Semi Prime hesitant fuzzy submodule

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ABSTRACT

In this paper, we study the definition of hesitant fuzzy set, with some of its properties and introduce the definition of semi prime hesitant fuzzy submodule with the result equivalent of definition . We proved the image and the inverse image of the concept above with respect to the homomorphism between modules , finally we proved all the relationships between these concepts.

Keywords: hesitant fuzzy set , hesitant fuzzy module , Prime hesitant fuzzy submodule, Semi Prime hesitant fuzzy submodul

1- Introduction

Torra (2010) proposed a new generalized type of fuzzy set called hesitant fuzzy set (HFS) and he defined the complement, union and intersection of HFS [12] . Xia and Xu (2011) originally gave the mathematical expressions of HFS, and some operational laws for HFS, such as the addition and multiplication operations [13] . Afterwards, Liao and Xu (2014a) introduced the subtraction and division operations over HFSs [14] . Deepak D and J. John (2014) introduced the notion of hesitant fuzzy subgroups[6] . Abbasi, et al. (2018) introduced the hesitant fuzzy ideal, Hesitant fuzzy bi-ideal, and hesitant fuzzy interior ideal in Γ-semi group [2] . Kim et al. (2019) defined hesitant fuzzy subgroupoid, hesitant fuzzy subring , Hesitant fuzzy ideal [9] . Xueyou Chen (2020) introduced the notion of rough hesitant fuzzy group, and investigate some of their properties [5] . Ali Abbas. J. and M. J. Mohammed (2021) introduced the notions of hesitant fuzzy ideal, hesitant fuzzy prime ideal , hesitant fuzzy strongly prime ideal and hesitant fuzzy 3- prime ideals of a ring R [1] . A. Fadhil.J (2021) introduced the definitions of hesitant fuzzy submodule , prime hesitant fuzzy sub module and strongly prime hesitant fuzzy submodule [7] . In this study, we shall introduce the notion of semi prime hesitant fuzzy submodule and we proved some properties and results about these concept . Also , we proved the relationship between the above concepts and other concepts .

2-Preliminaries

In this section, we are about to give the concept of hesitant fuzzy set with some basic definitions and properties about it which are used in the next section.

Definition (2.1) [12]:

Let X be a reference set a hesitant fuzzy set (HFS) A on X is defined in terms of a function $h_A(x)$ when applied to X returns a finite subset of [0 , 1] i.e. $A = \{(x , h_A(x)) \mid x \in X\}$ where $h_A(x)$ is a set of some different values in [0 , 1], representing the possible membership degrees of the element $x \in X$ *to* A for convenience, we call $h_A(x)$ a hesitant fuzzy element (HFE).

Definition (2.2) [13]:

Let X be a reference set, then we define some types of hesitant fuzzy set . empty set : $h^0(x) = \{(x,\{0\})\ \forall\, x \in X\} = \{0\}$, full set : $h^1(x) = \{(x,\{1\})\ \forall\, x \in X\} = \{1\}$, complete ignorance for $x \in X$ (all is possible) : $h^{[0,1]}(x) = \{(x,[0,1])\ \forall\, x \in X\} = [0,1]$ and set for a nonsense $h^{\emptyset}(x) = \{(x,\emptyset)\ \forall\, x \in X\} = \emptyset$ such that $h^{\emptyset}(x) \subseteq h(x) \subseteq h^{[0,1]}(x)$

Definition (2.3) [10]:

Let A , B *be a* hesitant fuzzy set of a set X and Y respectively and let $f : X \to Y$ be a mapping. Then the image of A under f , denoted by $f(A)$, is a hesitant fuzzy set in Y defined as follows: for each y ? Y

$$f(A)(y) = \begin{cases} \bigcup_{x \in f^{-1}(y)} A(x) & iF\ \ f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

the invers of B under f, denoted by $f^{-1}(B)$, is a hesitant fuzzy set in X defined as follow : for each $x \in X$, $f^{-1}(B)(x) = B\big(f(x)\big)$.

