Numerical modelling of the one-dimensional thermoplastic coupled problem for isotropic materials using deformation theory

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ABSTRACT

The coupled thermo-plastic dynamic boundary problem is formulated using the deformation theory of plasticity for small deformations. The explicit and implicit schemes of finite difference equations in one dimension case are constructed. The discreet equations are numerically solved using the explicit and implicit schemes. Comparison shows the coincidence of the numerical results received using two methods.

Keywords:

Introduction

The governing equations of the thermoplastic coupled boundary problems consist of the motion, modelling and heat conduction equations [1]. In this paper the incremental type of plasticity modelling equations for isotropic and transversely isotropic materials are constructed. Using the proposed plasticity modelling equations, we formulate the thermo plasticity boundary problem which consists of hyperbolic and parabolic type of differential equations. In 1D coupled problem, the equations of motion and heat conduction depend on the displacement and temperature parameters; the corresponding initial and boundary conditions are inserted. Then the discrete equations are constructed by a finite difference method. In the discretization process all the derivatives are approximated by their corresponding difference formulas and it turns out that two kinds of schemes appear, i.e. explicit and implicit schemes. The explicit scheme is solved with the help of recurrent formulas. For the solution of implicit scheme, the “consecutive” method is used [2]. Comparison of two results shows a good coincidence.

Formulation of the dynamic thermoplastic coupled boundary problem using deformation theory of plasticity

The coupled thermodynamic elastic plasticboundary value problem consists of the motion equations [1]

$$\sigma_{ij,j} + X_i = \rho \ddot{u}_i , \qquad (1)$$

non linear constitutive relation between the strains and stresses tensorsfor isotropic or anisotropic materials [2]

$$\sigma_{ij} = K(\theta - 3\alpha\vartheta)\delta_{ij} + \frac{\sigma_u}{\varepsilon_u} e_{ij} \qquad (2)$$

The heat equation for isotropic materials [1]

$$\lambda_0 T_{,ii} - c_\varepsilon \dot{T} - T_0 \cdot \alpha(3\lambda + 2\mu) \cdot \dot{\varepsilon}_{ij} = 0 \quad (3)$$

Cauchy relations [1]

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \quad (4)$$

withthe corresponding initial

$$u_i\big|_{t=t_0} = \varphi_i\,,\ \dot{u}_i\big|_{t=t_0} = \psi_i\,,\ T\big|_{t=t_0} = T_0 \quad (5)$$

and boundary conditions

$$u_i\big|_{\Sigma_1} = u_i^0\,,\quad T\big|_{\Sigma_1} = \bar{T}_0\,,\quad \sigma_{ij} n_j\big|_{\Sigma_2} = S_i^0 \quad (6)$$

where c_ε − heat at a constant temperature, α − thermal expansion coefficient, λ_0 − the heat flow coefficient.

Equations (1) - (6) in one dimension case take the form

$$\frac{\partial \sigma_{11}}{\partial x} + X_1 = \rho \frac{\partial^2 u}{\partial t^2} \quad (7)$$

$$\sigma_{11} = (\lambda + 2\mu - \frac{4}{3}(\mu - \mu')(1 - \frac{\varepsilon_u^*}{\varepsilon_u}))\varepsilon_{11} - \alpha(3\lambda + 2\mu)(T - T_0) \quad (8)$$

$$\varepsilon_{11} = \frac{\partial u}{\partial x} \quad (9)$$

Substituting the eq. (9) into eq. (8)

$$\sigma_{11} = (\lambda + 2\mu - \frac{4}{3}(\mu - \mu')(1 - \frac{\varepsilon_u^*}{\varepsilon_u}))\frac{\partial u}{\partial x} - \alpha(3\lambda + 2\mu)(T - T_0) \quad (10)$$

and obtained expression into eq. (7) allows to write the motion equation in terms of displacements

$$(\lambda + 2\mu - \frac{4}{3}(\mu - \mu'))\frac{\partial^2 u}{\partial x^2} - \alpha\gamma(T - T_0) = \rho \frac{\partial^2 u}{\partial t^2}\,,$$
$$\gamma = 3\lambda + 2\mu \quad (11)$$

