, hesitant fuzzy subgroup , hesitant fuzzy subring.

The remainder of the paper is organized as follows: in section two, we recall some definition along with some properties of Hesitant fuzzy set and some results. In section three, Hesitant fuzzy ideal of ring is presented. We establish some results on an operations of Hesitant fuzzy ideal of ring, in section four. Finally, Homomorphism on Hesitant fuzzy ideal of ring introduced in section five.

2. Preliminaries

In this section, we recall the following definitions and some results which needed in the next sections.

Definition 2-1[9,10]:- Let X be a fixed set, a hesitant fuzzy set (in short, HFS) on X is in terms of a function that when applied to X returns a subset of [0,1], that is $h: X \rightarrow \mathrm{P}[0,1]$. To be easily understood,Xia and Xu expressed the HFS by a mathematical symbol:$A = \{< x, h_A(x) > / x \in X\}$where $h_A(x)$ is a set of some values in [0,1], denoting the possible membership degrees of the element $x \in X$ to the set A. They called $h = h_A(x)$ is a hesitant fuzzy element (HFE). We will denote the set of all hesitant fuzzy sets in X as $HFS\ (X)$.

Example2-2:- Let $X=\{x_1, x_2, x_3\}$ be a reference set, and $h_A(x_1) = \{0.5,0.7,0.9\}$, $h_A(x_2) = \{0.2,0.5,0.6\}$ $h_A(x_3) = \{0.4,0.7,0.8\}$, then we can express the HFS A as:-

$A=\{< x_1, \{0.5,0.7,0.9\} >, < x_2, \{0.2,0.5,0.6\} >, < x_3, \{0.4,0.7,0.8\} >\}$.

Definition 2-3:-[5,6,7,8] Let h, $h_1, h_2 \in$ HFS(X) and $\{h_i / i \in I\} \subset$ HFS(X), then for each $x \in X$

1. We say that h_1 is a subset of h_2, denoted by $h_1 \subset h_2$, if $h_1(x) \subset h_2(x), \forall x \in X$.
2. The complement of h, denoted by h^c, is a hesitant fuzzy set in X defined as: $\forall x \in X$,
$h^c(x) = \{1-\gamma / \gamma \in h(x)\}$.
3. Lower bound: $h^-(x) = \min\{h(x)\}$.
4. α –lower bound: $h_\alpha^-(x) = \{\gamma \in h(x)/\gamma \le \alpha\}$.
5. Upper bound: $h^+(x) = \max\{h(x)\}$.
6. α –upper bound: $h_\alpha^+(x) = \{\gamma \in h(x)/\ge \alpha\}$.
7. Union: $(h_1 \cup h_2)(x) = \bigcup_{\gamma_1 \in h_1(x), \gamma_2 \in h_2(x)} \max\{\gamma_1, \gamma_2\}$.
8. Intersection: $(h_1 \cap h_2)(x) = \bigcup_{\gamma_1 \in h_1(x), \gamma_2 \in h_2(x)} \min\{\gamma_1, \gamma_2\}$.
9. $h^\lambda(x) = \bigcup_{\gamma \in h(x)}\{\gamma^\lambda\} \cong \{\gamma^\lambda / \gamma \in h(x)\}$.
10. $\lambda h(x) = \bigcup_{\gamma \in h(x)}\{1-(1-\gamma)^\lambda\}$.
11. $(\bigcup_{i \in I} h_i)(x) = \bigcup_{i \in I} h_i(x)$.
12. $(\bigcap_{i \in I} h_i)(x) = \bigcap_{i \in I} h_i(x)$.
13. $(h_1 \oplus h_2)(x) = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}$.
14. $(h_1 \otimes h_2)(x) = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{\gamma_1\gamma_2\}$.
15. $(h_1 \ominus h_2) = \{t / \gamma_1 \in h_1(x), \gamma_2 \in h_2(x)\}$.

Where
$$t = \begin{cases} \frac{\gamma_1-\gamma_2}{1-\gamma_2} & \text{if } \gamma_1 \ge \gamma_1, \gamma_2 \ne 1 \\ 0 & \text{other wise} \end{cases}$$

16 . $(h_1 \oslash h_2)(x = \{t / \gamma_1 \in h_1(x), \gamma_2 \in h_2(x)\}$.

Where
$$t = \begin{cases} \frac{\gamma_1}{\gamma_2} & \text{if } \gamma_1 \le \gamma_2, \gamma_2 \ne 0 \\ 1 & \text{other wise} \end{cases}$$

17. $(h_1 \circ h_2)(x) = \begin{cases} \bigcup_{yz=x}[h_1(y) \cap h_2(z)] & \text{if } yz = x \\ \Phi & \text{if otherwise} \end{cases}$

Definition 2-4:-[3]. Let $h \in$ HFS(X). Then h is called a hesitant fuzzy point with the support $x \in X$ and the value λ, denoted by x_λ, if $x_\lambda : X \to P[0, 1]$ is the mapping given by:

for each $y \in X$, $x_\lambda(y) = \begin{cases} \lambda \subset [0,1] & \text{if } y \ne x \\ \Phi & \text{other wise} \end{cases}$

We will denote the set of all hesitant fuzzy points in X as HFP(X).

Definition.2-5:[3] Let $h \in$ HS(X) and let $x_\lambda \in$ HFP(X). Then x_λ is said to be belong to h, denoted by $x_\lambda \in h$, if $\lambda \subset h(x)$.

Example.2-6:- Suppose that X={a,b},and let h_1, h_2 be two hesitant fuzzy sets given by: $h_1(a) = \{0,0.4,0.7\}, h_1(b) = [0,0.6)$ and let $\lambda = \{0,0.4,0.5\} \in P[0,1]$
,then $a_\lambda(a) = \emptyset$ and $a_\lambda(b) = \{0,0.4,0.5\}$, $b_\lambda(b) = \emptyset$ and $b_\lambda(a) = = \{0,0.4,0.5\}$

3.Hesitant Fuzzy Ideal Of Ring:

Definition3-1:-[1,2,3] Let G be a group and $h \in$ HFS(G).Then h is called a hesitant fuzzy subgroup in G, if it satisfies the following conditions: For any $x, y \in G$

1-$h(xy) \supseteq h(x) \cap h(y)$
2-$h(x^{-1}) \supseteq h(x)$

We will denote the set of all HFGs in G by HFG(G).

Example3-2:-Let (Z,+) be group and $h: Z \to P[0,1]$, difined as follows : For each h ∈Z

$$h(x) = \begin{cases} \left[\frac{1}{2}, \frac{4}{5}\right] & \text{if x is odd} \\ \left[\frac{1}{3}, \frac{2}{3}\right] & \text{if x is even} \\ [0,1] & \text{if } x = 0 \end{cases},$$

Then we can easily that h hesitant fuzzy ideal of Z.

proposition 3-3:- Let G be a group and $h \in$ HFG(G) , then
(1) $h(x^{-1}) = h(x)$, for each $x \in G$
(2) $h(e) \supseteq h(x)$, for each $x \in G$, where e is the identity of G

Definition3-4:-[3].Let (R,+,.) be a ring and $h \in$ HFS(R), $h \ne \emptyset$. Then h is called a hesitant fuzzy sub ring (in short, HFR), if satisfies the following conditions.
(1) $h \in$ HFG(R, +)
(2) $h \in$ HFG(R, .).
We will denote the set of all HFRs in R as HFR(R).