Definition (2.4) [10]:

Let h be an hesitant fuzzy set of a set X. Then h is called a hesitant fuzzy point with the support x ∈ X and the value λ , denoted by x_λ, if x_λ : X → P[0,1] is the mapping given by: for each y ∈ X, $x_\lambda(y)$

$$= \begin{cases} \lambda \subset [0,1] & \text{if} \quad y = x \\ \phi & \text{otherwise.} \end{cases}$$

We will denote the set of all hesitant fuzzy points in X as HFP(X).

Definition (2.5) [11+13]:

For any h_1 , h_2 in HFSs , some operations on them can be described as follows

(1) $(h_1 \cup h_2)(x) = = \cup_{y_1 \in h_1, y_2 \in h_2}\ max\,\{y_1, y_2\}$

(2) $(h_1 \cap h_2)(x) = \cup_{y_1 \in h_1, y_2 \in h_2}\ min\,\{y_1, y_2\}$

(3) ${h_1}^{\lambda}(x) = \{y^{\lambda} | y \in h_1(x)\}$, $\lambda h_1(x) = \{1 - (1-y)^{\lambda} \mid y \in h_1(x)\}$

(4) $(h_1 \otimes h_2)\,(x) = \{y_1 y_2 \mid y_1 \in h_1(x), y_2 \in h_2(x)\}$

(5) $(h_1 \oplus h_2)\,(x) = \{y_1 + y_2 - y_1 y_2 \mid y_1 \in h_1(x), y_2 \in h_2(x)\}$

Definition (2.6) [7] :

Let X be an anon empty set , h be a hesitant fuzzy set and $E \subseteq [0,1]$. Then, the $E-$ upper (*level set*) of h defined h_E=$\{x \in M: h(x) \supseteq E\}$.

Proposition (2.7) [7] :

Let A, B be a hesitant fuzzy sets over R-module M then

(1) $(A \cap B)_E = A_E \cap B_E$

(2) If $A \subseteq B$ *then* $A_E \subseteq B_E$

(3) $A_E \cup B_E \subseteq (A \cup B)_E$

Definition (2.8) [7] :

Let h be a hesitant fuzzy set over R−module M then h is said to be hesitant fuzzy submodule (in short, HFSM) over R-module M if for all $x, y \in M$, $r \in R$

(i) $h(x-y) \supseteq h(x) \cap h(y)$ (ii) $h(rx) \supseteq h(x)$.

Example (2.9) [7] :

Let h be a hesitant fuzzy set of Z_2-module $(Z_2, +_2)$ such that

$h = \{(0, h(0)), (1, h(1))\}$, $h(0) = \{0.1, 0.3, 0.6, 0.8\}$, $h(1) = \{0.1, 0.3\}$

Then h is a hesitant fuzzy submodule of Z_2-module Z_2

Proposition (2.10) [7] :

Let h_1, h_2 be a hesitant fuzzy module of R-module M , then ${h_1}^{\lambda}$, λh_1 , $h_1 \otimes h_2$ *and* $h_1 \oplus h_2$ are hesitant fuzzy module of R-module M .

Theorem (2.11) [7] :

Let h be a hesitant fuzzy set of left $R-$ module M. Then h is hesitant fuzzy submodule of R-module M iff h_E , $E \subseteq [0,1]$ is submodule of R-module M.

Proposition (2.12) [7] :

Let f be a homomorphism from R-module M into R-module N, if B hesitant fuzzy submodule of N then $f^{-1}(B)$ hesitant fuzzy submodule of M.

Proposition (2.13) [7]:

Let f be an epimorphism from R-module M into R-module N if A hesitant fuzzy submodule of M then $f(A)$ hesitant fuzzy submodule of R-module N.

Proposition (2.14) [7] :

Let M R-module . If $r_t \in HFP(R)$ and x_λ , $y_\beta \in HFP(M)$ then
$r_t \circ x_\lambda = (rx)_E$, $E = t \cap \lambda$.

Definition (2.15) [7] :

Let h be a hesitant fuzzy submodules of R-module M ,
$b_s \in HFP(M)$ and $r_t \in HFP(R)$. Then
$(h : b_s) = \cup \{a_E : a_E \in HFP(R)$ such that $a_E \circ b_s \subseteq h\}$
$(h : r_t) = \cup \{x_E : x_E \in HFP(M)$ such that $r_t \circ x_E \subseteq h\}$.