The heat equation in one dimension form takes the form

$$\lambda_0 \frac{\partial^2 T}{\partial x^2} - c_\varepsilon \frac{\partial T}{\partial t} - T_0 \alpha\gamma \frac{\partial^2 u}{\partial x \partial t} = 0\,, \quad (12)$$

with appropriate initial and boundary conditions

$$u(x,t)\big|_{t=0} = \varphi(x_i)\,,\ \frac{\partial u}{\partial t}\bigg|_{t=0} = \psi(x_i)\,,\ T(x,t)\big|_{t=0} = T_0 \quad (13)$$

$$u(x,t)\big|_{x=0} = u_0\,;\ u(x,t)\big|_{x=\ell} = \bar{u}_0\,;$$
$$T(x,t)\big|_{x=0} = T_1(t)\,;\ T(x,t)\big|_{x=\ell} = T_2(t) \quad (14)$$

where $\lambda,\ \mu, \mu', \alpha,\ c_\varepsilon,\ \lambda_0$ − the known values, ℓ − the length of the rod, $\varphi,\ \psi,\ T_0,\ T_1, T_2$ − the specified amounts.

The construction of the explicit and implicit finite difference equations

Considering in the area $t \ge 0, \quad 0 \le x \le l$ two sets of parallel lines $x = ih_1\ \ (i = \overline{0,n}),\ t = k\tau\ \ (k = 0,1,2,\ldots)$ replace the derivatives in eq. (11)-(12) by difference quotients, we obtain

$$(\lambda + 2\mu - \frac{4}{3}(\mu - \mu'))\frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2} - \alpha\gamma \frac{T_{i+1}^j - T_{i-1}^j}{2h} = \rho \frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{\tau^2} \quad (15)$$

$$\lambda_0 \frac{T_{i+1}^j - 2T_i^j + T_{i-1}^j}{h^2} + C_\varepsilon \frac{T_i^{j+1} - T_i^{j-1}}{2\tau} - \alpha\gamma T_0 \frac{u_{i+1}^{j+1} - u_{i-1}^{j+1} - u_{i+1}^{j-1} + u_{i-1}^{j-1}}{4h\tau} = 0 \quad (16)$$

Solving the differential equation (15) and (16) about u_i^{j+1} and T_i^{j+1} respectively, we get

$$u_i^{j+1} = \frac{\tau^2}{\rho}\left((\lambda + 2\mu - \frac{4}{3}(\mu - \mu'))\frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2} - \alpha(3\lambda + 2\mu)\frac{T_{i+1}^j - T_{i-1}^j}{2h}\right) + 2u_i^j - u_i^{j-1} \quad (17)$$

$$T_i^{j+1} = -\frac{2\tau}{C_\varepsilon}\left(\alpha\gamma T_i^j \frac{u_{i+1}^{j+1} - u_{i-1}^{j+1} - u_{i+1}^{j-1} + u_{i-1}^{j-1}}{4h\tau} - \lambda' \frac{T_{i+1}^j - 2T_i^j + T_{i-1}^j}{h^2}\right) - T_i^{j-1} \quad (18)$$

As can be seen, eq. (17) and eq. (18) allow to find the function values $u(x,t)$ and $T(x,t)$ at the layer t^{j+1} using the given value soft he se functions at the two previous layers. The values of $u(x,t)$ on two primary layers $j=0$ и $j=1$ we can findfrom the initial conditions and for the values of $T(x,t)$ and also find the following difference relations

$$u_i^1=\frac{1}{2}\left(\frac{\tau^2}{\rho}\left((\lambda+2\mu-\frac{4}{3}(\mu-\mu'))\frac{u_{i+1}^0-2u_i^0+u_{i-1}^0}{h^2}-\alpha\gamma\frac{T_{i+1}^0-T_{i-1}^0}{2h}\right)+2u_i^0+2\psi\tau\right) \qquad (19)$$

$$T_i^1=-\frac{2\tau}{C_\varepsilon}\left(\alpha\gamma T_0\frac{u_{i+1}^1-u_{i-1}^1-u_{i+1}^0+u_{i-1}^0}{2h\tau}-\lambda'\frac{T_{i+1}^0-2T_i^0+T_{i-1}^0}{h^2}\right)-T_i^0 \qquad (20)$$