Theorem 3-5:-[3] Let (R,+,.) be a ring and $h \in$ HFS(R), $h \ne \emptyset$. Then h is a hesitant fuzzy sub ring if and only if, for any x,y ∈ R .
1- $h(x-y) \supseteq h(x) \cap h(y)$
2- $h(xy) \supseteq h(x) \cap h(y)$

Definition3-6:-[3] Let (R,+,.) be a ring and h be hesitant fuzzy sub ring of R , $h \neq \emptyset$ then h is called:-

(1) A hesitant fuzzy left ideal (in short, HFLI) of R, if $h(xy) \supseteq h(y)$, for any x , y∈R.

(2) A hesitant fuzzy right ideal (in short, HFRI) of R, if $h(xy) \supseteq h(x)$, for any x,y ∈R.

(3) A hesitant fuzzy ideal (in short , HFI) of R, if it is both a HFLI and a HFRI of R .

We will denote the set of all HFI of R by HFI(R).

Proposition3-7:- Let (R,+,.) be a ring and $h \in HFS(R)$, $h\neq \emptyset$,then h∈HFI(R) [resp. HFLI(R) and HFRI(R)] if and only if, for any x , y ∈R.

1-$h(x-y) \supseteq h(x)\cap h(y)$

2-$h(xy) \supseteq h(x)\cup h(y)$

Proof:- Suppose h∈HFI(R) and for any $x, y \in R$, we get $h(x-y) \supseteq h(y) \cap h(y)$. And h is hesitant fuzzy left (right) ideal of ring R, implies $h(xy) \supseteq h(y)$ and $h(xy) \supseteq h(x)$.

So, $h(xy) \supseteq h(x) \cup h(y)$.

Suppose the conditions are hold. Since $h(xy) \supseteq h(x) \cup h(y) \supseteq h(x) \cap h(y)$ and $h(x-y) \supseteq h(x) \cap h(y)$. Thus h is hesitant fuzzy sub ring. It follows that $h(xy) \supseteq h(x) \cup h(y) \supseteq h(y)$, then h is hesitant fuzzy left ideal of ring and $h(xy) \supseteq h(x) \cup h(y) \supseteq h(x)$, hence h is hesitant fuzzy right ideal of ring .

So, h ∈HFI(R).

Example3-8:-Let $(Z_4$,+,.) be a ring where Z_4 ={0,1,2,3} and $h: Z_4 \rightarrow P[0,1]$ defined as follows: h(0)=[0.2,0.8), h(1)= (0.3,0.7)=h(3), h(2)= [0.2 ,0.5]

Then we can easily note that h hesitant fuzzy ideal of Z_4.

Theorem3-9:- Let $h \in HFI(R)$,Then for any $x \in R$

(1) $h(x) \subseteq h(0)$

(2) $h(-x) = h(x)$

proof:- Suppose $h \in HFI(R)$

1- Since $0 = x - x$ .So that $h(0) = h(x-x) \supset h(x)\cap h(x) = h(x)$. Thus $h(x) \subset h(0)$

2- $h(-x) = h(0-x) \supset h(0)\cap h(x)$, by condition (1), hence $h(0)\cap h(x) = h(x)$

implies that $h(-x) \supset h(x)$....(1)

Also, $h(x) = h\big(0-(-x)\big) \supset h(0)\cap h(-x)$ since $h(0)\cap h(-x) = h(-x)$ by condition (1) so that, $h(x) \supset h(-x)$.....(2). From (1) , (2) we get $h(-x) = h(x)$.

Theorem3-10:- Let R be a ring and $h \in HFI(R)$, if $h(x+y) = h(0)$, for any $x , y \in R$, then $h(x) = h(y)$.

Proof:- Suppose $h(x+y) = h(0)$ and $h \in HFI(R)$, for any $x , y \in R$

$h(x) = h(x+y-y) = h\big((x+y)-y\big) \supseteq h(x+y) \cap h(y) = h(0) \cap h(y) = h(y)$

Thus, $h(x) \supset h(y)$.......(1)

$h(y) = h(y+x-x) = h\big((y+x)-x\big) \supseteq h(x+y) \cap h(x) = h(0) \cap h(x) = h(x)$

Hence, $h(y) \supset h(x)$........(2)

From (1) and (2) we have $h(x) = h(y)$.

Theorem 3-11:-Let $h \in HFI(R)$, then $h^i(0) = h(0)$ for all integers $i \geq 1$

Proof :- We prove the result by induction

Clearly the result is true for $i = 1$, so $h^1(0) = h(0)$

Assume the result is true for $i = r$, so $h^r(0) = h(0)$

Now $h^{r+1}(0) = (h^r \circ h^1)(0) = \bigcup_{0=zw}[h^r(z) \cap h^1(w)]$

Since the supermom is attained when z=w=0 .

Thus $\bigcup_{0=0.0}[h(0) \cap h(0)] = h(0)$, this completes the proof.

Thus $h^i(0) = h(0)$.

Theorem3-12:-Let $h \in HFS(R)$ and $h(x) = h(0)$ for any $x \in R$, then $h \in HFI(R)$

Proof:-Suppose $h(x) = h(0)$ for any $x \in R$,let $x, y \in R$ implies that $x - y \in R$.

So $h(x-y) = h(0) \supseteq h(x) \cap h(y)$ [From Theorem (3-9) , condition (1)]

Thus $h(x-y) \supseteq h(x) \cap h(y)$

Since $x, y \in R$, thus $xy \in R$.

Then, $h(xy) = h(0) \supseteq h(x) \cup h(y)$ [From Theorem (3-9) ,condition (1)]

Hence $h(xy) \supseteq h(x) \cup h(y)$. Thus $h \in HFI(R)$.

Proposition 3-13:-Let $h \in HFI(R)$ and k be a positive integer, if $x_1, x_2, x_3, \dots x_k \in R$,

Then $h^k(x_1x_2x_3 \dots\dots\dots x_k) \supseteq h(x_1) \cap h(x_2) \cap h(x_3) \dots \cap h(x_k)$.

Proof :-

It clear, the result is true for k =1

Assume that it is true for $k = r$, so $h^r(x_1x_2x_3 \dots \dots \dots x_r) \supseteq h(x_1) \cap h(x_2) \cap h(x_3) \dots \cap h(x_r)$.
Now
$h^{r+1}(x_1x_2x_3 \dots \dots \dots x_{r+1})$
$= h^r \circ h^1(x_1x_2x_3 \dots \dots \dots x_{r+1})$
since $x_1x_2x_3 \dots \dots \dots x_{r+1} = (x_1x_2x_3 \dots \dots \dots x_r)(x_{r+1})$
So $h^r \circ h^1(x_1x_2x_3 \dots \dots \dots x_{r+1}) = \cup[h^r(x_1x_2x_3 \dots \dots \dots x_r) \cap h^1(x_{r+1})]$
$\supseteq h(x_1) \cap h(x_2) \cap \dots h(x_r) \cap h(x_{r+1}) = \bigcap_{i=1}^{r+1} h(x_i)$, this completes the proof.
Thus, $h^k(x_1x_2x_3 \dots \dots \dots x_k) \supseteq h(x_1) \cap h(x_2) \cap h(x_3) \dots \cap h(x_k)$.