Definition (2.16) [7] :

Let h be a hesitant fuzzy submodule of R-module M then h is said to be prime hesitant fuzzy submodule of M (PHFSM) if $x_\lambda \circ y_\beta \subseteq h$ for x_λ $HFP(R)$ and $y_\beta \in HFP(M)$ we have either $y_\beta \subseteq h$ or $x_\lambda \subseteq (h : b_s)$ for all $b_s \in HFP(M)$.

Definition (2.17) [7] :

Let h be a hesitant fuzzy submodule of an R –module M . Then h is said to be a strongly prime hesitant fuzzy submodule of M (S -PHFSM) if $x_\lambda \circ y_\beta \subseteq h$ for $x_\lambda \in HFP(R)$ and $y_\beta \in HFP(M)$ implies that $y_\beta \subseteq h$.

Theorem (2.18) [7] :

Let h *be a* hesitant *fuzzy submodules* of R-module M .Then h is a strongly prime hesitant fuzzy submodule of R-module M iff
$h(rx) = h(x)$ for all $r \in R, r \neq 0$ and $x \in M$

Proposition (2.19) [7] :

Every strongly prime hesitant fuzzy submodule of R –module M is prime hesitant fuzzy submodule of M **.**

Definition (2.20) [3]:

Let K be a submodule of an R-module M. Then K is said to be semi prime submodule of M if $r^2 x \in K$ for all $r \in R$, $x \in M$ then $r x \in K$.

3. Semi prime hesitant fuzzy submodule

In this section we define the concept semi prime hesitant fuzzy submodule and prove the result equivalent to the definition with prove some results about it .

Definition (3.1):

Let h be a hesitant fuzzy submodule of an R –module M . Than h is said to be a semi prime hesitant fuzzy submodule of M (SPHFSM) if ${x_\lambda}^2 \circ y_\beta \subseteq h$ for $x_\lambda \in HFP(R)$ and $y_\beta \in HFP(M)$ implies that $x_\lambda \circ y_\beta \subseteq h$.

Proposition (3.2) :

Let h be a hesitant fuzzy submodule of R –module M then h is SPHFSM of M iff h_E is semi prime submodule of M , $E \subseteq [0,1]$.

Proof :

Sppose that h is SPHFSM of M .
Let $r^2 a \in h_E$ such that $r \in R$, $a \in M$.
Thus $h(r^2 a) \supseteq E$, *so* $(r^2 a)_E \subseteq h$.
Implies ${r_E}^2 \circ a_E \subseteq h$.

Since h SPHFSM of M .
Then $r_E \circ a_E \subseteq h$.
Implies $(ra)_E \subseteq h$, thus $ra \in h_E$.
So , h_E is semi prime submodule of M .
Conversely , if h_E is semi prime submodule of M .
Let $r_\lambda{}^2 \circ x_\gamma \subseteq h$.
For $r_\lambda \in HFP(R)$ and $x_\gamma \in HFP(M)$.
Thus $(r^2x)_E \subseteq h$, where $E = \lambda \cap \gamma$, implies $r^2x \in h_E$.
Since h_E is semi prime submodule of M .
Thus $rx \in h_E$, sothat $(rx)_E \subseteq h$.
Implies $r_\lambda \circ x_\gamma \subseteq h$. Therefore , h is SPHFSM of M . □

Proposition (3.3) :

Let h , k be a semi prime hesitant fuzzy submodule of R –module M . Then $h \cap k$ is semi prime hesitant fuzzy submodule of M .

Proof :

Let $r_\alpha{}^2 \circ x_\lambda \subseteq h \cap k$, for $r_\alpha \in HFP(R)$ *and* $x_\lambda \in HFP(M)$.
Implies $r_\alpha{}^2 \circ x_\lambda \subseteq h \wedge r_\alpha{}^2 \circ x_\lambda \subseteq k$.
Since h , k are SPHFSM of M , thus $r_\alpha \circ x_\lambda \subseteq h \wedge r_\alpha \circ x_\lambda \subseteq k$.
Implies $r_\alpha \circ x_\lambda \subseteq h \cap k$, so $h \cap k$ is SPHFSM of M . □

Proposition (3.4) :

Let f be an homomorphism from R-module M into R-module N , if B is SPHFSM of R-module N then $f^{-1}(B)$ is SPHFSM of R- module M.

