On Real Aw*-Algebras with Abelian Skew-Hermitian Part

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ABSTRACT

It is known that, unlike to the complex case, in real C^*- algebras R their Hermitian part R_s and skew-hermitian part R_k are not connected by the relation $R_k = iR_s$. In [1] described up to *- isomorphism all real W^*-algebras with abekian Hermitian part. In paper this result is generalized for the real AW^*-algebras. Exactly, it is described up to *-isomorphism all real AW^*-algebras with abelian Hermitian part.

Keywords: real AW^*-algebra, skew-Hermitian part of real C*-algebras

Preliminaries

Recently, along with the theory of W^*- and C^*-algebras, the theory of real W^*- and C^*-algebras has also been developed quite well. It is known that, unlike to the complex case, in real C^*-algebras R their hermitian part R_s and skew-hermitian part R_k are not connected by the relation $R_k = iR_s$. In [1] described up to *-isomorphism all real W^*-algebras with abelian skew-hermitian part. In paper it is considered real AW^*-algebras with abelian skew-hermitian part.

Definition 1

Let A be a Banach *-algebra over the field C. The algebra A is called a C*-*algebra,* if $\|AA^*\| = \|A\|^2$, for any A∊A.

Definition 2

A real Banach *-algebra $\Re$ is called a real C^*-*algebra,* if $\|AA^*\| = \|A\|^2$ and an element $1 + AA^*$ is invertible for any $A \in \Re$

It is easy to see that $\Re$ is a real C^*-algebra if and only if a norn on $\Re$ can be extended onto the complexification A=R+iR of the algebra R so that algebra A is a C^*-algebra (see [2], [3] and [4.,5.1.1).

Let *B(H)* be the algebra of all bounded linear operators on a complex Hilbert space *H*. A weakly closed *-subalgebra *M* containing the identity operator **1** in *B(H)* is called a W^*-*algebra*. A real *-subalgebra $R \subset B(H)$ is called a *real* W^*-*algebra* if it is closed in the weak operator topology, $1 \in R$ and $R \cap iR = \{0\}$ (see [2], [3]).

The notion of AW^*-algebras was introduced by Kaplansky as an abstract generalization of weakly closed self-adjoint operator algebras on a complex Hilbert space (W^*-algebras). He showed that much of the “non-spatial theory” of W^*-algebras can be extended to AW^*-algebras. By an AW^*-algebra it is meant a C^*-algebra such that the left annihilator of any subset is aprincipal left ideal geerated by a projection, i.e. an idempotent self-adjoint element. Every W^*-algebra is an AW^*-algebra, but the converse is not true as was shown by Dixmier with an abelian example. Let A be real or complex *-algebra and let S be nonempty subset of A. Put

$R(S)=\{x\in A \mid sx=0 \text{ for all } s\in S\}$

And call $R(S)$ the *right-annihilator* of S. Similarly

$L(S)=\{x\in A \mid xs=0 \text{ for all } s\in S\}$

Denotes the *left-annihilator* of S. Following [5] we introduce the following notions.

Definition 3

A *-algebra A is called a *Baer *-algebra* if for any nonempty $S\subset A$, $R(S)=gA$ for an appropriate projection g.

Since $L(S)=(R(S^*))^*=(hA)^*=Ah$ the definition is symmetric and can be given in terms of the left-annihilator and a suitable projection h. $S^*=\{s^* \mid s\in S\}$

Defenition 4

A complex or real C^*-algebra, which is a Baer *-algebra is called an (complex or real, respectively) *AW^*-algebra*

As mentioned above in the paper [1] it was described up to *-isomorohism all real W^*-algebras with abelian skew-hermitian part. Here we have obtained some results from this work for real AW^*-algebras. The main result of this work is the following theorem.

Theorem

Let A be real AW^*-algebra whose skew-symmetric part A_k is abelian. Then

1) For any $x,y\in A_k$, the product xy is a center element of A, i.e. it commutes with every element of A.

2) If the JC-algebra A_s is abelian, then the real AW^*-algebra A is commutative.

Proof. Since A_k is abelian, we have $(xy)^*=xy\in A_s$, and xy commutes with every element of A_k. Further, since $xy\in A_s$, it follows that $[a,xy]\in A_k$ for any $a\in A_s$, and therefore $[a,xy]$ commutes with x and with y, and thus with xy, i.e. $[[a,xy]xy]=0$. Since the symmetric element xy is normal that $[a,xy]=0$ for any $a\in A_s$. Therefore, xy commutes with any element in $R=R_s+R_k$. There exists a central projection z is R_s such that zR_s is of type I_1 (i.e., an abelian JC-algebra) and $(1-z)R_s$ is a type I_2 JC-algebra. The central element z in R_s is automatically central in R. Indeed, for $x\in R_k$, the commutator $[z,x]$ is in R_s, and therefore $[z,[z,x]]=0$, and $[z,x]=0$, i.e., z commutes with each element of R_k as well. Thus, $R=zR\oplus(1-z)R$, where the real AW^*-algebra zR has the abelian symmetric part zR_s and the abelian skew-symmetric part $(zR)_k=zR_k$. The real W^*-algebra zR is abelian.

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