Theorem 3-14:-Every hesitant fuzzy ideal of a rig R is an hesitant fuzzy ring of R.
Proof:- Suppose h hesitant fuzzy ideal of a rig R.
We have , for any $x , y \in R$
(1) $h(x-y) \supseteq h(x) \cap h(y)$
(2) $h(xy) \supseteq h(x) \cup h(y) \supseteq h(x) \cap h(y)$ so that $h(xy) \supseteq h(x) \cap h(y)$
Hence, h is an hesitant fuzzy ring of a ring R.

The converse of Theorem (3-13) may not to be true as in the following counter example.
Example 3-15 :- let (R, +,.) be a ring of real numbers define :- $h(x) = \begin{cases} \{0.2, 0.4, 0.6\} & \text{if } x \text{ is rational} \\ \{0.1, 0.3, 0.4\} & \text{if } x \text{ is irrational} \end{cases}$
Then, we can easily see that h is an hesitant fuzzy ring of R.
But h is not hesitant fuzzy ideal of a rig R because
$h(2\sqrt{2}) = \{0.1, 0.3, 0.4\} \not\supseteq h(2) \cup h(\sqrt{2}) = \{0.2, 0.3, 0.4, 0.6\}$.

Proposition 3-16 :-Let h be an hesitant fuzzy ideal of a ring R and $h_* = \{x \in R: h(x) = h(0)\}$, then h_* ideal of a ring R.
Proof :- Suppose $h \in HFI(R)$and $x, y \in h_*$ so that $h(x) = h(0)$ and $h(y) = h(0)$
$h(x-y) \supseteq h(x) \cap h(y) = h(0) \cap h(0) = h(0)$
So $h(x-y) \supseteq h(0)$ and $h(0) = h(0(x-y)) \supseteq h(x-y)$ so that $h(x-y) = h(0)$
Thus $xy \in h_*$. Now let $x \in R$ and $y \in h_*$ implies $h(xy) \supseteq h(y) = h(0)$
Since $h \in HFI(R)$ and $xy \in R$. From theorem 3 − 9 condition (1). So that $h(xy) \subseteq h(0)$
Thus $h(xy) \subseteq h(0)$ SO $xy \in h_*$. Similarly we have $yx \in h_*$. Thus h_* is ideal of R.

Proposition3-17:-Let h_1, h_2 be hesitant fuzzy ideal of a ring R ,then $h_{1*} \cap h_{2*} \subseteq (h_1 \cap h_2)_*$
Proof :-Let $x \in h_{1*} \cap h_{2*}$ implies $h_1(x) = h_1(0)$ and $h_2(x) = h_2(0)$
Now$(h_1 \cap h_2)(x) = \bigcup_{t_1 \in h_1(x),\ t_2 \in h_2(x)} \min\{t_1, t_2\} = \bigcup_{t_1 \in h_1(0),\ t_2 \in h_2(0)} \min\{t_1, t_2\}$
$=(h_1 \cap h_2)(0)$, thus $x \in (h_1 \cap h_2)_*$.Hence $h_{1*} \cap h_{2*} \subseteq (h_1 \cap h_2)_*$
In general the equality in the above lemma need not hold, as shown in the following example.

Example 3-18:- Let R be a ring and let h_1, h_2 be hesitant fuzzy ideal of a ring R define as following
$h_1(x) = 0$ for all $x \in R$ and $h_2(x) = 0$ if $x \neq 0, h_2(0) = \{1\}$
Now $h_{1*} \cap h_{2*} = R \cap \{0\} = \{0\}$ and $(h_1 \cap h_2)_* = R$.

Proposition 3-19:-Let h_1, h_2 be hesitant fuzzy ideal of a ring R such that $h_1(0) = \{1\} = h_2(0)$, then $h_{1*} \cap h_{2*} = (h_1 \cap h_2)_*$.
Proof:-Let $x \in (h_1 \cap h_2)_*$ implies$(h_1 \cap h_2)(x) =(h_1 \cap h_2)(0)= \bigcup_{t_1 \in h_1(0), t_2 \in h_2(0)} \min\{t_1, t_2\}$
Thus $\bigcup_{t_1 \in h_1(x), t_2 \in h_2(x)} \min\{t_1, t_2\} = \bigcup_{t_1 \in h_1(0), t_2 \in h_2(0)} \min\{t_1, t_2\} = \{1\}$
Hence $h_1(x) = h_1(0) = \{1\}$and $h_2(x) = h_2(0) = \{1\}$ implies $x \in h_{1*} \cap h_{2*}$
Thus $(h_1 \cap h_2)_* \subseteq h_{1*} \cap h_{2*}$
Also $h_{1*} \cap h_{2*} \subseteq (h_1 \cap h_2)_*$ by Proposition 3-16
Then, $h_{1*} \cap h_{2*} = (h_1 \cap h_2)_*$.

Proposition3-20:-Let $\{h_i: i \in I\}$ be a family of hesitant fuzzy ideal of R such that $h_i(0) = \{1\}$
for all $i \in I$,then $\bigcap_{i \in I}(h_i)_* = (\bigcap_{i \in I} h_i)_*$.
Proof :-Let $x \in \bigcap_{i \in I}(h_i)_*$ implies $x \in (h_i)_*$ implies $h_i(x) = h_i(0) = \{1\}$
Implies $\bigcap_{i \in I} h_i(x) = \bigcap_{i \in I} h_i(0)$ implies $x \in (\bigcap_{i \in I} h_i)_*$

So, $\bigcap_{i\in I}(h_i)_* \subseteq (\bigcap_{i\in I} h_i)_*$

Let $x \in (\bigcap_{i\in I} h_i)_*$ implies $(\bigcap_{i\in I} h_i)(x) = (\bigcap_{i\in I} h_i)(0)$ implies $h_i(x) = h_i(0) = \{1\}$

Implies $x \in (h_i)_*$ implies $x \in \bigcap_{i\in I}(h_i)_*$ implies $(\bigcap_{i\in I} h_i)_* \subseteq \bigcap_{i\in I}(h_i)_*$

Hence, $\bigcap_{i\in I}(h_i)_* = (\bigcap_{i\in I} h_i)_*$.

Proposition3-21:-Let $h \in HFI(R)$, then $(h^i)_* \subseteq h_*$, for all integers $i \geq 1$
proof :-Let $x \in (h^i)_*$ implies $h^i(x) = h^i(0) = h(0)$ since $h^i(x) \subseteq h(x)$

Thus $h(0) \subseteq h(x)$. Since $h(x) \subseteq h(0)$. So $h(x) = h(0)$ hence $x \in h_*$. Thus $(h^i)_* \subseteq h_*$

Theorem 3-22:-Let (R,+,.) be a ring with unity and "0" be identity of R for "+" and "e" be the identity of R for "." , then h is hesitant fuzzy ideal of R , if and only if

$h(x) = h(e) \subseteq h(0)$, for each $x \in R, x \neq 0$

proof: (1) Suppose that $h \in HFI(R)$ and $x \in R , x \neq 0$

Hence $h(x) = h(xe) \supseteq h(x) \cup h(e)$, since $h \in HFG(G)$. So that $h(x) \cup h(e) = h(e)$.

Thus $h(x) \supseteq h(e)$……….(1)

Since $e = xx^{-1}$, so $h(e) = h(xx^{-1}) \supseteq h(x) \cup h(x^{-1})$, since $h \in HFG(G)$ [From Proposition 3-3] .