Proof :

Since $f^{-1}(B)$ hesitant fuzzy submodule of M . (by Proposition 2.12)
Let $r_\alpha{}^2 \circ x_\lambda \subseteq f^{-1}(B)$.
For $r_\alpha \in HFP(R)$ and $x_\lambda \in HFP(M)$.
Implies $(r^2 x)_E \subseteq f^{-1}(B)$, $E = \alpha \cap \lambda$, thus $r^2 x \in \left(f^{-1}(B)\right)_E$.
Hence $\left(f^{-1}(B)\right)(r^2 x) \supseteq E$.
by definition of the inverse image $B\left(f(r^2 x)\right) \supseteq E$.
Since f be an homomorphism , thus $r^2 f(x) \in B_E$.
Since B is SPHFSM of N then B_E semi prime submodule of N .
Since $f(x) \in N$. Hence $r f(x) \in B_E$.
Implies $B\left(f(r x)\right) \supseteq E$.
by definition of the inverse image $\left(f^{-1}(B)\right)(r x) \supseteq E$
Thus $(r x)_E \subseteq f^{-1}(B) \Rightarrow r_\alpha \circ x_\lambda \subseteq f^{-1}(B)$.
So , $f^{-1}(B)$ is SPHFSM of R-module M . □

Proposition (3.5) :

Let f be an epimorphism from R-module M into R-module N. If A is SPHFSM of R-module M then $f(A)$ is SPHFSM of R-module N.

Proof :

Since $f(A)$ hesitant fuzzy submodule of N . (by Proposition 2.13)
Let $r_\alpha{}^2 \circ y_\lambda \subseteq f(A)$.
For $r_\alpha \in HFP(R)$ and $y_\lambda \in HFP(N)$.
Implies $(r^2 y)_E \subseteq f(A)$, $E = \alpha \cap \lambda$, thus $r^2 y \in \left(f(A)\right)_E$
Since f is onto and $y \in$ N , Hence $\exists\, a \in M$ Such that $f(a) = y$.
Hence $(f(A))\left(r^2 f(a)\right) \supseteq E$.
Since f is homomorphism , thus $(f(A))\left(f(r^2 a)\right) \supseteq E$.
by definition of the inverse image $f^{-1}\left(f(A)\right)(r^2 a) = A(r^2 a) \supseteq E$

Since A is SPHFSM of M then A_E semi prime submodule of M .
Hence $r\ a \in A_E$, implies $A(ra) \supseteq E$.
$f^{-1}\left(f(A)\right)(ra) = A(ra) \supseteq E$, *therfor* $(f(A))\left(f(ra)\right) \supseteq E$
$(f(A))(ry) \supseteq E$, *implies* $(r^2\, y)_E \subseteq f(A)$
$r_\alpha \circ y_\lambda \subseteq f(A)$, *thus* $f(A)$ is SPHFSM of R-module N. ◻

Theorem (3.6) :

Let h be a hesitant fuzzy submodules of R-module M .Then h is a semi prime hesitant fuzzy submodule of R-module M iff
$h(r^2x) = h(rx) \quad \forall\ r \in R, r \neq 0$, *and* $x \in M$.