The difference equation (11) can be reduced to

$$a_iu_{i+1}^{j+1}+b_iu_i^{j+1}+c_iu_{i-1}^{j+1}=f_{ij} \qquad (21)$$

where

$$a_i=\frac{\lambda+2\mu-\frac{4}{3}(\mu-\mu')}{h^2},\ b_i=-\frac{2(\lambda+2\mu-\frac{4}{3}(\mu-\mu'))}{h^2}-\frac{\rho}{\tau^2},\ c_i=\frac{\lambda+2\mu-\frac{4}{3}(\mu-\mu')}{h^2}$$

$$f_{ij}=\alpha(3\lambda+2\mu)\frac{T_{i+1}^j-T_{i-1}^j}{2h}+\rho\frac{u_i^{j-1}-2u_i^j}{\tau^2}$$

Similarly, we can bring the difference equation (12) into the form

$$a_iT_{i+1}^{j+1}+b_iT_i^{j+1}+c_iT_{i-1}^{j+1}=f_{ij} \qquad (22)$$

where

$$a_i=\frac{\lambda_0}{h^2},\quad b_i=-2\frac{\lambda_0}{h^2}-\frac{C_\varepsilon}{2\tau},\quad c_i=\frac{\lambda_0}{h^2}$$

$$f_{ij}=\beta T_i^j\,\frac{u_{i+1}^{j+1}-u_{i-1}^{j+1}-u_{i+1}^{j-1}+u_{i-1}^{j-1}}{4h\tau}-c_\varepsilon\frac{T_i^{j-1}}{2\tau}$$

Calculating the values of the functions $u(x,t)$ and $T(x,t)$ on two primary layers at $j=0$ from the initial conditions and at $j=1$ from the eq. (15) and eq. (16) respectively. The values of these functions on the other layers can be calculated from the eq. (21) and eq. (22) using the boundary conditions of the “sweep” method [4].

The joint solution of the thermo-plasticity equations with the heat equation may describe more adequately the process of thermoplastic deformations under mechanical and thermal effects.

Numerical test.

As an example, the coupled dynamic thermoplastic problem has been solved (eq.7 – eq.10) by an explicit grid method and “sweep” method, with the following initial and boundary conditions and constants:

$$u(x,t)\big|_{t=0}=\sin\left(\frac{\pi x}{\ell}\right),\quad \frac{\partial u(x,t)}{\partial t}\bigg|_{t=0}=0,\quad T(x,t)\big|_{t=0}=T_0,$$

$$u(x,t)\big|_{x=0}=0,\quad u(x,t)\big|_{x=1}=0,\quad T(x,t)\big|_{x=0}=T_0,\quad T(x,t)\big|_{x=1}=T_0$$

$\lambda=1{,}2$, $\lambda_0=0{,}8$, $\alpha=0.05$, $\mu=0.5$, $\rho=0{,}9$, $Ce=3.5$, $T_0=90$, $h=0.1$, $\tau=0.01$, $\ell=1$.

The value of the displacement u(x, t) calculated the “sweep” method.

Table 1.

x / t	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1

0	0	0,3090	0,5877	0,8090	0,9510	1	0,9510	0,8090	0,5877	0,3090	0
0,01	0	0,3086	0,5870	0,8080	0,9499	0,9988	0,9499	0,8080	0,5870	0,3086	0
0,02	0	0,3075	0,5849	0,8051	0,9465	0,9952	0,9465	0,8051	0,5849	0,3075	0
0,03	0	0,3084	0,5750	0,8003	0,9408	0,9892	0,9408	0,8003	0,5703	0,3020	0
0,04	0	0,3043	0,5674	0,7936	0,9329	0,9809	0,9329	0,7936	0,5628	0,2979	0
0,05	0	0,2970	0,5647	0,7850	0,9229	0,9704	0,9229	0,7850	0,5647	0,2969	0
0,06	0	0,2929	0,5571	0,7746	0,9106	0,9575	0,9106	0,7746	0,5571	0,2928	0
0,07	0	0,2904	0,5424	0,7624	0,8963	0,9424	0,8963	0,7624	0,5379	0,2842	0
0,08	0	0,2834	0,5297	0,7485	0,8799	0,9252	0,8799	0,7484	0,5252	0,2772	0
0,09	0	0,2765	0,5214	0,7328	0,8616	0,9059	0,8615	0,7327	0,5213	0,2743	0
0,1	0	0,2713	0,5187	0,7156	0,8413	0,8845	0,8412	0,7154	0,5186	0,2712	0

The value of the displacement u(x, t) calculated the grid method.