Hence $h(x) \cup h(x^{-1}) \supseteq h(x) \cup h(x) = h(x)$

Thus $h(e) \supseteq h(x)$, from (1) and (2) we get $h(x) = h(e)$

Now we must prove $h(x) \subseteq h(0)$or $h(e) \subseteq h(0)$, since $0 = x - x$

So that $h(0) = h(x - x) \supseteq h(x) \bigcap h(x) = h(x)$.

Thus $h(x) \subseteq h(0)$…….(3)

From (1),(2) and (3) we get $h(x) = h(e) \subseteq h(0)$

(2) Suppose that $h(x) = h(e) \subseteq h(0)$, for each $x \in R , x \neq 0$

Let $c , d \in R$, We have four cases : $c \neq 0 , d \neq 0$ and $c \neq d$ or $c \neq 0 , d \neq 0$ and $c = d$ or $c \neq 0$,
$d = 0$ or $c = 0 , d \neq 0$

(1) Suppose $c \neq 0 , d \neq 0$ and $c \neq d$ so that $c - d \neq 0$ and $cd \neq 0$,
Thus $h(c - d) = h(e)$
$\supseteq h(c) \cap h(d)$ and $h(cd)$
$= h(e) \supseteq h(c) \cup h(d)$.

(2) Suppose $c \neq 0 , d \neq 0$ and $c = d$, implies $c - d = 0$ and $cd \neq 0$.
Thus $h(c - d) = h(0)$
$\supseteq h(c) \cap h(d)$ and $h(cd)$
$= h(e) \supseteq h(c) \cup h(d)$.

(3) Suppose $c \neq 0$ or , $d = 0$ .Then clearly, $c - d \neq 0$ and $cd = 0$. Thus by the hypothesis,
$h(c - d) = h(c) \supset h(c) \cap h(d)$ and $h(cd)$
$= h(0) \supset h(c) \cup h(d)$.

(4) Suppose $x = 0 , y \neq 0$ .Then the proof is similar to case (3)
So in all cases , $h \in HFI(R)$. This completes the proof.

Proposition 3-22:-Let R be a ring with unity and h be hesitant fuzzy ideal of R, then the set $R_h = \{x \in R : h(x) = h(e)\}$ *is ideal of R*

Proof:- Clearly $R_h \neq \phi$, let $x , y \in R_h$ *so that* $h(x) = h(e)$ *and* $h(y) = h(e)$

Since h be hesitant fuzzy ideal of R. Implies $h(x - y) \supseteq h(x) \cap h(y) = h(e) \cap h(e) = h(e)$

Thus $h(x - y) \supseteq h(e)$. Since $h(e) \supset h(x - y)$From Proposition $3 - 3$.

So $h(x - y) = h(e)$. *Thus* $x - y \in R_h$.

Now let $x \in R$ *and* $y \in R_h$ .So $h(xy) \supseteq h(y) = h(e)$,thus $h(xy) \supseteq h(e)$,

Since $h(e) \supseteq h(xy)$ From Proposition $3 - 3$, this implies $h(xy) = h(e)$. *So* $xy \in R_h$. Similarly we have $yx \in R_h$. Hence R_h *is ideal of R*.

4-Some operations of hesitant fuzzy ideal of R

Proposition 4-1:-Let h_1 and h_2 be two a hesitant fuzzy ideal of a ring R. Then $h_1 \cup h_2$ is hesitant fuzzy ideal of a ring R if $h_1 \subset h_2$ or $h_2 \subset h_1$.

Proof:- If $h_1 \subset h_2$ implise $h_1 \cup h_2 = h_2$, since h_2 is hesitant fuzzy ideal

So $h_1 U h_2$ is hesitant fuzzy ideal. If $h_2 \subset h_1$, so that $h_1 \cup h_2 = h_1$, since h_2 is hesitant fuzzy ideal

So $h_1 \cup h_2$ is hesitant fuzzy ideal. Thus $h_1 U h_2$ is hesitant fuzzy ideal of a ring R .

But $h_1 \cup h_2$ is not necessarily hesitant fuzzy ideal of a ring R where $h_1 \not\subseteq h_2$ or $h_2 \not\subseteq h_1$,The following example shows that.

Example 4-2:-let (Z_4 ,+,.) be a ring where Z_4={0,1,2,3} and h_1, h_2: $Z_4 \to P[0,1]$ defined as follows: $h_1(0)$=[0.2,0.8) , $h_1(1)$=(0.3,0.7)= h_1

(3) , h_1 (2)= [0.1 ,0.7], $h_2(0)$=[0.1,0.7] , $h_2(1)$=[0.4,0.7]
, h_2 (2)= [0.4 ,0.5] = h_2 (3). Then we can easily see that h_1, h_2 are an hesitant fuzzy ideal of R.

Then $(h_1 \cup h_2)(3-1) = (h_1 \cup h_2)(2) = (h_1)(2) \cup (h_2)(2) = [0.4,0.7]$

$(h_1 \cup h_2)(3) = h_1(3) \cup h_2(3)$=[0.4,0.7)

$(h_1 \cup h_2)(1) = h_1(1) \cup h_2(1)$=(0.3,0.7]

$(h_1 \cup h_2)(3) \cap (h_1 \cup h_2)(1) = (0.3\ ,0.7)$

Thus $(h_1 \cup h_2)(3-1) \not\supseteq (h_1 \cup h_2)(3) \cap (h_1 \cup h_2)(1)$.

Hence $h_1 \cup h_2$ is not hesitant fuzzy ideal of Z_4

Theorem 4-3:-Let $\{h_i\ /i \in I\}$ be a family of a hesitant fuzzy ideal of a ring R, then$(\bigcup_{i\in I} h_i)$is a hesitant fuzzy ideal of a ring R if the family is chain.

Proof:- For any x ,y ∈ R, we have

$(\bigcup_{i\in I} h_i)$(x-y)= $\bigcup_{i\in I} h_i$ (x-y)= $\bigcup_{t_i \in h_i(x-y)}$) max$\{t_i\}$ ⊇ $\bigcup_{t_i \in h_i(x) \cap t_i \in h_i(y)}$ max$\{t_i\}$, i ∈ I

=$\bigcup_{t_i \in h_i(x) \wedge t_i \in h_i(y)}$ max$\{t_I\}$,since $\{h_i\ /i\in I\}$ is chain so that for all h_i , $h_j \in\{h_i\ /i\in I\}$then either $h_i \subset h_j$or $h_j \subset h_i$ and has hesitant fuzzy ideal be an upper bound ,by Zorn's Lemma applyons then $\{h_i\ /i\in I\}$ has maximal hesitant fuzzy ideal. So

$\bigcup_{t_i \in h_i(x) \wedge t_i \in h_i(y)}$ max$\{t_I\}$=$\bigcup_{t_i \in h_i(x)}$ max$\{t_i\}$ ∩ $\bigcup_{t_i \in h_i(y)}$ max$\{t_i\}$

=$\bigcup_{t_i \in h_i(x)}$ max$\{t_i\}$ ∩ $\bigcup_{t_i \in h_i(y)}$ max$\{t_i\}$=$\bigcup_{i\in I} h_i$ (x) ∩ $\bigcup_{i\in I} h_i$ (y)

Thus $(\bigcup_{i\in I} h_i)$(x-y)= $\bigcup_{i\in I} h_i$ (x) ∩ $\bigcup_{i\in I} h_i$ (y)…………(1)