Proof :

suppose that $h(r^2x) = h(rx)$ the condition hold .
Let $r_\alpha{}^2 \circ x_\lambda \subseteq h$, for $r_\alpha \in HFP\ (R)$ and $x_\lambda \in HFP\ (M)$.
Thus $(r^2x)_E \subseteq h$, $E = \alpha \cap \lambda$, implies $r^2\, x \in h_E$
Implies $h(r^2x) \supseteq E$.
Since the condition hold .
Thus $h(r\,x) \supseteq E$, implies $r\ x \in h_E$, Hence $r_\alpha \circ x_\lambda \subseteq h$
So , h is semi prime hesitant fuzzy submodule of M .
Conversely ,
Suppose that h is semi prime hesitant fuzzy submodule of M .
Let $r \in R, r \neq 0$ *and* $x \in M$.
Suppose that $h(r^2x) = E$.
Then $(r^2x)_E \subseteq h$, implies $r^2{}_E \circ x_E \subseteq h$.
Since h is SPHFSM , Hence $r_E \circ x_E \subseteq h$.
Thus $(r\,x)_E \subseteq h$, implies $h(rx) \supseteq E = h(r^2x)$.
Since $h(r^2x) \supseteq h(rx)$, thus $h(r^2x) = h(rx)$.

Proposition (3.7) :

Let h be a hesitant fuzzy module of $R -$ module M and A submodule of M then A is a semi prime submodule of M iff

$$h(x) = \begin{cases} [0,1] & if \quad x \in A \\ \emptyset & otherwise \end{cases}$$

is SPHFSM of M .

Proof:

Suppose that A is semi prime submodule of M.
Let $r \in R, r \neq 0$ *and* $x \in M$, if $r^2a \in A$.
since A is semi prime submodule of M , *then* $ra \in A$
Hence $h\,(r^2a) = [0\,,1] = h\,(ra)$. if $r^2a \notin A$.
Since A is semiprime submodule of M , *then* $ra \notin A$.
Hence $h\,(r^2a) = \emptyset = h\,(ra)$.
Therefore , $h\,(r^2a) = h\,(ra)$.
Hence h is semiprime hesitant fuzzy submodule of M .
Conversely ,
Let h is semiprime hesitant fuzzy submodule of M .
Let $r^2a \in A$, *for* $r \in R$, $r \neq 0$ *and* $a \in M$.
Hence $h\ (r^2a\,) = [0,1]$.
Since h is semiprime hesitant fuzzy submodule of M.
Implies $h\,(r^2a) = h\ (ra) = [0,1]$, *so that* $ra \in A$.
Hence A is semiprime submodule of M. ◻

Proposition (3.8) :

Let h be a semi prime hesitant fuzzy submodule of M then λh is semi prime hesitant fuzzy submodule of M .

Proof :

Since λh hesitant fuzzy submodule of M . (by Proposition 2.10)

Let $r \in R$, $r \neq 0$ and $a \in M$.

$\lambda h(r^2\, x) = \{1-(1-y)^{\lambda} \mid y \in h(r^2\, x)\}$

Since h is semi prime hesitant fuzzy submodule of M .

Thus , $\lambda h(r^2\, x) = \{1-(1-y)^{\lambda} \mid y \in h(rx)\} = \lambda h(r\, x)$.

Thus , λh is semi prime hesitant fuzzy submodule of M . ◻

Proposition (3.9) :

Let h_1 and h_2 are semi prime hesitant fuzzy submodule of M . Then $h_1 \otimes h_2$ semi prime hesitant fuzzy submodule of M.

Proof :

Since $h_1 \otimes h_2$ hesitant fuzzy submodule of M . (by Proposition 2.10)

Let $r \in R$, $r \neq 0$ *and* $a \in M$

$(h_1 \otimes h_2)(r^2 a) = \{y_1 y_2 \mid y_1 \in h_1(r^2 a)\ ,\ y_2 \in h_2(r^2 a\ )\}$

Since h_1, h_2 are semi prime hesitant fuzzy submodule of M.

Thus $(h_1 \otimes h_2)(r^2 a) = \{y_1 y_2 \mid y_1 \in h_1(ra)\ ,\ y_2 \in h_2(ra)\}$.

Implies $(h_1 \otimes h_2)(r^2 a) = (h_1 \otimes h_2)(ra)$.

Thus $h_1 \otimes h_2$ is semi prime hesitant fuzzy submodule of M. ◻

Proposition (3.10) :

Let h be a semi prime hesitant fuzzy submodule of M. Then $A = \{x \in M \mid h(x) = h(0)\}$ is a semi prime prime submodule of M.

Proof:

Let $rx^2 \in A$ where $r \in R$, $r \neq 0$ and $x \in M$.