Table 2.

(x T	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1
0	0	0,309	0,5877	0,809	0,951	1	0,951	0,809	0,5877	0,309	0
0,01	0	0,3086	0,5871	0,808	0,9499	0,9988	0,9499	0,808	0,5871	0,3086	0
0,02	0	0,3076	0,5852	0,8051	0,9465	0,9952	0,9465	0,8051	0,5852	0,3076	0
0,03	0	0,3059	0,582	0,8003	0,9408	0,9892	0,9408	0,8003	0,582	0,3059	0
0,04	0	0,3036	0,5775	0,7936	0,9329	0,9809	0,9329	0,7936	0,5775	0,3036	0
0,05	0	0,3005	0,5718	0,785	0,9228	0,9703	0,9228	0,785	0,5718	0,3005	0
0,06	0	0,2967	0,5647	0,7745	0,9105	0,9574	0,9105	0,7745	0,5647	0,2967	0
0,07	0	0,2922	0,5564	0,7622	0,8961	0,9422	0,8961	0,7622	0,5564	0,2922	0
0,08	0	0,287	0,5468	0,7482	0,8796	0,9249	0,8796	0,7482	0,5468	0,287	0
0,09	0	0,2811	0,5359	0,7325	0,8611	0,9055	0,8611	0,7325	0,5359	0,2811	0
0,1	0	0,2745	0,5238	0,715	0,8407	0,884	0,8407	0,715	0,5238	0,2745	0

The temperature T(x, t) values found using the “sweep” method.

Table 3.

t \ x	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1
0	90	90	90	90	90	90	90	90	90	90	90
0,01	90	90,01	90,01	90	90	89,99	89,99	89,98	89,98	89,98	90
0,02	90	90,06	90,05	90,03	90,01	89,98	89,96	89,93	89,92	89,93	90
0,03	90	90,12	90,12	90,08	90,02	89,97	89,91	89,86	89,83	89,85	90
0,04	90	90,21	90,21	90,14	90,05	89,94	89,84	89,76	89,71	89,75	90
0,05	90	90,31	90,33	90,22	90,07	89,91	89,76	89,64	89,57	89,62	90
0,06	90	90,43	90,45	90,31	90,11	89,88	89,66	89,49	89,4	89,48	90
0,07	90	90,56	90,6	90,42	90,14	89,84	89,55	89,32	89,21	89,32	90
0,08	90	90,69	90,76	90,53	90,18	89,79	89,42	89,13	89	89,15	90
0,09	90	90,83	90,92	90,66	90,23	89,75	89,29	88,93	88,77	88,97	90
0,1	90	90,97	91,1	90,79	90,28	89,69	89,14	88,71	88,53	88,78	90

The temperature T(x, t) values found using the grid method.

Table 4.

t \ x	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1
0	90	90	90	90	90	90	90	90	90	90	90
0,01	90	90,02	90,017	90,012	90,006	90	89,993	89,987	89,982	89,979	90
0,02	90	90,077	90,07	90,051	90,026	90	89,973	89,948	89,929	89,922	90
0,03	90	90,161	90,155	90,113	90,059	90	89,94	89,886	89,844	89,838	90
0,04	90	90,267	90,269	90,2	90,105	90	89,894	89,799	89,73	89,732	90
0,05	90	90,392	90,407	90,308	90,163	90	89,836	89,691	89,592	89,607	90
0,06	90	90,531	90,568	90,435	90,231	90	89,768	89,564	89,431	89,468	90
0,07	90	90,683	90,747	90,58	90,31	90	89,689	89,419	89,252	89,316	90
0,08	90	90,845	90,941	90,74	90,398	90	89,601	89,259	89,058	89,154	90

0,09	90	91,016	91,149	90,912	90,494	90	89,505	89,087	88,85	88,983	90
0,1	90	91,194	91,367	91,095	90,596	90	89,403	88,904	88,632	88,805	90

In the tables 1 – 4, the values of $u(x,t)$ and $T(x,t)$ depend on the position– xand time – tare shown. It can be seen from the tables that the numerical results obtained using the explicit grid and the "sweep" methods are close enough.

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