Now we must prove $(\bigcup_{i\in I} h_i)$(xy)= $\bigcup_{i\in I} h_i$ (xy)= $\bigcup_{t_i \in h_i(xy)}$) max$\{t_i\}$

⊇ $\bigcup_{t_i \in h_i(x) \cup t_i \in h_i(y)}$ max$\{t_i\}$, i ∈ I

=$\bigcup_{t_i \in h_i(x) \vee t_i \in h_i(y)}$ max$\{t_i\}$=$\bigcup_{t_i \in h_i(x)}$ max$\{t_i\}$ ∪ $\bigcup_{t_i \in h_i(y)}$ max$\{t_i\}$

=$\bigcup_{t_i \in h_i(x)}$ max$\{t_i\}$ ∪ $\bigcup_{t_i \in h_i(y)}$ max$\{t_i\}$ =$\bigcup_{i\in I} h_i$ (x) ∪ $\bigcup_{i\in I} h_i$ (y)

Thus $(\bigcup_{i\in I} h_i)$(xy)= $\bigcup_{i\in I} h_i$ (x) ∪ $\bigcup_{i\in I} h_i$ (y)…………(2)

From (1) and (2) we get $(\bigcup_{i\in I} h_i)$ is a hesitant fuzzy ideal of a ring R.

Proposition 4-4:-Let h_1 and h_2 be two hesitant fuzzy ideal of a ring R.Then $h_1 \cap h_2$ is a hesitant fuzzy ideal of a ring R.

Proof:- for any x , y ∈ R

$1 - (h_1 \cap h_2)$(x-y)= $h_1(x-y) \cap h_2(x-y)$,

Since h_1 , h_2 hesitant fuzzy ideal of a ring R ,then

$h_i(x-y) \supseteq h_i(x) \cap h_i(y)$,where i=1,2.

So $(h_1 \cap h_2)(x - y) = h_1(x-y) \cap h_2(x-y) \supseteq h_1(x) \cap h_1(y) \cap h_2(x) \cap h_2(y)$

$= h_1(x) \cap h_2(x) \cap h_1(y) \cap h_2(y) = (h_1 \cap h_2)(x) \cap (h_1 \cap h_2)(y)$

Thus$(h_1 \cap h_2)$(x-y) ⊇ $(h_1 \cap h_2)(x) \cap (h_1 \cap h_2)(y)$

Now we must prove $(h_1 \cap h_2)(xy)$

$= h_1(xy) \cap h_2(xy)$

Since $h_1, h_2 \in$ HFI(R)so that $h_1, h_2 \in$ HFLI(R). Thus $h_1(xy) \supseteq h_1(x)$ and $h_2(xy) \supseteq h_2(x)$

Thus $h_1(xy) \cap h_2(xy) \supseteq h_1(x) \cap h_2(x)$……….(1)

Also $h_1, h_2 \in$ HFRI(R) so that $h_1(xy) \supseteq h_1(y)$ and $h_2(xy) \supseteq h_2(y)$

Thus $h_1(xy) \cap h_2(xy) \supseteq h_1(y) \cap h_2(y)$……….(2).From (1) and (2) we get

$h_1(xy) \cap h_2(xy) \supseteq h_1(x) \cap h_2(x) \cup h_1(y) \cap h_2(y)$=$(h_1 \cap h_2)(x) \cup (h_1 \cap h_2)(y)$

So $(h_1 \cap h_2)(xy) \supseteq (h_1 \cap h_2)(x) \cup (h_1 \cap h_2)(y)$. Thus $h_1 \cap h_2$ is a hesitant fuzzy ideal of a ring R.

Theorem 4-5:-Let $\{h_i\ /i\in I\}$ be a family of a hesitant fuzzy ideal of a ring R, then $\bigcap_{i\in I} h_i$is a hesitant fuzzy ideal of a ring R .

Proof:- For any x , y ∈ R, we have

$1 - (\bigcap_{i\in I} h_i)$(x-y) = $\bigcap_{i\in I} h_i$ (x-y) , since h_i is a hesitant fuzzy ideal of a ring R, for any i ∈ I,

So h_i (x-y) ⊇ $h_i(x) \cap h_i(y)$,Hence $(\bigcap_{i\in I} h_i)$(x-y)= $\bigcap_{i\in I} h_i$ (x-y) ⊇ $\bigcap_{i\in I}\{h_i(x) \cap h_i(y)\}$

= $\{\bigcap_{i\in I} h_i(x)\} \cap \{\bigcap_{i\in I} h_i(y)\}$,Thus $(\bigcap_{i\in I} h_i)$(x-y) ⊇$\{\bigcap_{i\in I} h_i(x)\} \cap \{\bigcap_{i\in I} h_i(y)\}$………..(1)

$2 - (\bigcap_{i\in I} h_i)$(xy)= $\bigcap_{i\in I} h_i$ (xy) , since h_i is a hesitant fuzzy ideal of a ring R,for any i∈I

So h_i (xy) ⊇ $h_i(x) \cup h_i(y)$,Hence $(\bigcap_{i\in I} h_i)$(xy)= $\bigcap_{i\in I} h_i$ (xy) ⊇ $\bigcap_{i\in I}\{h_i(x) \cup h_i(y)\}$ = $\{\bigcap_{i\in I} h_i(x)\} \cup \{\bigcap_{i\in I} h_i(y)\}$.Thus $(\bigcap_{i\in I} h_i)$(xy) ⊇$\{\bigcap_{i\in I} h_i(x)\} \cup \{\bigcap_{i\in I} h_i(y)\}$………(2)

From 1 and 2 ,we get $(\bigcap_{i\in I} h_i)$ is hesitant fuzzy ideal of a ring R.

Proposition 4-6:- Let h_1 and h_2 be two hesitant fuzzy ideal of a ring R. Then $h_1 \oplus h_2$ is a hesitant fuzzy ideal of a ring R.

Proof:- Since $(h_1 \oplus h_2)(x)=\{\gamma_1+\gamma_2-\gamma_1\gamma_2 : \gamma_1 \in h_1(x), \gamma_2 \in h_2(x)\}$

Hence $(h_1 \oplus h_2)(x-y)=\{\gamma_1+\gamma_2-\gamma_1\gamma_2 : \gamma_1 \in h_1(x-y), \gamma_2 \in h_2(x-y)\}$

Since h_1 and h_2 be two hesitant fuzzy ideal of a ring R

So $(h_1 \oplus h_2)(x-y)=\{\gamma_1+\gamma_2-\gamma_1\gamma_2 / \gamma_1 \in h_1(x-y), \gamma_2 \in h_2(x-y)\}$

$\supseteq \{\gamma_1+\gamma_2-\gamma_1\gamma_2 : \gamma_1 \in h_1(x) \cap h_1(y) \wedge \gamma_2 \in h_2(x) \cap h_2(y)\}$

$=\{\gamma_1+\gamma_2-\gamma_1\gamma_2 : \gamma_1 \in h_1(x) \wedge \gamma_1 \in h_1(y) \wedge \gamma_2 \in h_2(x) \wedge \gamma_2 \in h_2(y)\}$

$\{\gamma_1+\gamma_2-\gamma_1\gamma_2 : \gamma_1 \in h_1(x) \wedge \gamma_2 \in h_2(x) \wedge \gamma_1 \in h_2(y) \wedge \gamma_2 \in h_2(y)\}$