We get $h(\,rx^2) = h(0)$.

Since h is semi prime hesitant fuzzy submodule of M.

Thus $h(r\,x) = h(0)$, implies $rx \in A$.

Hence A is a semi prime submodule submodule of M. ◻

Proposition (3.11):

Every prime hesitant fuzzy submodule of R –module M is semi prime hesitant fuzzy submodule of M .

Proof :

Suppose that h is prime hesitant fuzzy submodule of M.

Let $r_\alpha{}^2 \circ x_\lambda \subseteq h$.

For $r_\alpha \in HFP\,(R)$ and $x_\lambda \in HFP\,(M)$.

Implies $r_\alpha \circ (r_\alpha \circ x_\lambda) \subseteq h$.

Since h is prime hesitant fuzzy submodule of M .

We have $r_\alpha \circ x_\lambda \subseteq h$ *or* $r_\alpha \subseteq (h{:}\, b_E)$ for all $b_E \in HFP\,(M)$.

So each case implies that $r_\alpha \circ x_\lambda \subseteq h$. Thus h *is* SPHFSM of M . ◻

Remark (3.12) :

Notice that the converse of proposition (3.11) is not true in general , for example .

Let $M = \{0\} \oplus 2Z$ as a $Z-$module and $h : Z \rightarrow P[0,1]$ defined by

$$h(x) = \begin{cases} [0,1] & if \quad x \in N = \{0\} \oplus 6Z \\ \emptyset & \text{otherwise} \end{cases}$$

Since N is *semi prime* submodule of M. (see 8 , example (3.2.15))

Thus h is semi prime hesitant fuzzy submodule of M .

Since $3_{[0.1\,,0.5]} \circ (0{,}2)_{[0.1\,,0.5]} = (0{,}6)_{[0.1\,,0.5]} \subseteq h$, because

$h\big((0{,}6)\big) = [0\ ,1] \supseteq [0.1\,,0.5]$, but

$(0{,}2)_{[0.1\,,0.5]} \nsubseteq h$, because $h(0{,}2) = \emptyset \nsupseteq [0.1\,,0.5]$

$3_{[0.1\,,0.5]} \nsubseteq (h{:}\, b_E\,)$, $b_E \in HFP\,(M)$

There exist $(0{,}3)_{[0.1\,,0.5]} \in HFP\,(M)$.

$3_{[0.1\,,0.5]} \circ (0{,}3)_{[0.1\,,0.5]} = (0{,}9)_{[0.1\,,0.5]} \nsubseteq h$, because

$h\big((0{,}9)\big) = \emptyset \nsupseteq [0.1\,,0.5]$

Thus h is not prime hesitant fuzzy submodule of M.

<u>Corollary (3. 13):</u>

Every strongly prime hesitant fuzzy submodule of R-module M is a *semi prime* hesitant fuzzy submodule of M.

<u>Proof :</u>

Let h is strongly prime hesitant fuzzy submodule of M.

Implies h is prime hesitant fuzzy submodule of M. (by proposition 2.18)

Hence h is semi prime hesitant fuzzy submodule of M. (by proposition 3.11) ◻

<u>Remark (3.14) :</u>

Notice that the the converse of Corollary (4.13) is not true in general , for example .

Let $M = Z$ as a $Z-$module and $h : Z \rightarrow P[0{,}1]$ defined by

$$h(x) = \begin{cases} [0{,}1] & if \quad x \in N = 6Z \\ \emptyset & \text{otherwise} \end{cases} .$$

Since N is semi prime submodule of M. (see 9 , example (3.8))

Thus h is semi prime hesitant fuzzy submodule of M .

Since $9_{[0\,,0.2]} \circ 4_{[0.3\,,0.7]} = 36_{[0\,,0.2]} \subseteq h$, because .

$h(36) = [0\ ,1] \supseteq [0\,,0.2]$, but

$4_{[0.3\,,0.7]} \nsubseteq h$, because $h(4) = \emptyset \nsupseteq [0.3\,,0.7]$

Hence h is not strongly prime hesitant fuzzy submodule of M .

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