$=\{\gamma_1+\gamma_2-\gamma_1\gamma_2 : \gamma_1 \in h_1(x), \gamma_2 \in h_2(x)\} \cap \{\gamma_1+\gamma_2-\gamma_1\gamma_2 : \gamma_1 \in h_1(y), \gamma_2 \in h_2(y)\}$

$=\{\gamma_1+\gamma_2-\gamma_1\gamma_2 : \gamma_1 \in h_1(x), \gamma_2 \in h_2(x)\} \cap \{\gamma_1+\gamma_2-\gamma_1\gamma_2 : \gamma_1 \in h_1(y), \gamma_2 \in h_2(y)\}$

Thus $(h_1 \oplus h_2)(x-y) \supseteq (h_1 \oplus h_2)(x) \cap (h_1 \oplus h_2)(y)$…………..(1)

Now $(h_1 \oplus h_2)(xy)=\{\gamma_1+\gamma_2-\gamma_1\gamma_2 / \gamma_1 \in h_1(xy), \gamma_2 \in h_2(xy)\}$

$\supseteq \{\gamma_1+\gamma_2-\gamma_1\gamma_2 : \gamma_1 \in h_1(x) \cup h_1(y) \wedge \gamma_2 \in h_2(x) \cup h_2(y)\}$

$=\{\gamma_1+\gamma_2-\gamma_1\gamma_2 : \gamma_1 \in h_1(x) \vee \gamma_1 \in h_1(y) \wedge \gamma_2 \in h_2(x) \vee \gamma_2 \in h_2(y)\}$

$=\{\gamma_1+\gamma_2-\gamma_1\gamma_2 : \gamma_1 \in h_1(x) \wedge \gamma_2 \in h_2(x) \vee \gamma_1 \in h_1(y) \wedge \gamma_2 \in h_2(y) \vee \gamma_1 \in h_1(x) \wedge \gamma_2 \in h_2(y) \vee \gamma_1 \in h_1(y) \wedge \gamma_2 \in h_2(x)\}$

$\supseteq \{\gamma_1+\gamma_2-\gamma_1\gamma_2 : \gamma_1 \in h_1(x) \wedge \gamma_2 \in h_2(x) \vee \gamma_1 \in h_1(y) \wedge \gamma_2 \in h_2(y)\}$

$=\{\gamma_1+\gamma_2-\gamma_1\gamma_2 : \gamma_1 \in h_1(x), \gamma_2 \in h_2(x)\} \cup \{\gamma_1+\gamma_2-\gamma_1\gamma_2 : \gamma_1 \in h_1(y), \gamma_2 \in h_2(y)\}$

$=\{\gamma_1+\gamma_2-\gamma_1\gamma_2 : \gamma_1 \in h_1(x), \gamma_2 \in h_2(x)\} \cup \{\gamma_1+\gamma_2-\gamma_1\gamma_2 : \gamma_1 \in h_1(y), \gamma_2 \in h_2(y)\}$

$=(h_1 \oplus h_2)(x) \cup (h_1 \oplus h_2)(y)$. Thus $(h_1 \oplus h_2)(xy) \supseteq (h_1 \oplus h_2)(x) \cup (h_1 \oplus h_2)(y)$………(2)

From 1 and 2 we get $(h_1 \oplus h_2)$ is hesitant fuzzy ideal of a ring R

Proposition 4-7:-Let h_1 and h_2 be two hesitant fuzzy ideal of a ring R . Then $h_1 \otimes h_2$ is a hesitant fuzzy ideal of a ring R.

Proof:-Since $(h_1 \otimes h_2)(x)=\{\gamma_1\gamma_2 : \gamma_1 \in h_1(x), \gamma_2 \in h_2(x)\}$

1- Hence $(h_1 \otimes h_2)(x-y)=\{\gamma_1\gamma_2 : \gamma_1 \in h_1(x-y), \gamma_2 \in h_2(x-y)\}$

Since h_1 and h_2 be two hesitant fuzzy ideal of a ring R.

So $(h_1 \otimes h_2)(x-y)=\{\gamma_1\gamma_2 : \gamma_1 \in h_1(x-y), \gamma_2 \in h_2(x-y)\} \supseteq \{\gamma_1\gamma_2 : \gamma_1 \in h_1(x) \cap h_1(y) \wedge \gamma_2 \in h_2(x) \cap h_2(y)\}$

$=\{\gamma_1\gamma_2 : \gamma_1 \in h_1(x) \wedge \gamma_1 \in h_1(y) \wedge \gamma_2 \in h_2(x) \wedge \gamma_2 \in h_2(y)\}$

$=\{\gamma_1\gamma_2 : \gamma_1 \in h_1(x) \wedge \gamma_2 \in h_2(x) \wedge \gamma_1 \in h_1(y) \wedge \gamma_2 \in h_2(y)\}$

$=\{\gamma_1\gamma_2 : \gamma_1 \in h_1(x) \wedge \gamma_2 \in h_2(x)\} \cap \{\gamma_1\gamma_2 : \gamma_1 \in h_1(y) \wedge \gamma_2 \in h_2(y)\}$

$=\{\gamma_1\gamma_2 : \gamma_1 \in h_1(x), \gamma_2 \in h_2(x)\} \cap \{\gamma_1\gamma_2 : \gamma_1 \in h_1(y), \gamma_2 \in h_2(y)\}$

$=(h_1 \otimes h_2)(x) \cap (h_1 \otimes h_2)(y)$

Hence $(h_1 \otimes h_2)(x-y) \supseteq (h_1 \otimes h_2)(x) \cap (h_1 \otimes h_2)(y)$…….(1)

2- Now we must prove $(h_1 \otimes h_2)(xy)=\{\gamma_1\gamma_2 : \gamma_1 \in h_1(xy), \gamma_2 \in h_2(xy)\}$

$\supseteq \{\gamma_1\gamma_2 : \gamma_1 \in h_1(x) \cup h_1(y) \wedge \gamma_2 \in h_2(x) \cup h_2(y)\}$

$=\{\gamma_1\gamma_2 : \gamma_1 \in h_1(x) \vee \gamma_1 \in h_1(y) \wedge \gamma_2 \in h_2(x) \vee \gamma_2 \in h_2(y)\}$

$=\{\gamma_1\gamma_2 : \gamma_1 \in h_1(x) \wedge \gamma_2 \in h_2(x) \vee \gamma_1 \in h_1(y) \wedge \gamma_2 \in h_2(y) \vee \gamma_1 \in h_1(x) \wedge \gamma_2 \in h_2(y) \vee \gamma_1 \in h_1(y) \wedge \gamma_2 \in h_2(x)\}$

$\supseteq \{\gamma_1\gamma_2 \backslash \gamma_1 \in h_1(x) \wedge \gamma_2 \in h_2(x) \vee \gamma_1 \in h_1(y) \wedge \gamma_2 \in h_2(y)\}$

$=\{\gamma_1\gamma_2 \backslash \gamma_1 \in h_1(x), \gamma_2 \in h_2(x) \vee \gamma_1 \in h_1(y), \gamma_2 \in h_2(y)\}$

$=\{\gamma_1\gamma_2 / \gamma_1 \in h_1(x), \gamma_2 \in h_2(x)\} \cup \{\gamma_1\gamma_2 / \gamma_1 \in h_1(y), \gamma_2 \in h_2(y)\}$

$=(h_1 \otimes h_2)(x) \cup (h_1 \otimes h_2)(y)$. Hence $(h_1 \otimes h_2)(xy) \supseteq (h_1 \otimes h_2)(x) \cup (h_1 \otimes h_2)(y)$………(2)

From 1 and 2 we get $(h_1 \otimes h_2)$ is hesitant fuzzy ideal of a ring R.

Proposition4-8:-Let h is a hesitant fuzzy ideal of a ring R, then λh is a hesitant fuzzy ideal of ring R.

Proof:- Since $\lambda h(x) = \{1-(1-\gamma)^\lambda : \gamma \in h(x)\}$ .Now we must prove

1- $\lambda h(x-y) = \{1-(1-\gamma)^\lambda : \gamma \in h(x-y)\} \supseteq \{1-(1-\gamma)^\lambda : \gamma \in h(x) \cap h(y)\}$

$=\{1-(1-\gamma)^{\lambda}: \gamma \in h(x) \wedge \gamma \in h(y)\}$ $=\{1-(1-\gamma)^{\lambda}: \gamma \in h(x)\} \cap \{1-(1-\gamma)^{\lambda}: \gamma \in h(y)\}$
$=\{1-(1-\gamma)^{\lambda}: \gamma \in h(x)\} \cap \{1-(1-\gamma)^{\lambda}: \gamma \in h(y)\} = \lambda h(x) \cap \lambda h(y)$
Thus $\lambda h(x-y) \supseteq \lambda h(x) \cap \lambda h(y)$........(1)
2 $-\lambda h(xy) =\{1-(1-\gamma)^{\lambda}: \gamma \in h(xy)\} \supseteq \{1-(1-\gamma)^{\lambda}: \gamma \in h(x) \cup h(y)\}$
$=\{1-(1-\gamma)^{\lambda}: \gamma \in h(x) \vee \gamma \in h(y)\} = \{1-(1-\gamma)^{\lambda}: \gamma \in h(x)\} \cup \{1-(1-\gamma)^{\lambda}: \gamma \in h(y)\}$
$=\{1-(1-\gamma)^{\lambda}: \gamma \in h(x)\} \cup \{1-(1-\gamma)^{\lambda}: \gamma \in h(y)\} = \lambda h(x) \cup \lambda h(y)$
Hence $\lambda h(xy) \supseteq \lambda h(x) \cup \lambda h(y)$ ....(2)
From 1 and 2 we get λh is hesitant fuzzy ideal of a ring.

Proposition 4-9:-Let h_1 and h_2 be two hesitant fuzzy ideal of a ring. Then $h_1 \ominus h_2$ is a hesitant fuzzy ideal of a ring R.
Proof:-Since $h_1(x) \ominus h_2(x) = \{t: \gamma_1 \in h_1(x), \gamma_2 \in h_2(x)\}, \forall x \in X$

where $t = \begin{cases} \frac{\gamma_1-\gamma_2}{1-\gamma_2} & \text{if } \gamma_1 \geq \gamma_2, \gamma_2 \neq 1 \\ 0 & \text{if other wise} \end{cases}$

Hence $(h_1 \ominus h_2)(x-y) = \{t : \gamma_1 \in h_1(x-y), \gamma_2 \in h_2(x-y)\}$
$\supseteq \{t : \gamma_1 \in h_1(x) \cap h_1(y) \wedge \gamma_2 \in h_2(x) \cap h_2(y)\}$
$=\{t : \gamma_1 \in h_1(x) \wedge \gamma_1 \in h_1(y) \wedge \gamma_2 \in h_2(x) \wedge \gamma_2 \in h_2(y)\}$
$=\{t : \gamma_1 \in h_1(x) \wedge \in h_2(x) \wedge \gamma_1 \in h_1(y) \wedge \gamma_2 \in h_2(y)\}$
$=\{t : \gamma_1 \in h_1(x), \gamma_2 \in h_2(x)\} \cap \{t: \gamma_1 \in h_1(y), \gamma_2 \in h_2(y)\}$
$=(h_1 \ominus h_2)(x) \cap (h_1 \ominus h_2)(y)$
Thus $(h_1 \ominus h_2)(x-y) \supseteq (h_1 \ominus h_2)(x) \cap (h_1 \ominus h_2)(y)$ ......(1)
Now we must prove $(h_1 \ominus h_2)(xy) = \{t : \gamma_1 \in h_1(xy), \gamma_2 \in h_2(xy)\} \supseteq \{t : \gamma_1 \in h_1(x) \cup h_1(y) \wedge \gamma_2 \in h_2(x) \cup h_2(y)\}$ $=\{t : \gamma_1 \in h_1(x) \vee \gamma_1 \in h_1(y) \wedge \gamma_2 \in h_2(x) \vee \gamma_2 \in h_2(y)\}$
$= \{t: \gamma_1 \in h_1(x) \wedge \gamma_2 \in h_2(x) \vee \gamma_1 \in h_1(y) \wedge \gamma_2 \in h_2(y) \vee \gamma_1 \in h_1(x) \wedge \gamma_2 \in h_2(y) \vee \gamma_1 \in h_1(y) \wedge \gamma_2 \in h_2(x)\} \supseteq \{t : \gamma_1 \in h_1(x) \wedge \gamma_2 \in h_2(x) \vee \gamma_1 \in h_1(y) \wedge \gamma_2 \in h_2(y)\}$
$=\{t : \gamma_1 \in h_1(x), \gamma_2 \in h_2(x) \vee \gamma_1 \in h_1(y), \gamma_2 \in h_2(y)\}$
$=\{t : \gamma_1 \in h_1(x), \gamma_2 \in h_2(x)\} \cup \{t: \gamma_1 \in h_1(y), \gamma_2 \in h_2(y)\}$
$=(h_1 \ominus h_2)(x) \cup (h_1 \ominus h_2)(y)$.Thus $(h_1 \ominus h_2)(x-y) \supseteq (h_1 \ominus h_2)(x) \cup (h_1 \ominus h_2)(y)$ ..(2)
From 1 and 2 we get $(h_1 \ominus h_2)$ is hesitant fuzzy ideal of a ring R.

Proposition 4-10:-Let h_1 and h_2 be two hesitant fuzzy ideal of a ring R . Then $h_1 \oslash h_2$ is a hesitant fuzzy ideal of a ring R.
Proof: Since $h_1(x) \oslash h_2(x) = \{t : \gamma_1 \in h_1(x), \gamma_2 \in h_2(x)\} \forall x \in X$

where $t = \begin{cases} \frac{\gamma_1}{\gamma_2} & \text{if } \gamma_1 \leq \gamma_2, \gamma_2 \neq 0 \\ 1 & \text{if other wise} \end{cases}$

Hence $(h_1 \oslash h_2)(x-y) = \{t : \gamma_1 \in h_1(x-y), \gamma_2 \in h_2(x-y)\}$
$\supseteq \{t : \gamma_1 \in h_1(x) \cap h_1(y) \wedge \gamma_2 \in h_2(x) \cap h_2(y)\}$
$=\{t : \gamma_1 \in h_1(x) \wedge \gamma_1 \in h_1(y) \wedge \gamma_2 \in h_2(x) \wedge \gamma_2 \in h_2(y)\}$
$=\{t : \gamma_1 \in h_1(x), \wedge \gamma_2 \in h_2(x) \wedge \gamma_1 \in h_1(y) \wedge \gamma_2 \in h_2(y)\}$
$=\{t : \gamma_1 \in h_1(x), \gamma_2 \in h_2(x)\} \cap \{t: \gamma_1 \in h_1(y), \gamma_2 \in h_2(y)\}$
$=(h_1 \oslash h_2)(x) \cap (h_1 \oslash h_2)(y)$
Thus $(h_1 \oslash h_2)(x-y) \supseteq (h_1 \oslash h_2)(x) \cap (h_1 \oslash h_2)(y)$ ......(1)
Now we must prove $(h_1 \oslash h_2)(xy) = \{t : \gamma_1 \in h_1(xy), \gamma_2 \in h_2(xy)\}$
$\supseteq \{t : \gamma_1 \in h_1(x) \cup h_1(y) \wedge \gamma_2 \in h_2(x) \cup h_2(y)\}$
$=\{t : \gamma_1 \in h_1(x) \vee \gamma_1 \in h_1(y) \wedge \gamma_2 \in h_2(x) \vee \gamma_2 \in h_2(y)\}$
$= \{t: \gamma_1 \in h_1(x) \wedge \gamma_2 \in h_2(x) \vee \gamma_1 \in h_1(y) \wedge \gamma_2 \in h_2(y) \vee \gamma_1 \in h_1(x) \wedge \gamma_2 \in h_2(y) \vee \gamma_1 \in h_1(y) \wedge \gamma_2 \in h_2(x)\}$
$\supseteq \{t : \gamma_1 \in h_1(x) \wedge \gamma_2 \in h_2(x) \vee \gamma_1 \in h_1(y) \wedge \gamma_2 \in h_2(y)\}$
$=\{t : \gamma_1 \in h_1(x), \gamma_2 \in h_2(x)\} \cup \{t: \gamma_1 \in h_1(y), \gamma_2 \in h_2(y)\}$
$=(h_1 \oslash h_2)(x) \cup (h_1 \oslash h_2)(y)$
Thus $(h_1 \oslash h_2)(x-y) \supseteq (h_1 \oslash h_2)(x) \cup (h_1 \oslash h_2)(y)$ ......(2)
From 1 and 2 we get $(h_1 \oslash h_2)$ is hesitant fuzzy ideal of a ring R.

5-Homomorphism on hesitant fuzzy ideal of R.

Theorem 5-1:-Let $f: R \to R^{\star}$be onto homomorphism from a ring R to a ring $R^{\star}$, let h_R is hesitant fuzzy ideal of a ring R then $f(h_R)$ is hesitant fuzzy ideal of a ring $R^{\star}$.

Proof:-Let $y \in R^{\star}$,by def. $f(h_R)(y) = \cup_{x \in f^{-1}(y)} h_X(x)$ if $f^{-1}(y) \neq \emptyset$,Where$f^{-1}(y)$ $=\{ x: f(x) = y \}$

Now, we must prove

1.Let $x = x_1 - x_2$, $x_1, x_2 \in R$ since f homomorphism and onto there exists a,b $\in R^{\star}$ such that $f(x_1) = a$, $f(x_2) = b$,since a ,b $\in R^{\star}$and $f^{-1}(a) \neq \emptyset$, $f^{-1}(b) \neq \emptyset$ so that $f^{-1}(a-b) \neq \emptyset$, $f(h_R)(a-b)= \cup_{x \in f^{-1}(a-b)} h_R(x) \supseteq \cup_{f(x_1)=a, f(x_2)=b} h_R(x_1 - x_2)$ $\supseteq \cup_{f(x_1)=a, f(x_2)=b} h_R(x_1) \cap h_R(x_2) = \cup_{f(x_1)=a} h_R(x_1) \cap \cup_{f(x_2)=b} h_R(x_2)$ $= f(h_R)(a) \cap f(h_R)(b)$.Thus $f(h_R)(a-b) \supseteq f(h_R)(a) \cap f(h_R)(b)$

Similarly

$f(h_R)(a.b)= \cup_{f(x_1)=a, f(x_2)=b} h_R(x_1 x_2) \supseteq \cup_{f(x_1)=a, f(x_2)=b} h_X(x_1) \cup h_X(x_2)$ $= \cup_{f(x_1)=a, f(x_2)=b} h_R(x_1) \vee h_R(x_2) = \cup_{f(x_1)=a} h_R(x_1) \cup \cup_{f(x_2)=b} h_R(x_2)$ $= \cup_{f(x_1)=a} h_R(x_1) \cup \cup_{f(x_2)=b} h_R(x_2) = f(h_R)(a) \cup f(h_R)(b)$

Hence $f(h_R)(a.b) \supseteq f(h_R)(a) \cup f(h_R)(b)$,Thus$f(h_R)$ is hesitant fuzzy ideal of a ring R.

Theorem 5-2:- Let $f: R \to R^{\star}$be homomorphism from a ring R to a ring $R^{\star}$ .If h_R is hesitant fuzzy ideal of a ring R and has the sub-property then $f(h_R)$ is hesitant fuzzy ideal of a ring $R^{\star}$.

Proof: Since h_R is hesitant fuzzy ideal of a ring R and has the sub-property

There is $x_\circ \in f^{-1}_{(y)}$ such that $h_R(x_\circ) = \cup_{t \in f^{-1}_{(y)}} h_R(t)$

Thus $f(h_R)(x-y) = \cup_{t \in f^{-1}(x-y)} h_R(t) \supseteq h_R(x-y) \supseteq h_R(x) \cap h_R(y)$

So $f(h_R)(x-y) \supseteq h_R(x) \cap h_R(y)$,Hence,$f(h_R)(xy) = \cup_{t \in f^{-1}(xy)} h_R(t) \supseteq h_R(xy) \supseteq h_R(x) \cup h_R(y)$

So $f(h_R)(xy) \supseteq h_R(x) \cup h_R(y)$.Thus $f(h_R)$ is hesitant fuzzy ideal of a ring $R^{\star}$.

Theorem 5-3:-Let $f: R \to R^{\star}$be homomorphism from a ring R to a ring $R^{\star}$,let h_R is hesitant fuzzy ideal of a ring R and $h_{R^{\star}}$is hesitant fuzzy ideal of a ring $R^{\star}$, then $f^{-1}(h_{R^{\star}})$ is hesitant fuzzy ideal of a ring R.

Proof: Since $h_{R^{\star}}$is hesitant fuzzy ideal of a ring $R^{\star}$,

Thus $f^{-1}(h_{R^{\star}})(x-y) = h_{R^{\star}}(f(x-y))$
$= h_{R^{\star}}(f(x) - f(y))$
$\supseteq f^{-1}(h_{R^{\star}})(x) \cap f^{-1}(h_{R^{\star}})(y)$

Thus $f^{-1}(h_{R^{\star}})(x-y) \supseteq f^{-1}(h_{R^{\star}})(x) \cap f^{-1}(h_{R^{\star}})(y)$

Hence $f^{-1}(h_{R^{\star}})(xy) = h_{R^{\star}}(f(xy))$
$= h_{R^{\star}}(f(x) f^{-1}(y))$
$\supseteq h_{R^{\star}}(f(x)) \cup h_{R^{\star}}(f(y))= f^{-1}(h_{R^{\star}})(x) \cup f^{-1}(h_{R^{\star}})(y)$

Thus $f^{-1}(h_{R^{\star}})(xy) \supseteq f^{-1}(h_{R^{\star}})(x) \cup f^{-1}(h_{R^{\star}})(y)$

Thus $f^{-1}(h_{R^{\star}})$ is hesitant fuzzy ideal of a ring R.

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